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Characterization of Strong Stability of Stationary Solutions of Nonlinear Programs with a Finite Number of Equality Constraints and an Abstract Convex Constraint

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B-418 Characterization of strong stability of stationary solutions of nonlinear programs with a finite number of equality constraints and an abstract convex constraint

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Abstract.

In this report we treat nonlinear programs $\mathbf{Pro}(f, h; K)$ having an objective function f, a finite number of equality constraints $h(\boldsymbol{x}) = (h_1(\boldsymbol{x}), \cdots, h_\ell(\boldsymbol{x})) = \mathbf{0}$, and an abstract convex constraint $\boldsymbol{x} \in K$ with its convex set K. Our particular interest is an algebraic criterion for a locally isolated stationary solution to be strong stable, in the sense of Kojima, under a Linear Independence Constraint Qualification condition defined to those programs. First, we introduce a simple sufficient condition for semismoothness of the Euclidean projector ρ_K^+ onto K. Semismoothness of the Euclidean projectors onto closed convex cones pointed at $\mathbf{0}$ follows directly from this sufficiency. Secondly, under the condition of semismoothness of ρ_K^+ and what we call the regular boundary condition for K, we characterize strong stability of a locally isolated stationary solution $\bar{\boldsymbol{x}}^+$, with $(\bar{\boldsymbol{x}}, \bar{\lambda})$ its associate stationary point, in terms of B-subderivative $\partial_B \psi(\bar{\boldsymbol{x}}, \bar{\lambda}; f, h)$ of some appropriately defined map $\psi(\boldsymbol{x}, \lambda; f, h)$ for programs $\mathbf{Pro}(f, h; K)$. This result is a generalization of the theory that Kojima developed in his famous paper. Thirdly, we state an explicit formula for the Jacobian of the Euclidean projector onto any closed convex set satisfying the C^2 stratification, and interpret the regular boundary condition in terms of principal curvatures of the stratum.

Key words.

Nonlinear programming, Stationary solution, Strong stability, Lipschitz map, Nonsmooth analysis, Variational inequality, Semismooth map, Inverse function theorem

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1 Introduction.

Through this study, we investigate strong stability of a locally isolated stationary solution of the following nonlinear programs with a finite number of equality constraints and an abstract convex constraint $\boldsymbol{x} \in K$, which we refer to as NPAC:

$$\mathbf{Pro}(f, h; K) \quad \left\| \begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in K \\ & h_i(\boldsymbol{x}) = 0 \quad (i = 1, \cdots, \ell) \end{array} \right\},$$

where K is a closed convex set in $\boldsymbol{R}^n$ and f, h_i $(i = 1, \cdots, \ell)$ are C^2 functions on $\boldsymbol{R}^n$. Then, $\bar{\boldsymbol{x}} \in K$ is called a stationary solution of program $\mathbf{Pro}(f, h; K)$ if both $-D_{\boldsymbol{x}} f(\bar{\boldsymbol{x}}) \in \boldsymbol{R} D_{\boldsymbol{x}} h(\bar{\boldsymbol{x}}) + \sigma_K(\bar{\boldsymbol{x}})^T$ and $h_i(\bar{\boldsymbol{x}}) = 0$ $(i = 1, \cdots, \ell)$ hold. Here, $D_{\boldsymbol{x}} f(\bar{\boldsymbol{x}})$ (resp. $D_{\boldsymbol{x}} h_i(\bar{\boldsymbol{x}})$) denotes the Jacobian of f (resp. h_i) at $\bar{\boldsymbol{x}}$, $\boldsymbol{R} D_{\boldsymbol{x}} h(\bar{\boldsymbol{x}})$ denotes the affine space spanned by $\{D_{\boldsymbol{x}} h_i(\bar{\boldsymbol{x}}) : i = 1, \cdots, \ell\}$, $\sigma_K(\bar{\boldsymbol{x}})$ is the normal cone of K at $\bar{\boldsymbol{x}}$, and $\sigma_K(\bar{\boldsymbol{x}})^T$ denotes the transpose of $\sigma_K(\bar{\boldsymbol{x}})$, i.e., $\sigma_K(\bar{\boldsymbol{x}})^T = \{\boldsymbol{w} : \boldsymbol{w}^T \in \sigma_K(\bar{\boldsymbol{x}})\}$. The stationary solution $\bar{\boldsymbol{x}}$ is defined to be strongly stable if there exist $\delta > 0$ such that, for any small perturbation (f', h') of (f, h), there exists a unique stationary solution $\boldsymbol{x}(f', h')$ of $\mathbf{Pro}(f', h'; K)$ that satisfies $\|\boldsymbol{x}(f', h') - \bar{\boldsymbol{x}}\| \leq \delta$, and the correspondence $(f', h') \mapsto \boldsymbol{x}(f', h')$ is continuous at (f, h).

Let ρ_K^+ denote the Euclidean projector onto K, $\boldsymbol{x}^+ = \rho_K^+(\boldsymbol{x})$, and $\boldsymbol{x}^- = \boldsymbol{x} - \boldsymbol{x}^+$. Given a map $F : \boldsymbol{R}^n \to \boldsymbol{R}^n$, the variational inequality VI(K, F) is to find a vector $\boldsymbol{x} \in \boldsymbol{R}^n$ such that $\mathbf{F}_K^{\mathrm{nor}}(\boldsymbol{x}) = 0$ with $\mathbf{F}_K^{\mathrm{nor}}(\boldsymbol{x}) = F(\boldsymbol{x}^+) + \boldsymbol{x}^-$. Under the assumption that ρ_K^+ is semismooth, it follows from Theorem 3 of [6] that a locally isolated solution $\bar{\boldsymbol{x}}$ of VI(K, F) is strongly stable if and only if the coherent orientation property holds for the B-subderivative $\partial_B \mathbf{F}_K^{\mathrm{nor}}(\bar{\boldsymbol{x}})$, i.e., sgn det$A$ is nonzero constant for any $A \in \partial_B \mathbf{F}_K^{\mathrm{nor}}(\bar{\boldsymbol{x}})$. Such a condition that is described in terms of B-subderivative is referred to as algebraic.

For any triplet $(f, h; K)$, there exists a map F and a closed convex set $\tilde{K}$ such that stationary solutions of $\mathbf{Pro}(f, h; K)$ and solutions of VI$(\tilde{K}, F)$ correspond bijectively. However, we cannot simply derive an algebraic criterion to characterize the stability for $\mathbf{Pro}(f, h; K)$ from the above criterion for VI$(\tilde{K}, F)$ (see Remark 4.3 for details).

In one paper [20], an analog of Kojima's approach [14] was constructed under the Linear Independence Constraint Qualification (abb. LICQ) condition defined to $\mathbf{Pro}(f, h; K)$ and an additional condition that we call the regular boundary condition for K. As a result, we derived that for every stongly stable stationary solution $\bar{\boldsymbol{x}}^+$ of $\mathbf{Pro}(f, h; K)$ with its associate stationary point $(\bar{\boldsymbol{x}}, \bar{\lambda})$, the B-subdifferential $\partial_B \psi(\bar{\boldsymbol{x}}, \bar{\lambda}; f, h)$ satisfies the coherent orientation property. In this study, we deduce from the implicit function theorem of Gowda [6] that this coherent orientation property is an algebraic criterion for stability in the case that the Euclidean projector ρ_K^+ is semismooth.

We also introduce a simple sufficient condition for semismoothness of ρ_K^+. Consequently, we prove that ρ_K^+ is semismooth for any closed convex cone K pointed at $\mathbf{0}$, especially both for the cone $S_+(n)$ of positive semidefinite symmetric matrices, and for any polyhedral cone pointed at $\mathbf{0}$. On the other hand, because we know from [19] that the regular boundary condition holds for $S_+(n)$, the above coherent orientation property determines the stability of a locally isolated stationary solution of $\mathbf{Pro}(f, h; S_+(n))$. A similar result holds for any polyhedral cone pointed at $\mathbf{0}$ because the Euclidean projector onto any polyhedron is semismooth.

In the final section, we investigate the regular boundary condition for any closed convex set $K \subset \boldsymbol{R}^n$ such that every stratum naturally defined from convexity of K is a C^2 submanifold of $\boldsymbol{R}^n$. We deduce an explicit formula of B-subderivative $\partial_B \rho_K^+(\bar{\boldsymbol{x}})$; we also interpret the regular boundary condition in more geometric terms, i.e., principal curvatures of the stratum.

In section 2,

- we define stationary solutions and strong stability and prepare a series of elementary results and facts; and
- under LICQ conditions defined to program $\mathbf{Pro}(f, h; K)$, we state one theorem that provides a necessary and sufficient condition for strong stability of stationary solutions by virtue of one-to-one maps.

In section 3,

- we prepare several kinds of derivatives for sections 4 and 5 and known results about them;
- we propose a simple condition sufficient for the Euclidean projector ρ_K^+ onto K to be semismooth; and as a result we deduce the semismoothness of ρ_K^+ in case of any closed convex cone K pointed at $\mathbf{0}$.

In section 4, under LICQ condition,

- we deduce the one necessary condition, which we call the coherent semiorientation property, for any stationary solution of $\mathbf{Pro}(f, h; K)$ to be strongly stable:
- in the case that the Euclidean projector ρ_K^+ is semismooth, we prove that the coherent orientation property is sufficient for a locally isolated stationary solution of $\mathbf{Pro}(f, h; K)$ to be strongly stable:
- in the case that the Euclidean projector ρ_K^+ is semismooth and the regular boundary condition holds for K, we prove that the coherent orientation property is an algebraic criterion for the stability of a locally isolated stationary solution.

In section 5,

- we treat a closed convex set $K \subset \boldsymbol{R}^n$ such that every stratum naturally defined from convexity of K is a C^2 submanifold of $\boldsymbol{R}^n$; we also deduce an explicit formula of B-subderivative $\partial_B \rho_K^+(\bar{\boldsymbol{x}})$, and
- we provide the more geometric interpretation of the regular boundary condition than the way of its definition.

2 Preliminaries.

In this preliminary discussion we define strong stability in the sense of Kojima and we prepare a series of elementary results and facts. For their preparation, we list notations used throughout this report:

$\boldsymbol{R}$	:	the field of all real numbers,
$\boldsymbol{R}^n$	:	the space of n dimensional real column vectors,
$\boldsymbol{R}_+$	$=$	$\{t \in \boldsymbol{R} :\ t \geq 0\ \}$,
$\boldsymbol{R}^n_+$	$=$	$\{(t_1, \cdots, t_n) \in \boldsymbol{R}^n :\ t_i \geq 0\ (1 \leq i \leq n)\ \}$,
$M(m,n)$	:	the set of all $m \times n$ real matrices,
$M(n)$	:	the set of all $n \times n$ real matrices,
$S(n)$	:	the set of all $n \times n$ symmetric real matrices,
$S_+(n)$	:	the set of all $n \times n$ positive semidefinite symmetric real matrices,
$S_{r,s}(n)$	:	the set of all $n \times n$ symmetric real matrices with r positive eigenvalues and s negative eigenvalues,
$S^*(n)$	:	the set of all $n \times n$ nonsingular symmetric real matrices,
$GL_+(n)$	:	the set of all $n \times n$ nonsingular real matrices with positive determinants,
$GL_-(n)$	:	the set of all $n \times n$ nonsingular real matrices with negative determinants,
$O(n)$	:	the set of all $n \times n$ orthogonal matrices,
$\boldsymbol{I}_r$	:	the $r \times r$ identity matrix, i.e., the identity map on $\boldsymbol{R}^r$,
$\boldsymbol{O}$	:	the zero matrix of an appropriate size,
$\langle \boldsymbol{x}, \boldsymbol{y} \rangle$	:	the standard inner product of $\boldsymbol{x},\ \boldsymbol{y} \in \boldsymbol{R}^n$,
$\mathbf{0}$	:	the zero vector of appropriate size,
$\boldsymbol{x}^T$	:	the transpose of the vector $\boldsymbol{x}$,
Z^T	$=$	$\{\boldsymbol{x}^T : \boldsymbol{x} \in Z\}$ for a set Z of vectors,
$\det C$	:	the determinant of a matrix C,
$\mathrm{sgn}\ t$	$=$	$\begin{cases} 1 & (t > 0) \\ 0 & (t = 0) \\ -1 & (t < 0) \end{cases}$,
$\mathrm{conv}(A)$	:	the convex hull of a subset A of a vector space V,
$\Vert \boldsymbol{x} \Vert$	$=$	$\sqrt{\sum_{i=1}^{n} \vert x_i \vert^2}$ for $\boldsymbol{x} = (x_1, \cdots, x_n)^T \in \boldsymbol{R}^n$, i.e., the Euclidean norm of $\boldsymbol{R}^n$,
$A^\perp$	$=$	$\{\boldsymbol{u} \in \boldsymbol{R}^n : \langle \boldsymbol{u}, \boldsymbol{a} \rangle = 0$ for any $\boldsymbol{a} \in A\}$ for any subset A of $\boldsymbol{R}^n$,
$A \perp B$	:	$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = 0$ for any $\boldsymbol{a} \in A,\ \boldsymbol{b} \in B,$ where $A,\ B \subset \boldsymbol{R}^n$,
$A \amalg B$	:	the disjoint union of subsets $A,\ B \subset \boldsymbol{R}^n$,
$A \setminus B$	$=$	$\{\boldsymbol{a} \in A :\ \boldsymbol{a} \notin B\}$ for subsets $A,\ B \subset \boldsymbol{R}^n$,
$A - B$	$=$	$\{\boldsymbol{a} - \boldsymbol{b} \in \boldsymbol{R}^n :\ \boldsymbol{a} \in A,\ \boldsymbol{b} \in B\}$ for subsets $A,\ B \subset \boldsymbol{R}^n$,
$\boldsymbol{R}A$	:	the linear subspace of $\boldsymbol{R}^n$ generated by A for a subset $A \subset \boldsymbol{R}^n$,
$int(A)$	:	the interior of A in $\boldsymbol{R}^n$ for a subset $A \subset \boldsymbol{R}^n$,
$c\ell(A)$	:	the closure of A in $\boldsymbol{R}^n$ for a subset $A \subset \boldsymbol{R}^n$,
$re\ell int(A)$	:	the interior of A in $\boldsymbol{R}A$ for a subset $A \subset \boldsymbol{R}^n$,
$\mathcal{F}$	$=$	$\{(f,h) = (f, h_1, \cdots, h_\ell) : f, h_1, \cdots, h_\ell \in C^2(\boldsymbol{R}^n)\}$,

where $C^2(\boldsymbol{R}^n)$ is the set of all functions on $\boldsymbol{R}^n$ of C^2 class,

$F|A$: the restriction of a map F to a subset A of the domain where F is defined,

$E_F = \{\boldsymbol{x} \in U : F$ is differentiable at $\boldsymbol{x}$, i.e., $D_{\boldsymbol{x}}F(\boldsymbol{x})$ exists at $\boldsymbol{x}\}$ for $F : U \to \boldsymbol{R}^m$, where U is the domain of F.

Throughout this report, K denotes a closed convex subset of $\boldsymbol{R}^n$ that is fixed. Let $\sigma_K(\boldsymbol{x})$ be the normal cone of K at $x \in K$, i.e., $\sigma_K(\boldsymbol{x}) = \{\boldsymbol{v} \in \boldsymbol{R}^n : \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{v}\rangle \le 0 \ (\forall \boldsymbol{y} \in K)\}$, and $\rho_K^+ : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the Euclidean projector onto K. Denote $\boldsymbol{x}^+ = \rho_K^+(\boldsymbol{x})$ and define $\boldsymbol{x}^-$ by $\boldsymbol{x}^- = \boldsymbol{x} - \boldsymbol{x}^+$. It is readily inferred that $\boldsymbol{x}^- \in \sigma_K(\boldsymbol{x}^+)$.

Definition 2.1. Let $(f, h) \in \mathcal{F}$. $D_{\boldsymbol{x}}f(\boldsymbol{x})$ and $D_{\boldsymbol{x}}h(\boldsymbol{x})$ respectively denote the Jacobians of $f(\boldsymbol{x})$ and $h(\boldsymbol{x})$. In addition, $\boldsymbol{R}D_{\boldsymbol{x}}h(\boldsymbol{x}) = \sum_{i=1}^{\ell} \boldsymbol{R}D_{\boldsymbol{x}}h_i(\boldsymbol{x})$ denotes the affine space spanned by $\{D_{\boldsymbol{x}}h_i(\boldsymbol{x}) : i = 1, \cdots, \ell\}$. Then $(\bar{\boldsymbol{x}}, \bar{\lambda})$ is called a stationary point of program $\mathbf{Pro}(f, h; K)$ if both $D_{\boldsymbol{x}}f(\bar{\boldsymbol{x}}^+) + \sum_{i=1}^{\ell} \bar{\lambda}_i D_{\boldsymbol{x}}h_i(\bar{\boldsymbol{x}}^+) + (\bar{\boldsymbol{x}}^-)^T = \mathbf{0}$ and $h_i(\bar{\boldsymbol{x}}^+) = 0$ $(i = 1, \cdots, \ell)$ hold. If $(\bar{\boldsymbol{x}}, \bar{\lambda})$ is a stationary point, then $\bar{\boldsymbol{x}}^+$ is called a stationary solution of $\mathbf{Pro}(f, h; K)$.

Following are some notations used in the remainder of this report. For $(f, h) \in \mathcal{F}$, we define $L(\cdot, \cdot; f, h) : \boldsymbol{R}^{n+\ell} \to \boldsymbol{R}$, $\psi(\cdot, \cdot; f, h) : \boldsymbol{R}^{n+\ell} \to \boldsymbol{R}^{n+\ell}$, $\Omega \subset \boldsymbol{R}^{n+\ell} \times \mathcal{F}$, $\Xi \subset \boldsymbol{R}^n \times \mathcal{F}$ and $\chi : \Omega \to \Xi$ as follows.

$$
\begin{aligned}
L(\boldsymbol{x}, \lambda; f, h) &= f(\boldsymbol{x}) + \sum_{i=1}^{\ell} \lambda_i h_i(\boldsymbol{x}), \\
\psi(\boldsymbol{x}, \lambda; f, h) &= (D_{\boldsymbol{x}}L(\boldsymbol{x}^+, \lambda; f, h) + (\boldsymbol{x}^-)^T, D_{\lambda}L(\boldsymbol{x}^+, \lambda; f, h)) \\
&= (D_{\boldsymbol{x}}f(\boldsymbol{x}^+) + \sum_{i=1}^{\ell} \lambda_i D_{\boldsymbol{x}}h_i(\boldsymbol{x}^+) + (\boldsymbol{x}^-)^T, h(\boldsymbol{x}^+)), \\
\Omega &= \{(\boldsymbol{x}, \lambda, f, h) \in \boldsymbol{R}^{n+\ell} \times \mathcal{F} : (\boldsymbol{x}, \lambda) \text{ be a stationary point of } \mathbf{Pro}(f, h; K)\} \\
&= \{(\boldsymbol{x}, \lambda, f, h) \in \boldsymbol{R}^{n+\ell} \times \mathcal{F} : \psi(\boldsymbol{x}, \lambda, f, h) = \mathbf{0}\}, \\
\Xi &= \{(\boldsymbol{x}, f, h) \in \boldsymbol{R}^n \times \mathcal{F} : \boldsymbol{x} \text{ is a stationary solution of } \mathbf{Pro}(f, h; K)\}, \\
\chi(\boldsymbol{x}, \lambda, f, h) &= (\boldsymbol{x}^+, f, h), \textit{i.e.}, \chi : \Omega \to \Xi \text{ is a natural projection.}
\end{aligned}
$$

For $f \in C^2(\boldsymbol{R}^n)$ and a subset $B \subset \boldsymbol{R}^n$, a norm $\|f\|_B$ is defined as

$$\|f\|_B = \sup\{|f(\boldsymbol{x})|, \|D_{\boldsymbol{x}}f(\boldsymbol{x})\|, \|D_{\boldsymbol{x}}^2 f(\boldsymbol{x})\| : \boldsymbol{x} \in B\}.$$

For $(f, h) \in \mathcal{F}$ and a subset $B \subset \boldsymbol{R}^n$, a norm $\|\cdot\|_B$ is defined as

$$\|(f, h)\|_B = \max\{\|f(\boldsymbol{x})\|_B \ , \ \|h_i(\boldsymbol{x})\|_B : 1 \le i \le \ell\}.$$

We denote by $\mathcal{F}_B$ the space $\mathcal{F}$ with $\|\cdot\|_B$-topology and $B_\delta((f, h); B)$ denotes a closed ball centered at (f, h) with radius δ in $\mathcal{F}_B$, i.e., $B_\delta((f, h); B) = \{(f', h') \in \mathcal{F} : \|(f', h') - (f, h)\|_B \le \delta\}$.

In general, given a normed vector space V with its norm $\|\cdot\|$, we define a closed ball and an open ball by $B_\delta(\boldsymbol{x}) = \{\boldsymbol{y} \in V : \|\boldsymbol{y} - \boldsymbol{x}\| \le \delta\}$ and $int(B_\delta(\boldsymbol{x})) = \{\boldsymbol{y} \in V : \|\boldsymbol{y} - \boldsymbol{x}\| < \delta\}$ for $\boldsymbol{x} \in V$ and a positive real number $\delta > 0$.

Definition 2.2. (see [11], [14]) Let $\bar{\boldsymbol{x}} \in \boldsymbol{R}^n$ be a stationary solution of $\mathbf{Pro}(\bar{f}, \bar{h}; K)$. $\bar{\boldsymbol{x}}$ is said to be strongly stable if there exist neighborhoods $U = B_\delta(\bar{\boldsymbol{x}})$ of $\bar{\boldsymbol{x}}$ in $\boldsymbol{R}^n$ and V of $(\bar{f}, \bar{h})$ in $\mathcal{F}_U$ such that the natural projection $pr : \Xi \bigcap (U \times V) \to V$ is bijective and $pr^{-1} : V \to \Xi \bigcap (U \times V)$ is continuous at $(\bar{f}, \bar{h})$.

We refer to the following condition as the LICQ condition 2.3 because, under the condition, each stationary solution corresponds to a unique stationary point and this condition takes a role in program $\mathbf{Pro}(f, h; K)$ just as the LICQ condition does in the setting of [14].

Condition 2.3.

(i) For any $\boldsymbol{x} \in \boldsymbol{R}^n$, $D_{\boldsymbol{x}} h_i(\boldsymbol{x})$ $(1 \leq i \leq \ell)$ are linearly independent.

(ii) For any $\boldsymbol{x} \in K$ with $h(\boldsymbol{x}) = \boldsymbol{0}$, $\boldsymbol{R} D_{\boldsymbol{x}} h(\boldsymbol{x}) \bigcap \boldsymbol{R} \sigma_K(\boldsymbol{x})^T = \{\boldsymbol{0}\}$.

Under the LICQ condition 2.3, the following proposition holds.

Proposition 2.4. *(see [18]) Under the LICQ condition 2.3, for any subset $U \subset \boldsymbol{R}^n$, $\chi : \Omega \bigcap ((\rho_K^+)^{-1}(U) \times \boldsymbol{R}^\ell \times \mathcal{F}_U) \to \Xi \bigcap (U \times \mathcal{F}_U)$ is a homeomorphism.*

Definition 2.5. Under the LICQ condition 2.3, we refer to a stationary point $(\boldsymbol{x}, \lambda)$ of $\mathbf{Pro}(f, h; K)$ as strongly stable if and only if $\boldsymbol{x}^+$ is a strongly stable stationary solution of $\mathbf{Pro}(f, h; K)$.

We can prove the following theorem that gives an equivalent condition for strong stability under the LICQ condition 2.3. This theorem plays an important role to prove the sufficiency of a condition for stability in proof of Theorems 4.2.

Theorem 2.6. *(see [20]) Suppose that the LICQ condition 2.3 holds. Let $(\bar{f}, \bar{h}) \in \mathcal{F}$ and $(\bar{\boldsymbol{x}}, \bar{\lambda}) \in \boldsymbol{R}^{n+\ell}$ be a stationary point of $\boldsymbol{Pro}(\bar{f}, \bar{h}; K)$. Then the following (i) and (ii) are equivalent.*

(i) $(\bar{\boldsymbol{x}}, \bar{\lambda})$ is strongly stable.

(ii) There exist neighborhoods $U = B_{\delta^*}(\bar{\boldsymbol{x}}^+)$ of $\bar{\boldsymbol{x}}^+$ in $\boldsymbol{R}^n$ and $W = B_\delta((\bar{\boldsymbol{x}}, \bar{\lambda}))$ of $(\bar{\boldsymbol{x}}, \bar{\lambda})$ with $W \subset (\rho^+)^{-1}(U) \times \boldsymbol{R}^\ell$ satisfying the following two conditions.

- *(a) $\bar{\boldsymbol{x}}^+$ is a unique stationary solution in U for $\boldsymbol{Pro}(\bar{f}, \bar{h}; K)$.*
- *(b) $V = \{(f, h) \in \mathcal{F} : \psi(\cdot, \cdot; f, h)$ is one-to-one on $W\}$ is a neighborhood of $(\bar{f}, \bar{h})$ in $\mathcal{F}_U$.*

3 Several Kinds of Derivatives and a Sufficient Condition for Semismoothness of Euclidean Projectors.

In this section, we introduce several kinds of derivatives and known results about them in preparation for later sections. In the last part of this section, we introduce a simple condition about the B-subderivative that suffices semismoothness for locally Lipschitz maps. This condition is, in some sense, sufficiently reasonable that, in the case that K is a closed convex cone K pointed at $\mathbf{0}$, the Euclidean projector ρ_K^+ satisfies the condition and therefore follows semismoothness of ρ_K^+. As an example, semismoothness of $\rho_{S_+(n)}^+$, which is well known by Lemma 4.12 of [24] follows immediately because $S_+(n)$ is a closed convex cone pointed at $\boldsymbol{O}$.

Definition 3.1. Let V_1 and V_2 be normed vector spaces with their norms denoted by $\|\cdot\|$, U are an open subset of V_1, and $f : U \to V_2$ be a map.

(i) In that case, f is called Lipschitz continuous if there exists a constant M such that $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| \leq M\|\boldsymbol{x} - \boldsymbol{y}\|$ for any $\boldsymbol{x},\ \boldsymbol{y} \in U$. This constant M is called a modulus of a Lipschitz continuous map f.

(ii) f is called locally Lipschitz continuous if for any $\boldsymbol{x} \in U$ there exists an open neighborhood $W \subset U$ of $\boldsymbol{x}$ such that $f|W : W \to V_2$ is Lipschitz continuous.

(iii) If f is Lipschitz in a neighborhood of $\bar{\boldsymbol{x}}$ and f^{-1} is Lipschitz in a neighborhood of $f(\bar{\boldsymbol{x}})$, then f is called Lipschitz homeomorphic around $\bar{\boldsymbol{x}}$.

The operator ρ_K^+ is well known as Lipschitz continuous with its modulus 1. It satisfies the inequality $\|\rho_K^+(\boldsymbol{x}) - \rho_K^+(\boldsymbol{y})\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$ (see [4], [26]).

Definition 3.2. Let U be an open subset of $\boldsymbol{R}^m$ and $f : U \to \boldsymbol{R}^n$ be locally Lipschitz continuous. f is called Bouligand-differentiable (B-differentiable) at $\bar{\boldsymbol{x}}$ if there exists a positively homogeneous map $f'(\bar{\boldsymbol{x}};\cdot) : \boldsymbol{R}^m \to \boldsymbol{R}^n$ such that $\lim\limits_{\boldsymbol{x}\to\bar{\boldsymbol{x}}} \dfrac{f(\boldsymbol{x}) - f(\bar{\boldsymbol{x}}) - f'(\bar{\boldsymbol{x}};\boldsymbol{x}-\bar{\boldsymbol{x}})}{\|\boldsymbol{x}-\bar{\boldsymbol{x}}\|} = \mathbf{0}$. In this case, $f'(\bar{\boldsymbol{x}};\cdot)$ is revealed as locally Lipschitz continuous in the second variable and is called the B-derivative of f at $\bar{\boldsymbol{x}}$.

Definition 3.3. Let U be an open subset of $\boldsymbol{R}^m$ and $f : U \to \boldsymbol{R}^n$.

(1) f is called directionally differentiable at $\bar{\boldsymbol{x}}$ in the direction $\boldsymbol{v}$ if $\lim\limits_{t\to+0} \dfrac{f(\bar{\boldsymbol{x}}+t\boldsymbol{v}) - f(\bar{\boldsymbol{x}})}{t}$ exists. That limit is called the directional derivative of f at $\bar{\boldsymbol{x}}$ in the direction $\boldsymbol{v}$.

(2) f is called directionally differentiable at $\bar{\boldsymbol{x}}$ if f is directionally differentiable at $\bar{\boldsymbol{x}}$ in the direction $\boldsymbol{v}$ for any $\boldsymbol{v} \in \boldsymbol{R}^n$. In this case, $(df|_{\bar{\boldsymbol{x}}})(\cdot)$ denotes the map $\boldsymbol{v} \mapsto \lim\limits_{t\to+0} \dfrac{f(\bar{\boldsymbol{x}}+t\boldsymbol{v}) - f(\bar{\boldsymbol{x}})}{t}$ and is called the directional derivative of f at $\bar{\boldsymbol{x}}$.

Remark 3.4. (see [23]) Let U be an open subset of $\boldsymbol{R}^m$ and $f : U \to \boldsymbol{R}^n$ be locally Lipschitz continuous. Then it is provable with little difficulty that f is B-differentiable at $\bar{\boldsymbol{x}}$ if and only if f is directionally differentiable at $\bar{\boldsymbol{x}}$. In this case, $f'(\bar{\boldsymbol{x}};\cdot) = (df|_{\bar{\boldsymbol{x}}})(\cdot)$ holds.

Before we state the next definition, we remark that any locally Lipschitz continuous map is differentiable almost everywhere in the sense of Lebesgue measure using Rademacher's Theorem (see [5]).

Definition 3.5. Let U be an open neighborhood of $\bar{\boldsymbol{x}}$ in $\boldsymbol{R}^n$ and f be a locally Lipschitz continuous map from U to $\boldsymbol{R}^m$. Then the B-subderivative $\partial_B f(\bar{\boldsymbol{x}})$ of f at $\bar{\boldsymbol{x}}$ is defined by $\partial_B f(\bar{\boldsymbol{x}}) = \{\ \lim_{k\to\infty} D_{\boldsymbol{x}} f(\boldsymbol{x}_k) : \boldsymbol{x}_k \in E_f, (k = 1, 2, \cdots) \text{ and } \lim_{k\to\infty} \boldsymbol{x}_k = \bar{\boldsymbol{x}}\ \}$.

The following is well known.

Fact 3.6. *(see [3], [6]) If f is a locally Lipschitz map, then $\partial_B f(\boldsymbol{x})$ is a compact set; furthermore, $\boldsymbol{x} \mapsto \partial_B f(\boldsymbol{x})$ is upper semicontinuous at every $\boldsymbol{x}$. That is, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\partial_B f(B_\delta(\boldsymbol{x})) = \bigcup\{\partial_B f(\boldsymbol{x}') : \boldsymbol{x}' \in B_\delta(\boldsymbol{x})\} \subset \partial_B f(\boldsymbol{x}) + B_\epsilon(\boldsymbol{0})$ holds.*

There exists another concept of derivative which is called the generalized Jacobian in the sense of Clarke.

Definition 3.7. The generalized Jacobian $\partial f(\bar{\boldsymbol{x}})$ of f at $\bar{\boldsymbol{x}}$ is defined to be a convex hull of the B-subderivative, i.e., $\partial f(\bar{\boldsymbol{x}}) = \text{conv}\ \partial_B f(\bar{\boldsymbol{x}})$.

The generalized Jacobian is "blind" to sets of Lebesgue measure zero as described in the following remark. On the other hand we do not know whether this blindness holds for the B-subderivative.

Remark 3.8. (see [3], [25]) For any locally Lipschitz map defined on U and any subset $\mathcal{N} \subset U$ of Lebesgue measure zero, the following equality holds:

$$\partial f(\bar{\boldsymbol{x}}) = \{\ \lim_{k\to\infty} D_{\boldsymbol{x}} f(\boldsymbol{x}_k) : \boldsymbol{x}_k \in E_f \setminus \mathcal{N}, (k = 1, 2, \cdots) \text{ and } \lim_{k\to\infty} \boldsymbol{x}_k = \bar{\boldsymbol{x}}\ \}.$$

In the following remark, we state some corrections to our previous documents, [19] and [20], that do not markedly affect their results.

Remark 3.9. (i) The following type of chain rule for generalized Jacobians is readily inferred.

> *Let U and V be open subsets of $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$ respectively, and $f : V \to \boldsymbol{R}^\ell$ and $g : U \to V$. If f is differentiable at $\bar{\boldsymbol{y}} = g(\bar{\boldsymbol{x}})$, then the chain rule for generalized Jacobians of f and g holds at $\bar{\boldsymbol{x}}$. That is, $\partial(f \circ g)(\bar{\boldsymbol{x}}) = D_y f(\bar{\boldsymbol{y}})\partial g(\bar{\boldsymbol{x}})$ holds.*

(ii) In Remark 4.3 of [20] we stated that the chain rule of generalized Jacobians holds, but that is false. Nevertheless, all assertions of [20] are still correct because some kind of chain rule holds for generalized Jacobians, as explained in (i). Therefore, we present the following relation

$$\partial_{(\boldsymbol{x},\lambda)}\psi(\boldsymbol{x}, \lambda; f, h) = \left\{ \begin{pmatrix} D_{\boldsymbol{x}}^2 L(\boldsymbol{x}^+, \lambda; f, h)C_+ + C_- & (D_{\boldsymbol{x}} h(\boldsymbol{x}^+))^T \\ D_{\boldsymbol{x}} h(\boldsymbol{x}^+)C_+ & \boldsymbol{O} \end{pmatrix} : C \in \partial_{\boldsymbol{x}}\rho(\boldsymbol{x}) \right\}$$

because of

$$\begin{aligned} \psi(\boldsymbol{x}, \lambda; f, h) &= (D_{\boldsymbol{x}} f(\boldsymbol{x}^+) + \textstyle\sum_{i=1}^{\ell} \lambda_i D_{\boldsymbol{x}} h_i(\boldsymbol{x}^+) + \boldsymbol{x}^-, h(\boldsymbol{x}^+)) \\ &= (\boldsymbol{x}, \boldsymbol{0}) + (D_{\boldsymbol{x}} f(\boldsymbol{x}^+) + \textstyle\sum_{i=1}^{\ell} \lambda_i D_{\boldsymbol{x}} h_i(\boldsymbol{x}^+) - \boldsymbol{x}^+, h(\boldsymbol{x}^+)). \end{aligned}$$

We require the concept of strict differentiability in section 5.

Definition 3.10. Let U be an open subset of $\boldsymbol{R}^n$ and $F : U \to \boldsymbol{R}^m$ be a locally Lipschitz map. Then F is called strictly differentiable at $\bar{\boldsymbol{x}} \in U$ if there exists a matrix $A \in M(m, n)$ such that $\lim_{\substack{\boldsymbol{u}\to 0 \\ t\downarrow 0}} \dfrac{\rho_K^+(\bar{\boldsymbol{x}}+\boldsymbol{u}+t\boldsymbol{v}) - \rho_K^+(\bar{\boldsymbol{x}}+\boldsymbol{u})}{t} = A\boldsymbol{v}$ holds for any $\boldsymbol{v} \in \boldsymbol{R}^n$.

It is readily inferred that strict differentiability implies directional differentiability. It follows from Proposition 2.2.4 of [3] that strict differentiability can be characterized using the B-subderivative.

Proposition 3.11. *(see [3]) Let U be an open subset of $\boldsymbol{R}^n$, $F : U \to \boldsymbol{R}^m$ be a locally Lipschitz map, and $\bar{\boldsymbol{x}} \in U$. Then F is strictly differentiable at $\bar{\boldsymbol{x}}$ if and only if $\partial_B F(\bar{\boldsymbol{x}})$ is a singleton, i.e., a set consisting of only one element.*

The concept of degree plays an important role in study of stability.

Definition 3.12. (see [9]) Let U be an open subset of $\boldsymbol{R}^n$ and $\boldsymbol{x} \in U$. For a continuous map $f : U \to \boldsymbol{R}^n$ satisfying that there exists a positive real number $\delta > 0$ such that $B_\delta(\boldsymbol{x}) \bigcap f^{-1}(f(\boldsymbol{x})) = \{\boldsymbol{x}\}$, the map degree of Brouwer $\deg(\boldsymbol{x}; f)$ is definable.

The following property of degree is important.

Fact 3.13. *(see pp.130-132 of [4]) Let U be an open subset of $\boldsymbol{R}^n$, and $f : U \to \boldsymbol{R}^n$ be a continuous map, and $\bar{\boldsymbol{x}} \in U$. Suppose that $f^{-1}(f(\bar{\boldsymbol{x}})) = \{\bar{\boldsymbol{x}}\}$ and that f is differentiable at $\bar{\boldsymbol{x}}$ and that* $\det D_{\boldsymbol{x}} f(\bar{\boldsymbol{x}}) \neq 0$. *In that case,* $\deg(\bar{\boldsymbol{x}}; f) = \mathrm{sgn}\,\det\, D_{\boldsymbol{x}} f(\bar{\boldsymbol{x}})$ *holds.*

We require the concept of semismooth maps in the sense of Gowda.

Definition 3.14. (see [6]) Let U be an open subset of $\boldsymbol{R}^n$ and $f : U \to \boldsymbol{R}^m$ be a locally Lipschitz continuous map. In that situation, f is called semismooth at $\bar{\boldsymbol{x}} \in U$ if $f(\boldsymbol{x}_k) - f(\bar{\boldsymbol{x}}) - A_k(\boldsymbol{x}_k - \bar{\boldsymbol{x}}) = o(\|\boldsymbol{x}_k - \bar{\boldsymbol{x}}\|)$ holds for any sequence $\boldsymbol{x}_k \to \bar{\boldsymbol{x}}$ and any $A_k \in \partial_B f(\boldsymbol{x}_k)$.

To prove Theorem 4.2 we need the following important result of Gowda [6] for semismooth maps referred to as Proposition 3.15.

Proposition 3.15. *(see [6]) Let W be an open subset of $\boldsymbol{R}^n$ and $\bar{\boldsymbol{x}} \in W$. Suppose that $f : W \to \boldsymbol{R}^n$ is semismooth and that* $\deg(\bar{\boldsymbol{x}}; f) = 1$ *(or,* -1*) and* $\mathrm{sgn}\,\det A = \deg(\bar{\boldsymbol{x}}; f)$ *holds for any $A \in \partial_B f(\bar{\boldsymbol{x}})$. Then f has a local semismooth inverse at $\bar{\boldsymbol{x}}$, i.e., there exists an open neighborhood $U \subset W$ of $\bar{\boldsymbol{x}}$ such that $V = f(U)$ is an open subset of $\boldsymbol{R}^n$, $f|U : U \to V$ is bijective, and the inverse $(f|U)^{-1} : V \to U$ is semismooth on V. Moreover, $\partial_B f^{-1}(f(\bar{\boldsymbol{x}})) = \{A^{-1} : A \in \partial_B f(\bar{\boldsymbol{x}})\}$ holds.*

For a locally Lipschitz map f to be semismooth, the following condition is sufficient. We define $\partial_B f(\boldsymbol{x})\boldsymbol{y}$ as $\partial_B f(\boldsymbol{x})\boldsymbol{y} = \{C\boldsymbol{y} : C \in \partial_B f(\boldsymbol{x})\}$ and $o(\cdot)$ stands for Landau's small o.

Condition 3.16. $\partial_B f(\boldsymbol{x})\boldsymbol{x} = \{f(\boldsymbol{x})\}$ *for any $\boldsymbol{x} \in \boldsymbol{R}^n$.*

We can deduce the following proposition.

Proposition 3.17. *Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be a locally Lipschitz map that satisfies Condition 3.16. Then, f is semismooth on $\boldsymbol{R}^n$.*

Proof: We have the following calculation:

$$\begin{aligned} f(\boldsymbol{x}) - f(\boldsymbol{y}) - \partial_B f(\boldsymbol{x})(\boldsymbol{x}-\boldsymbol{y}) &= f(\boldsymbol{x}) - f(\boldsymbol{y}) - \partial_B f(\boldsymbol{x})\boldsymbol{x} + \partial_B f(\boldsymbol{x})\boldsymbol{y} \\ &= -f(\boldsymbol{y}) + \partial_B f(\boldsymbol{x})\boldsymbol{y} \\ &= (\partial_B f(\boldsymbol{x}) - \partial_B f(\boldsymbol{y}))\boldsymbol{y}. \end{aligned}$$

With the above identity $f(\boldsymbol{x}) - f(\boldsymbol{y}) - \partial_B f(\boldsymbol{x})(\boldsymbol{x}-\boldsymbol{y}) = (\partial_B f(\boldsymbol{x}) - \partial_B f(\boldsymbol{y}))\boldsymbol{y}$, semismoothness of f follows from upper semicontinuity of $\partial_B f(\boldsymbol{x})$ stated in Fact 3.6. ∎

Definition 3.18. A subset $K \subset \boldsymbol{R}^n$ is called a closed convex cone pointed at $\boldsymbol{0}$ if it is a closed convex set satisfying the condition of $t\boldsymbol{x} \in K$ for any $\boldsymbol{x} \in K$ and $t \geq 0$.

Proposition 3.19. *Let $K \subset \boldsymbol{R}^n$ be a closed convex cone pointed at $\boldsymbol{0}$. Then the Euclidean projector ρ_K^+ onto K satisfies Condition 3.16. Therefore, ρ_K^+ is semismooth on $\boldsymbol{R}^n$.*

Proof: K is a closed convex cone pointed at $\boldsymbol{0}$. For that reason, $\rho_K^+(t\boldsymbol{x}) = t\rho_K^+(\boldsymbol{x})$ holds for any $\boldsymbol{x} \in \boldsymbol{R}^n$ and $t \geq 0$. Let $\bar{\boldsymbol{x}} \in E_{\rho_K^+}$. Then $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}})\bar{\boldsymbol{x}} = \lim_{t\to+0} \frac{\rho_K^+(\bar{\boldsymbol{x}}+t\bar{\boldsymbol{x}})-\rho_K^+(\bar{\boldsymbol{x}})}{t} = \lim_{t\to+0} \frac{\rho_K^+((1+t)\bar{\boldsymbol{x}})-\rho_K^+(\bar{\boldsymbol{x}})}{t} = \lim_{t\to+0} \frac{(1+t)\rho_K^+(\bar{\boldsymbol{x}})-\rho_K^+(\bar{\boldsymbol{x}})}{t} = \rho_K^+(\bar{\boldsymbol{x}})$. Next, presume that $\bar{\boldsymbol{x}} \notin E_{\rho_K^+}$ and $C_+ \in \partial_B\rho_K^+(\bar{\boldsymbol{x}})$. Then, from definition of B-subderivative, there exists a sequence $\boldsymbol{x}_k \in E_{\rho_K^+}$ $(k = 1, 2, \cdots)$ converging $\bar{\boldsymbol{x}}$ with $\lim_{k\to\infty} D_{\boldsymbol{x}}\rho_K^+(\boldsymbol{x}_k) = C_+$. From continuity of the matrix multiplication we can calculate $C_+\bar{\boldsymbol{x}} = \lim_{k\to\infty} D_{\boldsymbol{x}}\rho_K^+(\boldsymbol{x}_k) \cdot \lim_{k\to\infty} \boldsymbol{x}_k = \lim_{k\to\infty} D_{\boldsymbol{x}}\rho_K^+(\boldsymbol{x}_k)\boldsymbol{x}_k = \lim_{k\to\infty} \rho_K^+(\boldsymbol{x}_k) = \rho_K^+(\bar{\boldsymbol{x}})$. Therefore, ρ_K^+ satisfies Condition 3.16. In addition, it follows from Proposition 3.17 that ρ_K^+ is semismooth. ∎

$S_+(n)$ is a closed convex cone pointed at $\boldsymbol{O}$. Therefore, it follows from Proposition 3.19 that $\rho_{S_+(n)}^+$ is semismooth.

4 Characterization for Strong Stability of the Stationary Solution of Pro$(f, h; K)$.

In this section, we investigate the relation between strong stability of a locally isolated stationary solution of **Pro**$(f, h; K)$ and B-subderivative $\partial_B\psi(\boldsymbol{x}, \lambda; f, h)$ of $\psi(\cdot, \cdot; f, h)$ under the LICQ condition 2.3. We deduce one necessary condition for stability of $(\bar{\boldsymbol{x}}, \bar{\lambda})$ in terms of $\partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; f, h)$. On the other hand, if the Euclidean projector ρ_K^+ is semismooth, we deduce another sufficient condition for the stability. Moreover, under the LICQ condition 2.3 and an additional condition that we will call the regular boundary condition 4.4, we prove that this sufficient condition is necessary for stability, i.e., it stands for an algebraic criterion to the stability. In either the case where K is a polyhedral cone pointed at $\boldsymbol{0}$, or the case where $K = S_+(n)$, the Euclidean projector ρ_K^+ is semismooth and the regular boundary condition holds for K. Therefore, this criterion is effective for these two cases. This characterization is a complete generalization of Kojima's theory stated in Theorem 3.3 and Corollary 4.3 of [14].

Where $\bar{\boldsymbol{x}}^+$ is a stationary solution of $\mathbf{Pro}(\bar{f}, \bar{h}; K)$ with $(\bar{\boldsymbol{x}}, \bar{\lambda})$ the associate stationary point, we consider the relation between the following statements. We notice that for a locally isolated stationary solution $\bar{\boldsymbol{x}}^+$ of $\mathbf{Pro}(\bar{f}, \bar{h}; K)$ and the associate stationary point $(\bar{\boldsymbol{x}}, \bar{\lambda})$, we can introduce $\deg((\bar{\boldsymbol{x}}, \bar{\lambda}); \psi(\cdot, \cdot; \bar{f}, \bar{h}))$. We abbreviate this degree as $\deg(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$.

(a) $\bar{\boldsymbol{x}}^+$ is a strongly stable stationary solution of $\mathbf{Pro}(\bar{f}, \bar{h}; K)$:

(b1) sgn $\det A = \deg(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h}) \neq 0$ for any $A \in \partial\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$:

(b2) sgn det $A = \deg(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h}) \neq 0$ for any $A \in \partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$:

(b3) sgn $\det A = 0$ or, $= \deg(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$ for any $A \in \partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$:

(c) $\psi(\boldsymbol{x}, \lambda; \bar{f}, \bar{h})$ is one-to-one in the neighborhood of $(\bar{\boldsymbol{x}}, \bar{\lambda})$:

(d) $\bar{\boldsymbol{x}}^+$ is a locally isolated stationary solution of $\mathbf{Pro}(\bar{f}, \bar{h}; K)$.

We refer to (b1) as the nonsingularity of $\partial\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$, to (b2) as the coherent orientation property of $\partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$, and to (b3) as the coherent semiorientation property of $\partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$.

Remark 4.1. It is readily inferred that the implications (b1)⇒(b2)⇒(b3) always hold. The implication (b1)⇒(a) follows from the Implicit Function Theorem proved by Jongen, Klatte, and Tammer [10]. The implication (c)⇒(d) always holds. On the other hand, the implications (b1)⇔(b2) (i.e., the equivalence of (b1) and (b2)) also holds in case $K = \boldsymbol{R}^m_+ \times \boldsymbol{R}^{n-m} = \{\boldsymbol{x} \in \boldsymbol{R}^n : x_i \geq 0 \ (i = 1, \cdots, m)\}$ by Theorem 3.1 of [10] or Corollary 3.5 of [11].

We use, for simplicity, the notation (b2)+(d) to the effect that both (b2) and (d) are satisfied. Similarly, we use the notation (b3)+(c). Under the LICQ condition 2.3 the following theorem states the relation of (a), (b2)+(d), and (b3)+(c). In the proof of the following theorem, we deduce from Theorem 2.6 that the implication (a)⇒(b3)+(c) holds in general. Supposing the semismoothness of ρ^+_K, we also deduce the implication (b2)+(d)⇒(a) by virtue of Proposition 3.15.

Theorem 4.2. *Suppose that the LICQ condition 2.3 holds. Let* $(\bar{\boldsymbol{x}}, \bar{\lambda})$ *be a stationary point of* $\boldsymbol{Pro}(\bar{f}, \bar{h}; K)$*. Consequently, (a)⇒(b3)+(c) always holds. On the other hand, (b2)+(d)⇒(a) also holds under the assumption that* ρ^+_K *(therefore,* $\psi(\boldsymbol{x}, \lambda; \bar{f}, \bar{h})$*) is semismooth.*

Proof: (a)⇒(b3)+(c): We will prove this implication by deduction of a contradiction about values of $\deg(\boldsymbol{x}, \lambda; \bar{f}, \bar{h})$. It follows immediately from Theorem 2.6 that (a)⇒(c). Let $s = \deg(\boldsymbol{x}, \lambda; \bar{f}, \bar{h})$ $(= \pm 1)$. Suppose that (b3) does not hold, i.e., that there exists $A \in \partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$ with sgn $\det A = -s$. If that were true, then there exists $(\boldsymbol{x}', \lambda')$ sufficiently near $(\bar{\boldsymbol{x}}, \bar{\lambda})$ such that sgn det $D_{\boldsymbol{x}}\psi(\boldsymbol{x}', \lambda'; \bar{f}, \bar{h}) = -s$. From Fact 3.13, it follows that $\deg(\boldsymbol{x}', \lambda'; \bar{f}, \bar{h}) = -s$, which contradicts that $\deg(\boldsymbol{x}, \lambda; \bar{f}, \bar{h}) = s$.
(b2)+(d)⇒(a): It follows from (b2) that sgn $\det A = s$ for any $A \in \partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$. Let

$p = \begin{cases} + & (s=1) \\ - & (s=-1) \end{cases}$. Then $\partial_B \psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h}) \subset GL_p(n+\ell)$ holds. Because $\partial_B \psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$ is compact as a result of local Lipschitzness of $\psi(\boldsymbol{x}, \lambda; \bar{f}, \bar{h})$ and because $GL_p(n+\ell)$ is an open set of $M(n)$ and $\partial_B \psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h}) \subset GL_p(n+\ell)$, there exists $\epsilon > 0$ such that

$$\partial_B \psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h}) + B_\epsilon(\mathbf{0}) \subset GL_p(n+\ell). \tag{1}$$

As stated in Fact 3.6, it follows from local Lipschitzness of $\psi(\boldsymbol{x}, \lambda; \bar{f}, \bar{h})$ that $(\boldsymbol{x}, \lambda) \mapsto \partial_B \psi(\boldsymbol{x}, \lambda; \bar{f}, \bar{h})$ is upper semicontinuous at every $(\boldsymbol{x}, \lambda)$, i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that $\partial_B \psi(B_\delta((\bar{\boldsymbol{x}}, \bar{\lambda})); \bar{f}, \bar{h}) \subset \partial_B \psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h}) + B_{\frac{1}{2}\epsilon}(\mathbf{0})$ holds. Setting $W = B_\delta((\bar{\boldsymbol{x}}, \bar{\lambda}))$ we can restate this inclusion as

$$\partial_B \psi(W; \bar{f}, \bar{h}) \subset \partial_B \psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h}) + B_{\frac{1}{2}\epsilon}(\mathbf{0}). \tag{2}$$

Set $\delta^* = \delta$ and let $U = B_{\delta^*}(\bar{\boldsymbol{x}}^+)$. Then, because ρ_K^+ is Lipschitz continuous with its modulus 1, $\rho_K^+(B_{\delta^*}(\bar{\boldsymbol{x}})) \subset U$ holds, from which it follows that $W \subset (\rho_K^+)^{-1}(U) \times \boldsymbol{R}^\ell$. On the other hand, it is readily inferred that $(f, h) \mapsto \partial_B \psi(W; f, h)$ is also upper semicontinuous at every (f, h) with respect to the topology of $\mathcal{F}_U$, i.e., there exists $\delta_1 > 0$ such that

$$\partial_B \psi(W; B_{\delta_1}((\bar{f}, \bar{h}); U)) \subset \partial_B \psi(W; \bar{f}, \bar{h}) + B_{\frac{1}{2}\delta}(\mathbf{0}). \tag{3}$$

With the inclusion $\partial_B \psi(W; B_\delta((\bar{f}, \bar{h}); U)) \subset GL_p(n+\ell)$ that is readily deduced from (1), (2), and (3), Proposition 3.15 asserts that for any $(f, h) \in B_{\delta_1}((\bar{f}, \bar{h}); U)$ $\psi(\boldsymbol{x}, \lambda; f, h)$ has the semismooth inverse on a neighborhood of W. By Theorem 2.6 we can conclude that $\bar{\boldsymbol{x}}^+$ is a strongly stable solution of $\mathbf{Pro}(\bar{f}, \bar{h}; K)$. ∎

One remark must be made before the description of Theorem 4.6.

Remark 4.3. (see [4]) We investigate the relation between strong stability of locally isolated stationary solutions of NPAC and strong stability of locally isolated solutions of variational inequality in this remark.

Let K be a closed convex subset of $\boldsymbol{R}^n$, $\boldsymbol{x} \mapsto \boldsymbol{x}^+$ a Euclidean projector onto K, and $F : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a locally Lipschitz map. Define $\mathbf{F}_K^{\text{nor}}(\boldsymbol{x}) = F(\boldsymbol{x}^+) + \boldsymbol{x}^-$. Then the variational inequality, denoted as VI(K, F), is to find $\boldsymbol{x} \in \boldsymbol{R}^n$ satisfying $\mathbf{F}_K^{\text{nor}}(\boldsymbol{x}) = \mathbf{0}$. Let SOL$(K, F)$ denote the set of solutions of VI(K, F), i.e., SOL$(K, F) = \{\boldsymbol{x} \in \boldsymbol{R}^n : F(\boldsymbol{x}^+) + \boldsymbol{x}^- = \mathbf{0}\}$. From Theorem 3 of [6] it is provable without difficulty that a locally isolated solution $\bar{\boldsymbol{x}}$ of VI(K, F) is strongly stable if and only if the coherent orientation property holds for $\partial_B \mathbf{F}_K^{\text{nor}}(\bar{\boldsymbol{x}})$.

Let SOL$(f, h; K)$ denote the set of stationary points of $\mathbf{Pro}(f, h; K)$, i.e., SOL$(f, h; K) = \{(\boldsymbol{x}, \lambda) \in \boldsymbol{R}^{n+\ell} : \psi(\boldsymbol{x}, \lambda; f, h) = \mathbf{0}\}$. Let $\tilde{K} = K \times \boldsymbol{R}^\ell$, $F(\boldsymbol{x}, \lambda) = D^T_{(\boldsymbol{x},\lambda)} L(\boldsymbol{x}, \lambda; f, h) = (D_{(\boldsymbol{x},\lambda)} L(\boldsymbol{x}, \lambda; f, h))^T$, and $\boldsymbol{y} = (\boldsymbol{x}, \lambda)$. Then, it is readily inferred that $\boldsymbol{y}^+ = \rho_{\tilde{K}}^+(\boldsymbol{y}) = (\boldsymbol{x}^+, \lambda)$, $\boldsymbol{y}^- = \rho_{\tilde{K}}^-(\boldsymbol{y}) = (\boldsymbol{x}^-, \mathbf{0})$, and $\psi(\boldsymbol{x}^+, \lambda; f, h)^T = F(\boldsymbol{y}^+) + \boldsymbol{y}^-$ because $\psi(\boldsymbol{x}, \lambda; f, h) = (D_{\boldsymbol{x}} L(\boldsymbol{x}^+, \lambda; f, h) + (\boldsymbol{x}^-)^T, D_\lambda L(\boldsymbol{x}^+, \lambda; f, h)) = D_{(\boldsymbol{x},\lambda)} L(\boldsymbol{x}^+, \lambda; f, h) + ((\boldsymbol{x}^-)^T, \mathbf{0})$. This fact implies that SOL$(f, h; K) =$ SOL$(\tilde{K}, F)$.

First, consider the case of NPAC that has no equality constraints: i.e., $\ell = 0$. In this case, the concept of strong stability for SOL$(f, \emptyset; K)$ is the same for SOL$(K, D_{\boldsymbol{x}}{}^T f)$ because the perturbation of f that affects solutions is nothing but the perturbation of ${D_{\boldsymbol{x}}}^T f$. Therefore, in this case, we have an algebraic criterion for stability of a locally isolated stationary solution $\bar{\boldsymbol{x}}^+$ of $\mathbf{Pro}(f, \emptyset; K)$ that the coherent orientation property holds for $\partial_B \psi(\bar{\boldsymbol{x}}; f)$.

Next, consider the case in which program NPAC has some equality constraints, i.e. $\ell \neq 0$. In this case, however the concept of strong stability for $\mathrm{SOL}(f, h; K)$ is weaker than that for $\mathrm{SOL}(\tilde{K}, F)$ because perturbations of (f, h) comprise only a part of the perturbations of F. Therefore, we cannot merely apply the result of strong stability of locally isolated solutions of variational inequality $\mathrm{VI}(\tilde{K}, D^T_{(\boldsymbol{x},\lambda)} L(\boldsymbol{x}, \lambda; f, h))$ to locally isolated stationary solutions of $\mathbf{Pro}(f, h; K)$.

We must construct an analog of Kojima's approach to overcome this difference of perturbations stated in Remark 4.3 and to prove necessity for strong stability, i.e., to prove (a)⇒(b2)+(c) just as Jongen et al. [10] attributed proof of necessity to a result of Kojima [14]. We can construct an analog of Kojima's approach for where the following condition 4.4 holds (see [19], [20]). We refer to it as the regular boundary condition 4.4 for K. In the statement of the condition, $V(C_+; = 0)$ denotes the kernel of the operator C_+, i.e., $V(C_+; = 0) = \{\boldsymbol{u} \in \boldsymbol{R}^n : C_+ \boldsymbol{u} = 0\}$.

Condition 4.4. *$V(C_+; = 0) \subset \boldsymbol{R}\sigma_K(\boldsymbol{x}^+)$ for any $C_+ \in \partial_B \rho_K^+(\boldsymbol{x})$.*

The above regular boundary condition 4.4 is equivalent to the condition that $V(C_+; = 0) \subset \boldsymbol{R}\sigma_K(\boldsymbol{x}^+)$ for $C_+ \in \partial \rho_K^+(\boldsymbol{x})$ introduced in [20], because $\mathrm{conv}(\partial_B \rho_K^+(\boldsymbol{x})) = \partial \rho_K^+(\boldsymbol{x})$ and $\partial \rho_K^+(\boldsymbol{x}) \subset S_+(n)$ stated in Proposition 3.9 of [20]. Therefore, it follows from Lemma 4.15 of [19] that the regular boundary condition 4.4 is always fulfilled in case $K = S_+(n)$.

In the remainder of this section, we assume that both the LICQ condition 2.3 and the regular boundary condition 4.4 hold. Under these two conditions, we can prove that (a)⇒(b2)+(c) in Proposition 4.5. In [20] antecedent to [19], we constructed an analog of Kojima's approach [14] for programs $\mathbf{Pro}(f, h; S_+(n))$ under the LICQ condition 2.3. However, this construction is in fact applicable to $\mathbf{Pro}(f, h; K)$ for any closed convex set K satisfying the LICQ condition 2.3 and the regular boundary condition 4.4 (see [20]). In fact, the following proposition 4.5 is deduced immediately from the proof of Proposition 4.7 of [19].

Proposition 4.5. *(see [20]) Presume that the LICQ condition 2.3 holds and that the regular boundary condition 4.4 holds for K. Let $(\bar{\boldsymbol{x}}, \bar{\lambda})$ be a stationary point of $\boldsymbol{Pro}(\bar{f}, \bar{h}; K)$. Then (a)⇒(b2)+(c) holds.*

Under the LICQ condition 2.3 and the regular boundary condition 4.4, the following theorem provides several conditions that are equivalent to strong stability. We remark that the semismoothness of ρ_K^+ is required only for proof of the implication (4) ⇒ (2) of the theorem.

Theorem 4.6. *Suppose that the LICQ condition 2.3 holds and that the regular boundary condition 4.4 holds for K. Then the following (i) and (ii) hold.*

(i) The following (1)–(4) are equivalent if ρ_K^+ is semismooth.

(1) $\bar{\boldsymbol{x}}^+$ is a strongly stable stationary solution of $\boldsymbol{Pro}(\bar{f}, \bar{h}; K)$.

(2) $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *is locally Lipschitz homeomorphism around* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(3) (b2)+(c) holds for $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *at* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(4) (b2)+(d) holds for $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *at* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(ii) If ρ_K^+ *is semismooth and directionally differentiable at* $\bar{\boldsymbol{x}}$*, and if the inclusion* $\partial_B\rho_K^+(\bar{\boldsymbol{x}}) \subset \partial_B(d\rho_K^+|_{\bar{\boldsymbol{x}}})(\mathbf{0})$ *holds, the above (1)–(4), and the following (5)–(7) are all equivalent.*

(5) $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *is globally Lipschitz homeomorphism.*

(6) (b2)+(c) holds for $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *at* $(\mathbf{0},\mathbf{0})$.

(7) (b2)+(d) holds for $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *at* $(\mathbf{0},\mathbf{0})$.

Proof:
(i): Implication (4)⇒(2) follows from Corollary 4 of [6] for semismooth maps. Also, (2)⇒(1) follows from Theorem 2.6. (1)⇒(3) follows from Proposition 4.5. (3)⇒(4) is readily inferred. Therefore, (1), (2), (3), and (4) are equivalent.
(ii): (2)⇒(5) follows from Proposition 2.2 of [17] about locally Lipschitz maps that are directionally differentiable at a point. (5)⇒(6) is a special case of (2)⇒(3) that is proved without the assumption of semismoothness. (6)⇒(7) is readily inferred. Therefore, we have only to prove implication (7)⇒(4).

(b2) of (4) immediately follows from (b2) of (7) because we have the inclusion $\partial_B\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h}) \subset \partial_B d\psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})(\mathbf{0},\mathbf{0})$ from the assumption $\partial_B\rho_K^+(\bar{\boldsymbol{x}}) \subset \partial_B(d\rho_K^+|_{\bar{\boldsymbol{x}}})(\mathbf{0})$. Consider statement (d). Suppose that (d) of (4) does not hold, i.e., there exists a sequence $(\boldsymbol{x}_k,\lambda_k)$, $(k=1,2,\cdots)$ such that $\lim_{k\to\infty}(\boldsymbol{x}_k,\lambda_k) = (\bar{\boldsymbol{x}},\bar{\lambda})$ and $\psi(\boldsymbol{x}_k,\lambda_k;\bar{f},\bar{h}) = \mathbf{0}$ $(k = 1,2,\cdots)$. Taking a subsequence, we can assume that $\lim_{k\to\infty}\frac{(\boldsymbol{x}_k-\bar{\boldsymbol{x}},\ \lambda_k-\bar{\lambda})}{\|\boldsymbol{x}_k-\bar{\boldsymbol{x}}\|+\|\lambda_k-\bar{\lambda}\|} = (\boldsymbol{u},\xi) \neq (\mathbf{0},\mathbf{0})$. Therefore, we have

$$\lim_{k\to\infty}\frac{\psi(\boldsymbol{x}_k,\lambda_k;\bar{f},\bar{h}) - \psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})}{\|\boldsymbol{x}_k-\bar{\boldsymbol{x}}\|+\|\lambda_k-\bar{\lambda}\|} = \mathbf{0}.$$

On the other hand, because $\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})$ is directionally differentiable at $(\bar{\boldsymbol{x}},\bar{\lambda})$ and $\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})$ is locally Lipschitz continuous, it is readily inferred that

$$\lim_{k\to\infty}\frac{\psi(\boldsymbol{x}_k,\lambda_k;\bar{f},\bar{h}) - \psi(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})}{\|\boldsymbol{x}_k-\bar{\boldsymbol{x}}\|+\|\lambda_k-\bar{\lambda}\|} = \psi'(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})(\boldsymbol{u},\xi).$$

Therefore, we have deduced $\psi'(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})(\boldsymbol{u},\xi) = \mathbf{0}$ for $(\boldsymbol{u},\xi) \neq (\mathbf{0},\mathbf{0})$, which contradicts (b2) of (7) because $\psi'(\bar{\boldsymbol{x}},\bar{\lambda};\bar{f},\bar{h})(t\boldsymbol{u},t\xi) = \mathbf{0}$ for any $t > 0$. ∎

The following corollary readily follows from Proposition 3.19 and Theorem 4.6.

Corollary 4.7. *Let* K *be a closed convex cone pointed at* $\mathbf{0}$. *Presume that the LICQ condition 2.3 holds and that the regular boundary condition 4.4 holds for* K. *Then the following (i) and (ii) hold.*

(i) The following (1)–(4) are equivalent.

(1) $\bar{\boldsymbol{x}}^+$ *is a strongly stable stationary solution of* $\boldsymbol{Pro}(\bar{f},\bar{h};K)$.

(2) $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *is a locally Lipschitz homeomorphism around* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(3) (b2)+(c) holds for $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *at* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(4) (b2)+(d) holds for $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *at* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(ii) If ρ_K^+ *is directionally differentiable at* $\bar{\boldsymbol{x}}$*, and if the inclusion* $\partial_B\rho_K^+(\bar{\boldsymbol{x}}) \subset \partial_B(d\rho_K^+|_{\bar{\boldsymbol{x}}})(\mathbf{0})$ *holds, the (1)–(4) above, and the following (5)–(7) are all equivalent.*

(5) $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *is a globally Lipschitz homeomorphism.*

(6) (b2)+(c) holds for $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *at* $(\mathbf{0},\mathbf{0})$.

(7) (b2)+(d) holds for $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *at* $(\mathbf{0},\mathbf{0})$.

Definition 4.8. A closed convex set of $\boldsymbol{R}^n$ is called polyhedral or a polyhedron if it is defined by a finite number of linear inequalities. A cone that is polyhedral is called a polyhedral cone.

As already noted above, $S_+(n)$ satisfies the regular boundary condition 4.4 and $\rho^+_{S_+(n)}$ is semismooth from Proposition 3.19. Moreover, we know from Theorem 4.7 of [24] and Lemma 11 of [21] that $\rho^+_{S_+(n)}$ is directionally differentiable and that $\partial_B\rho^+_{S_+(n)}(\bar{\boldsymbol{x}}) = \partial_B(d\rho^+_{S_+(n)}|_{\bar{\boldsymbol{x}}})(\mathbf{0})$ holds. Similarly, for a polyhedral cone K pointed at $\mathbf{0}$, it follows from Proposition 3.19 that the Euclidean projector ρ_K^+ is semismooth. It is also readily inferred that both the regular boundary condition 4.4 and $\partial_B\rho_K^+(\bar{\boldsymbol{x}}) = \partial_B(d\rho_K^+|_{\bar{\boldsymbol{x}}})(\mathbf{0})$ hold for any polyhedron K. Consequently, we have the following corollary.

Corollary 4.9. *Let* K *be either* $S_+(n)$ *or a polyhedral cone pointed at* $\mathbf{0}$*. Then under the LICQ condition 2.3, the following (1)–(7) are equivalent.*

(1) $\bar{\boldsymbol{x}}^+$ *is a strongly stable stationary solution of* $\boldsymbol{Pro}(\bar{f},\bar{h};K)$.

(2) $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *is a locally Lipschitz homeomorphism around* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(3) (b2)+(c) holds for $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *at* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(4) (b2)+(d) holds for $\psi(\cdot,\cdot;\bar{f},\bar{h})$ *at* $(\bar{\boldsymbol{x}},\bar{\lambda})$.

(5) $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *is a globally Lipschitz homeomorphism.*

(6) (b2)+(c) holds for $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *at* $(\mathbf{0},\mathbf{0})$.

(7) (b2)+(d) holds for $(d\psi(\boldsymbol{x},\lambda;\bar{f},\bar{h})|_{(\bar{\boldsymbol{x}},\bar{\lambda})})(\cdot,\cdot)$ *at* $(\mathbf{0},\mathbf{0})$.

This corollary contains the result of Kojima's theory stated in Theorem 3.3 and Corollary 4.3 of [14] because the programs treated in [14] are reformulated to the programs $\mathbf{Pro}(\bar{f},\bar{h};\boldsymbol{R}^m_+\times\boldsymbol{R}^n)$ by transformation of a coordinate system. This indicates that the theory Kojima had perceived in the classical setting of [14] seems to have universal validity for more general nonlinear programs.

5 Calculation of $D_{\boldsymbol{x}}\rho_K^+(\boldsymbol{x})$ and Interpretation of the Regular Boundary Condition 4.4.

This section is intended to advance an interpretation of the regular boundary condition 4.4 geometrically. We deduce an explicit formula of $D_{\boldsymbol{x}}\rho_K^+(\boldsymbol{x})$ under the assumption that stratification of K, which is defined naturally for any closed convex set K is of C^2 class, and interpret the regular boundary condition 4.4 in terms of principal curvatures (or radii of principal curvature) at $\bar{\boldsymbol{x}}^+$ of the stratum. We use the following notation.

$$\begin{cases} K_r &= \{\boldsymbol{x} \in K : \dim \boldsymbol{R}\sigma_K(\boldsymbol{x}) = n - r\} \\ V_r &= \coprod\{\sigma_K(\boldsymbol{x}) : \boldsymbol{x} \in K_r\} \end{cases}$$

It is readily inferred that $K = \coprod_{r=0}^n K_r$. We consider this decomposition as a stratification of K. Similarly, $\boldsymbol{R}^n = \coprod_{r=0}^n V_r$ and $int(V_r) = \coprod\{re\ell int(\sigma_K(\boldsymbol{x})) : \boldsymbol{x} \in K_r\}$ hold. We assume the following condition throughout this section, which ensures that $K = \coprod_{r=0}^n K_r$ is a C^2 stratification.

Condition 5.1. *In the case that $K_r \neq \emptyset$, K_r is an r dimensional C^2 submanifold of $\boldsymbol{R}^n$ for $r = 0, 1, 2, \cdots, n$.*

We can prove the following theorem, which provides an explicit formula of $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}})$ under Condition 5.1.

Theorem 5.2. *Suppose that a closed convex set $K \subset \boldsymbol{R}^n$ has C^2 stratification 5.1. Then $\amalg_{r=0}^n int(V_r) \subset E_{\rho_K^+}$ holds. An explicit formula of $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}})$ for $\bar{\boldsymbol{x}} \in int(V_r)$ is given as follows.*

Let $\bar{\boldsymbol{x}} \in int(V_r)$ and $T_{\bar{\boldsymbol{x}}^+}K_r$ be the tangent space of K_r at $\bar{\boldsymbol{x}}^+$, $\boldsymbol{u}_i$ $(1 \leq i \leq r)$ be any orthonormal basis of $T_{\bar{\boldsymbol{x}}^+}K_r$, $\boldsymbol{u}_i$ $(r + 1 \leq i \leq n)$ be any orthonormal basis of $(T_{\bar{\boldsymbol{x}}^+}K_r)^\perp$, $P_1 = (\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r)$, $P_2 = (\boldsymbol{u}_{r+1}, \cdots, \boldsymbol{u}_r)$, and $P = (\, P_1 \quad P_2\,) = (\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r) \in O(n)$. Let U be an open subset of $\boldsymbol{R}^r$ of $\boldsymbol{0}$ and let $\boldsymbol{c} : U \to K_r$ be a C^2 coordinate system of K_r around $\bar{\boldsymbol{x}}$ such that $\boldsymbol{c}(\boldsymbol{0}) = \bar{\boldsymbol{x}}^+$ and $D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{0}) = P_1$.

Then $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}}) = P \begin{pmatrix} \left(\boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\boldsymbol{0}) \right)^{-1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} P^T$ *holds.*

Proof: It is readily inferred that $\rho_K^+(\boldsymbol{x}) = \rho_{K_r}^+(\boldsymbol{x})$ holds for any $\boldsymbol{x} \in int(V_r)$. Because K_r is a C^2 manifold, it is also inferred that $\rho_{K_r}^+(\boldsymbol{x})$ is C^1 on $int(V_r)$. Therefore, $\amalg_{r=0}^n int(V_r) \subset E_{\rho_K^+}$ holds. Consequently, below we have only to prove the explicit formula stated in the theorem.

Denote $W_1 = T_{\bar{\boldsymbol{x}}^+}K_r$ and $W_2 = (T_{\bar{\boldsymbol{x}}^+}K_r)^\perp$. Then $W_1 \perp W_2$ holds. Let $\boldsymbol{u}'_i$ $(1 \leq i \leq r)$ be another orthonormal basis of W_1 and let $\boldsymbol{u}'_i$ $(r + 1 \leq i \leq n)$ be another orthonormal basis of W_2. Also let $Q_1 = (\boldsymbol{u}'_1, \cdots, \boldsymbol{u}'_r)$, $Q_2 = (\boldsymbol{u}'_{r+1}, \cdots, \boldsymbol{u}'_n)$, $Q = (\, Q_1 \quad Q_2\,) \in O(n)$, and $\boldsymbol{c}' : U' \to K_r$ be a C^2 coordinate system of K_r around $\bar{\boldsymbol{x}}$ such that $\boldsymbol{c}'(\boldsymbol{0}) = \bar{\boldsymbol{x}}^+$ and $D_{\boldsymbol{t}}\boldsymbol{c}'(\boldsymbol{0}) = Q_1$. It is readily inferred that there exist $G_1 \in O(r)$ and $G_2 \in O(n - r)$ such that $P_1 = Q_1G_1$, $P_2 = Q_2G_2$, and $P = QG$ with $G = \begin{pmatrix} G_1 & \boldsymbol{O} \\ \boldsymbol{O} & G_2 \end{pmatrix} \in O(n)$. Let

$\Phi : \boldsymbol{R}^r \to \boldsymbol{R}^r$ be defined around $\mathbf{0}$ such that $\Phi(\boldsymbol{t}) = \boldsymbol{c}'^{-1}(\boldsymbol{c}(\boldsymbol{t}))$. Then $\boldsymbol{c}(\boldsymbol{t}) = \boldsymbol{c}'(\Phi(\boldsymbol{t}))$ and $D_{\boldsymbol{t}}\Phi(\mathbf{0}) = G_1$ hold. We use the following Taylor's expansions.

$$\begin{aligned} \boldsymbol{c}(\boldsymbol{t}) &= \boldsymbol{c}(\mathbf{0}) + D_{\boldsymbol{t}}\boldsymbol{c}(\mathbf{0})\boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T D_{\boldsymbol{t}}^2 \boldsymbol{c}(\theta\boldsymbol{t})\boldsymbol{t} \\ &= \bar{\boldsymbol{x}}^+ + P_1\boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T D_{\boldsymbol{t}}^2 \boldsymbol{c}(\theta\boldsymbol{t})\boldsymbol{t}, \qquad (4) \\ \boldsymbol{c}'(\boldsymbol{t}) &= \boldsymbol{c}'(\mathbf{0}) + D_{\boldsymbol{t}}\boldsymbol{c}'(\mathbf{0})\boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T D_{\boldsymbol{t}}^2 \boldsymbol{c}'(\theta'\boldsymbol{t})\boldsymbol{t} \\ &= \bar{\boldsymbol{x}}^+ + Q_1\boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T D_{\boldsymbol{t}}^2 \boldsymbol{c}'(\theta'\boldsymbol{t})\boldsymbol{t}, \qquad (5) \\ \Phi(\boldsymbol{t}) &= \Phi(\mathbf{0}) + D_{\boldsymbol{t}}\Phi(\mathbf{0})\boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T D_{\boldsymbol{t}}^2 \Phi(\theta''\boldsymbol{t})\boldsymbol{t} \\ &= G_1\boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T D_{\boldsymbol{t}}^2 \Phi(\theta''\boldsymbol{t})\boldsymbol{t}. \qquad (6) \end{aligned}$$

Because $\boldsymbol{c}(\boldsymbol{t}),\ \boldsymbol{c}'(\boldsymbol{t}),$ and $\Phi(\boldsymbol{t})$ are not necessary functions, these equalities do not hold simply. However, because each component of $\boldsymbol{c}(\boldsymbol{t}),\ \boldsymbol{c}'(\boldsymbol{t}),$ and $\Phi(\boldsymbol{t})$ is a function, $\theta,\ \theta',$ and θ'' $(0 \le \theta,\ \theta',\ \theta'' \le 1)$ can be taken independently for the component. In this sense, $\theta,\ \theta',$ and θ'' in the above equations are written symbolically. We remark that $\boldsymbol{t}^T D_{\boldsymbol{t}}^2 \boldsymbol{c}(\theta\boldsymbol{t})\boldsymbol{t}$, $D_{\boldsymbol{t}}^2 \boldsymbol{c}'(\theta'\boldsymbol{t})\boldsymbol{t}$, and $\boldsymbol{t}^T D_{\boldsymbol{t}}^2 \Phi(\theta\boldsymbol{t})\boldsymbol{t}$ are considered as elements of $M(n) \otimes \boldsymbol{R}^n$.

Calculating $\boldsymbol{c}'(\Phi(\boldsymbol{t}))$ from (4) and (6), and comparing the result with (5) and taking a limit $\boldsymbol{t} \to \mathbf{0}$, we can deduce

$$D_{\boldsymbol{t}}^2(Q_1\Phi)(\mathbf{0}) + G_1^T D_{\boldsymbol{t}}^2 \boldsymbol{c}'(\mathbf{0}) G_1 = D_{\boldsymbol{t}}^2 \boldsymbol{c}(\mathbf{0}). \qquad (7)$$

From $Q_1\Phi(\boldsymbol{x}) \in W_1$ and $\bar{\boldsymbol{x}}^- \in W_2$ it follows that $\langle Q_1\Phi(\boldsymbol{x}), \bar{\boldsymbol{x}}^-\rangle = 0$ for any $\boldsymbol{x}$ that makes $\langle D_{\boldsymbol{t}}^2(Q_1\Phi)(\mathbf{0}), \bar{\boldsymbol{x}}^-\rangle = 0$. Therefore, after operating $\langle\ \cdot\ , \bar{\boldsymbol{x}}^-\rangle$ to (7) we can deduce

$$G_1^T \langle D_{\boldsymbol{t}}^2 \boldsymbol{c}'(\mathbf{0}), \bar{\boldsymbol{x}}^-\rangle G_1 = \langle G_1^T D_{\boldsymbol{t}}^2 \boldsymbol{c}'(\mathbf{0}) G_1, \bar{\boldsymbol{x}}^-\rangle = \langle D_{\boldsymbol{t}}^2 \boldsymbol{c}(\mathbf{0}), \bar{\boldsymbol{x}}^-\rangle. \qquad (8)$$

Here again, we remark that $G_1^T D_{\boldsymbol{t}}^2 \boldsymbol{c}'(\mathbf{0}) G_1$ of this equation is considered in $M(n) \otimes \boldsymbol{R}^n$ and the inner product $\langle\ ,\ \rangle$ is that on $\boldsymbol{R}^n$. It follows from (8) and $P_1 G_1 = Q_1$ that

$$Q_1\Big(\ \boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}'(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\mathbf{0})\ \Big)Q_1^T = P_1\Big(\ \boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\mathbf{0}) - D_{\boldsymbol{t}}^2(P_1\Phi)(\mathbf{0})\ \Big)P_1^T. \qquad (9)$$

We will later prove that $\boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\mathbf{0})$ is nonsingular for some $\boldsymbol{c}(\boldsymbol{t})$. Then we know from (9) that $\boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\mathbf{0})$ is nonsingular for any $\boldsymbol{c}'(\boldsymbol{t})$. As a result, we can deduce that $Q_1\Big(\boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}'(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\mathbf{0})\Big)^{-1} Q_1^T = P_1\Big(\boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\mathbf{0})\Big)^{-1} P_1^T$ and that

$$Q\begin{pmatrix} \Big(\boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\mathbf{0})\Big)^{-1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} Q^T = P\begin{pmatrix} \Big(\boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\mathbf{0})\Big)^{-1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} P^T. \qquad (10)$$

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ be the standard basis of $\boldsymbol{R}^n$, i.e., $\boldsymbol{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ $(1 \le i \le n)$ where the only nonzero entry is 1 in the i-th position. Then, from (9), we might assume that $W_1 = \sum_{i=1}^r \boldsymbol{R}\boldsymbol{e}_i,\ W_2 = \sum_{i=r+1}^n \boldsymbol{R}\boldsymbol{e}_i$ and $\bar{\boldsymbol{x}}^- = -\|\bar{\boldsymbol{x}}^-\|\boldsymbol{e}_n$. We use the coordinate system (of C^2 class) ($\boldsymbol{c} : \boldsymbol{R}^r \to K_r$) of K_r around $\bar{\boldsymbol{x}}^+$ such as $\boldsymbol{c}(\boldsymbol{t}) = \bar{\boldsymbol{x}}^+ + \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{v}(\boldsymbol{t}) \end{pmatrix}$ (precisely in fact, $\boldsymbol{c}$ is defined on an appropriate open neighborhood of $\boldsymbol{0} \in \boldsymbol{R}^r$). To complete the proof of this theorem, we must prove nonsingularity of $\boldsymbol{I}_r - (D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle)(\boldsymbol{0})$ and $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}}) = \begin{pmatrix} \left(\boldsymbol{I}_r - {D_{\boldsymbol{t}}}^2\langle \boldsymbol{c}, \bar{\boldsymbol{x}}^-\rangle(\boldsymbol{0})\right)^{-1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}$. Because $c_n(\boldsymbol{t}) = \langle \boldsymbol{c}(\boldsymbol{t}), \boldsymbol{e}_n\rangle \geq 0$ for any $\boldsymbol{t}$ in a neighborhood of $\boldsymbol{0}$ and $c_n(\boldsymbol{0}) = 0$, we can deduce that

$$ {D_{\boldsymbol{t}}}^2 c_n(\boldsymbol{0}) = ({D_{\boldsymbol{t}}}^2\langle \boldsymbol{c}, \boldsymbol{e}_n\rangle)(\boldsymbol{0}) \text{ is positive semidefinite.} \tag{11} $$

Let $W_2'(\boldsymbol{x}^+) = \boldsymbol{R}\sigma_K(\boldsymbol{x}^+) = (T_{\boldsymbol{x}^+}K_r)^\perp$ for $\boldsymbol{x}^+ \in K_r$. Then there exists a positive real number $\delta > 0$ and a C^1 map $\boldsymbol{d} : \boldsymbol{R}^r \to \bar{\boldsymbol{x}} + W_1$ (precisely in fact, $\boldsymbol{d}$ is defined on an appropriate open neighborhood of $\boldsymbol{0} \in \boldsymbol{R}^r$) such that $(\bar{\boldsymbol{x}} + W_1) \bigcap (\boldsymbol{c}(\boldsymbol{t}) + W_2'(\boldsymbol{c}(\boldsymbol{t}))) = \{\boldsymbol{d}(\boldsymbol{t})\}$ holds for $\|\boldsymbol{t}\| < \delta$ because of transversality (see [13]). Moreover, it is readily inferred that $\rho_K^+(\boldsymbol{d}(\boldsymbol{t})) = \boldsymbol{d}(\boldsymbol{t})^+ = \boldsymbol{c}(\boldsymbol{t})$. Therefore, $\boldsymbol{d}(\boldsymbol{t})^- \in W_2'(\boldsymbol{c}(\boldsymbol{t})) = (T_{\boldsymbol{c}(\boldsymbol{t})}K_r)^\perp$, which engenders $\langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \boldsymbol{d}(\boldsymbol{t})^-\rangle = \boldsymbol{0}$. Represent $\boldsymbol{d}(\boldsymbol{t})$ as $\boldsymbol{d}(\boldsymbol{t}) = \bar{\boldsymbol{x}} + \begin{pmatrix} \boldsymbol{a}(\boldsymbol{t}) \\ \boldsymbol{0} \end{pmatrix}$ with $a_i(\boldsymbol{t})$ $(1 \leq i \leq r)$ of C^1 class. Then $\boldsymbol{d}(\boldsymbol{t})^- = \boldsymbol{d}(\boldsymbol{t}) - \boldsymbol{d}^+(\boldsymbol{t}) = \boldsymbol{d}(\boldsymbol{t}) - \boldsymbol{c}(\boldsymbol{t}) = \bar{\boldsymbol{x}}^- - \begin{pmatrix} \boldsymbol{t} - \boldsymbol{a}(\boldsymbol{t}) \\ \boldsymbol{v}(\boldsymbol{t}) \end{pmatrix}$ and $D_{\boldsymbol{t}}\boldsymbol{d}(\boldsymbol{t})^- = -\begin{pmatrix} \boldsymbol{I}_r - D_{\boldsymbol{t}}\boldsymbol{a}(\boldsymbol{t}) \\ D_{\boldsymbol{t}}\boldsymbol{v}(\boldsymbol{t}) \end{pmatrix}$. It is readily inferred that $D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}) = \begin{pmatrix} \boldsymbol{I}_r \\ D_{\boldsymbol{t}}\boldsymbol{v}(\boldsymbol{t}) \end{pmatrix}$. We can calculate the following:

$$
\begin{aligned}
\langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \boldsymbol{d}(\boldsymbol{t})^-\rangle &= \langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \boldsymbol{d}(\boldsymbol{t})^- - \bar{\boldsymbol{x}}^-\rangle + \langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle \\
&= -\begin{pmatrix} \boldsymbol{I}_r \\ D_{\boldsymbol{t}}\boldsymbol{v}(\boldsymbol{t}) \end{pmatrix}^T \begin{pmatrix} \boldsymbol{t} - \boldsymbol{a}(\boldsymbol{t}) \\ \boldsymbol{v}(\boldsymbol{t}) \end{pmatrix} + \langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle \\
&= -\begin{pmatrix} \boldsymbol{I}_r & (D_{\boldsymbol{t}}\boldsymbol{v}(\boldsymbol{t}))^T \end{pmatrix} \begin{pmatrix} \boldsymbol{t} - \boldsymbol{a}(\boldsymbol{t}) \\ \boldsymbol{v}(\boldsymbol{t}) \end{pmatrix} + \langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle \\
&= -\boldsymbol{t} + \boldsymbol{a}(\boldsymbol{t}) - (D_{\boldsymbol{t}}\boldsymbol{v}(\boldsymbol{t}))^T\boldsymbol{v}(\boldsymbol{t}) + \langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle \\
&= -\boldsymbol{t} + \boldsymbol{a}(\boldsymbol{t}) - \langle D_{\boldsymbol{t}}\boldsymbol{v}(\boldsymbol{t}), \boldsymbol{v}(\boldsymbol{t})\rangle_2 + \langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle \\
&= -\boldsymbol{t} + \boldsymbol{a}(\boldsymbol{t}) - \frac{1}{2}D_{\boldsymbol{t}}\langle \boldsymbol{v}(\boldsymbol{t}), \boldsymbol{v}(\boldsymbol{t})\rangle_2 + D_{\boldsymbol{t}}\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle,
\end{aligned}
$$

where $\langle\ ,\ \rangle_2$ is the standard inner product on W_2, which is the restriction of $\langle\ ,\ \rangle$ to W_2. From $\langle D_{\boldsymbol{t}}\boldsymbol{c}(\boldsymbol{t}), \boldsymbol{d}^-(\boldsymbol{t})\rangle = \boldsymbol{0}$ and the above equation, we have

$$ \boldsymbol{a}(\boldsymbol{t}) = \boldsymbol{t} - D_{\boldsymbol{t}}\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle + \frac{1}{2}D_{\boldsymbol{t}}\langle \boldsymbol{v}(\boldsymbol{t}), \boldsymbol{v}(\boldsymbol{t})\rangle_2. \tag{12} $$

Differentiation of the equation (12) with respect to $\boldsymbol{t}$ leads to

$$ D_{\boldsymbol{t}}\boldsymbol{a}(\boldsymbol{t}) = \boldsymbol{I}_r - D_{\boldsymbol{t}}^2\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle + \frac{1}{2}D_{\boldsymbol{t}}^2\langle \boldsymbol{v}(\boldsymbol{t}), \boldsymbol{v}(\boldsymbol{t})\rangle_2. \tag{13} $$

With an equality $-\langle \boldsymbol{c}(\boldsymbol{t}), \bar{\boldsymbol{x}}^-\rangle = \|\bar{\boldsymbol{x}}^-\|\langle \boldsymbol{c}(\boldsymbol{t}), \boldsymbol{e}_n\rangle$ and (11), it is readily inferred that

$$ -D_{\boldsymbol{t}}^2\langle \boldsymbol{c}, \bar{\boldsymbol{x}}^-\rangle(\boldsymbol{0}) \text{ is positive semidefinite.} \tag{14} $$

Let $\pi_2(\cdot)$ be the orthogonal projection from $\boldsymbol{R}^n$ to W_2, i.e., $\pi_2(\boldsymbol{x}) = P_2 P_2^T \boldsymbol{x} = Q_2 Q_2^T \boldsymbol{x}$. Then, because $P_i^T P_j = Q_i^T Q_j = P_i^T Q_j = Q_i^T P_j = \boldsymbol{O}$ $(i \neq j)$ we can deduce from (4) that $\langle \pi_2(\boldsymbol{c}(\boldsymbol{t}) - \bar{\boldsymbol{x}}^+), \pi_2(\boldsymbol{c}(\boldsymbol{t}) - \bar{\boldsymbol{x}}^+) \rangle_2 = o(\|\boldsymbol{t}\|^4)$. It follows that

$$D_{\boldsymbol{t}}^2 \langle \pi_2(\boldsymbol{c} - \bar{\boldsymbol{x}}^+), \pi_2(\boldsymbol{c} - \bar{\boldsymbol{x}}^+) \rangle_2(\boldsymbol{0}) \quad = \quad \boldsymbol{O}. \tag{15}$$

From (13), (14), (15), and $\pi_2(\boldsymbol{c}(\boldsymbol{t}) - \bar{\boldsymbol{x}}^-) = \boldsymbol{v}(\boldsymbol{t})$, we can conclude that

$$D_{\boldsymbol{t}} \boldsymbol{a}(\boldsymbol{0}) \quad = \quad \boldsymbol{I}_r - D_{\boldsymbol{t}}^2 \langle \boldsymbol{c}, \bar{\boldsymbol{x}}^- \rangle(\boldsymbol{0}) \ \text{ is positive definite,} \tag{16}$$

and

$$(D_{\boldsymbol{t}} \boldsymbol{a}(\boldsymbol{0}))^{-1} \quad = \quad (\ \boldsymbol{I}_r - D_{\boldsymbol{t}}^2 \langle \boldsymbol{c}, \bar{\boldsymbol{x}}^- \rangle(\boldsymbol{0})\ )^{-1}. \tag{17}$$

Therefore, there exists inverse $\boldsymbol{a}^{-1} : \boldsymbol{R}^r \to \boldsymbol{R}^r$ of the map $\boldsymbol{a}(\boldsymbol{t})$ locally around $\boldsymbol{t} = \boldsymbol{0} \in \boldsymbol{R}^r$.

Let $F_1 : \boldsymbol{R}^r \to \bar{\boldsymbol{x}} + W_1$ be defined as $\boldsymbol{y} \mapsto F_1(\boldsymbol{y}) = \bar{\boldsymbol{x}} + \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{0} \end{pmatrix}$ and $F_2 : \boldsymbol{R}^{n-r} \to \bar{\boldsymbol{x}} + W_2$ by $\boldsymbol{z} \mapsto F_2(\boldsymbol{z}) = \bar{\boldsymbol{x}} + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{z} \end{pmatrix}$. The following are readily inferred.

$$D_{\boldsymbol{y}} F_1(\boldsymbol{y}) \quad = \quad \begin{pmatrix} \boldsymbol{I}_r \\ \boldsymbol{O} \end{pmatrix} \ \text{and}\ D_{\boldsymbol{z}} F_2(\boldsymbol{z}) \quad = \quad \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{I}_{n-r} \end{pmatrix} \tag{18}$$

Through simple calculation, we can deduce that $\boldsymbol{c}(\boldsymbol{t}) = \rho_K^+ \circ F_1 \circ \boldsymbol{a}(\boldsymbol{t})$ for any $\boldsymbol{t} \in \boldsymbol{R}^r$ sufficiently near $\boldsymbol{0}$. Therefore,

$$\rho_K^+ \circ F_1(\boldsymbol{y}) = \boldsymbol{c} \circ \boldsymbol{a}^{-1}(\boldsymbol{y}) \ \text{for any}\ \boldsymbol{y} \in \boldsymbol{R}^r \ \text{sufficiently near}\ \boldsymbol{0}. \tag{19}$$

Similarly,

$$\rho_K^+ \circ F_2(\boldsymbol{z}) = \bar{\boldsymbol{x}}^+ \ \text{for any}\ \boldsymbol{z} \in \boldsymbol{R}^{n-r} \ \text{sufficiently near}\ \boldsymbol{0}. \tag{20}$$

Differentiating (19) and (20), we have $D_{\boldsymbol{x}} \rho_K^+(F_1(\boldsymbol{y})) D_{\boldsymbol{y}} F_1(\boldsymbol{y}) = (D_{\boldsymbol{t}} \boldsymbol{c})(\boldsymbol{a}^{-1}(\boldsymbol{y}))(D_{\boldsymbol{y}} \boldsymbol{a}^{-1})(\boldsymbol{y})$ and $D_{\boldsymbol{x}} \rho_K^+(F_2(\boldsymbol{z})) D_{\boldsymbol{z}} F_2(\boldsymbol{z}) = \boldsymbol{O}$. Then, from $D_{\boldsymbol{t}} \boldsymbol{c}(\boldsymbol{t}) = \begin{pmatrix} \boldsymbol{I}_r \\ D_{\boldsymbol{t}} \boldsymbol{v}(\boldsymbol{t}) \end{pmatrix}$, (18), and (17), and substituting $\boldsymbol{y} = \boldsymbol{0}$ and $\boldsymbol{z} = \boldsymbol{0}$, then we can derive

$$\begin{cases} D_{\boldsymbol{a}} \rho_K^+(\bar{\boldsymbol{x}}) \begin{pmatrix} \boldsymbol{I}_r \\ \boldsymbol{O} \end{pmatrix} & = \begin{pmatrix} \boldsymbol{I}_r \\ \boldsymbol{O} \end{pmatrix} (\ \boldsymbol{I}_r - D_{\boldsymbol{t}}{}^2 \langle \boldsymbol{c}, \bar{\boldsymbol{x}}^- \rangle(\boldsymbol{0})\ )^{-1} \ = \ \begin{pmatrix} (\ \boldsymbol{I}_r - D_{\boldsymbol{t}}{}^2 \langle \boldsymbol{c}, \bar{\boldsymbol{x}}^- \rangle(\boldsymbol{0})\ )^{-1} \\ \boldsymbol{O} \end{pmatrix}, \\ D_{\boldsymbol{x}} \rho_K^+(\bar{\boldsymbol{x}}) \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{I}_{n-r} \end{pmatrix} & = \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{O} \end{pmatrix}. \end{cases}$$

We have proved the formula $D_{\boldsymbol{x}} \rho_K^+(\bar{\boldsymbol{x}}) = \begin{pmatrix} \left(\boldsymbol{I}_r - D_{\boldsymbol{t}}{}^2 \langle\ \boldsymbol{c}\ , \bar{\boldsymbol{x}}^- \rangle(\boldsymbol{0}) \right)^{-1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}$. ∎

With an explanation stated in the following remark, we can understand the geometric meaning of the regular boundary condition 4.4.

Remark 5.3. Let $\boldsymbol{n} = -\frac{\bar{\boldsymbol{x}}^-}{\|\bar{\boldsymbol{x}}^-\|}$. Suppose that λ_i $(1 \le i \le r)$ is an eigenvalue of $A = -D_{\boldsymbol{t}}{}^2\langle \boldsymbol{c}, \bar{\boldsymbol{x}}^-\rangle(\boldsymbol{0}) = \|\bar{\boldsymbol{x}}^-\| D_{\boldsymbol{t}}{}^2\langle \boldsymbol{c}, \boldsymbol{n}\rangle(\boldsymbol{0})$ and that $\boldsymbol{w}_i$ $(1 \le i \le r)$ is an eigenvector belonging to λ_i satisfying $\|\boldsymbol{w}_i\| = 1$. Define $f_i(s) = \langle \boldsymbol{c}(\bar{\boldsymbol{x}} + s\boldsymbol{w}_i), \boldsymbol{n}\rangle = \langle \rho_K^+(\bar{\boldsymbol{x}} + s\boldsymbol{w}_i), \boldsymbol{n}\rangle = \langle (\bar{\boldsymbol{x}} + s\boldsymbol{w}_i)^+, \boldsymbol{n}\rangle$ for $s \in \boldsymbol{R}$ sufficiently near 0. Let $H_i = \bar{\boldsymbol{x}}^+ + \boldsymbol{R}\boldsymbol{w}_i + \boldsymbol{R}\boldsymbol{n} \subset \boldsymbol{R}^n$ and $\pi_i : \boldsymbol{R}^n \to H_i$ be the orthogonal projection. Then $c(s) = \pi_i(\boldsymbol{c}(\bar{\boldsymbol{x}} + s\boldsymbol{w}_i)) = \bar{\boldsymbol{x}}^+ + s\boldsymbol{w}_i + f_i(s)\boldsymbol{n} \in H_i$ holds. Here, $\boldsymbol{c}(\bar{\boldsymbol{x}} + s\boldsymbol{w}_i)$ is a curve on K_r, whose projection to H_i is a curve $c_i(s)$ on H. From the representation of $c_i(s) = \bar{\boldsymbol{x}}^+ + s\boldsymbol{w}_i + f_i(s)\boldsymbol{n}$, we can deduce that curvature of the curve c_i at $c_i(0) = \bar{\boldsymbol{x}}^+$ is $\frac{d^2 f_i}{ds^2}(0) = \lambda_i$ and that the curvature radius is $r_i = \frac{1}{\lambda_i}$. The λ_i $(1 \le i \le r)$ are called principal curvatures, and r_i $(1 \le i \le r)$ are the radii of principal curvatures. Because of $(\ \boldsymbol{I}_r - D_{\boldsymbol{t}}{}^2\langle\ \boldsymbol{c}\ , \bar{\boldsymbol{x}}^-\rangle(\boldsymbol{0})\ )^{-1} = (\ \boldsymbol{I}_r + \|\bar{\boldsymbol{x}}^-\|A\ )^{-1}$ it is readily inferred that eigenvalues of $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}})$ are $\frac{1}{1+\|\bar{\boldsymbol{x}}^-\|\lambda_i} = \frac{r_i}{\|\bar{\boldsymbol{x}}^-\| + r_i}$ $(1 \le i \le r)$ and $(n-r)$ 0's. From this relation between eigenvalues of $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}})$ and principal curvatures, we define principal curvatures of K at $\bar{\boldsymbol{x}}$ as λ_i $(1 \le i \le r)$ and $\lambda_i = 0$ $(r+1 \le i \le n)$. We additionally define radii of principal curvatures of K at $\bar{\boldsymbol{x}}$ to be $r_i = \frac{1}{\lambda_i}$ $(1 \le i \le r)$ and $r_i = \infty$ $(r+1 \le i \le n)$.

The following corollary asserts that for each r $(0 \le r \le n)$, $(D_{\boldsymbol{x}}\rho_K^+)|int(V_r) = D_{\boldsymbol{x}}(\rho_K^+|int(V_r))$ expands continuously to V_r under Condition 5.1.

Corollary 5.4. *Suppose that a closed convex set $K \subset \boldsymbol{R}^n$ has C^2 stratification 5.1. Then the following (i), (ii), and (iii) hold.*

(i) $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}}) = D_{\boldsymbol{x}}\rho_{K_r}^+(\bar{\boldsymbol{x}})$ *holds for any* $\bar{\boldsymbol{x}} \in E_{\rho_K^+} \bigcap V_r$.

(ii) $D_{\boldsymbol{x}}\rho_{K_r}^+(\bar{\boldsymbol{x}}) \in \partial_B\rho_K^+(\bar{\boldsymbol{x}})$ *holds for any* $\bar{\boldsymbol{x}} \in V_r$.

(iii) $\lim_{\substack{\boldsymbol{x} \in int(V_r) \\ \boldsymbol{x} \to \bar{\boldsymbol{x}}}} D_{\boldsymbol{x}}\rho_K^+(\boldsymbol{x}) = P \begin{pmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} P^T$ *holds for* $\bar{\boldsymbol{x}} \in K_r$*, where* $P = (\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r) \in O(n)$ *is defined such that* $\boldsymbol{u}_i$ $(1 \le i \le r)$ *is any orthonormal basis of* $T_{\bar{\boldsymbol{x}}}K_r$ *and* $\boldsymbol{u}_i$ $(r+1 \le i \le n)$ *is any orthonormal basis of* $(T_{\bar{\boldsymbol{x}}}K_r)^\perp$.

Proof: (i): Suppose that ρ_K^+ is differentiable at $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}}^+ \in K_r$. It is readily inferred either differential geometrically or from the proof of Theorem 5.2 that there exists an open neighborhood $U \subset \boldsymbol{R}^n$ of $\bar{\boldsymbol{x}}$ where the Euclidean projector $\rho_{K_r}^+$ is defined and $\rho_{K_r}^+$ is a C^1 map from U to K_r. From $\rho_{K_r}^+|(U \bigcap V_r) = \rho_K^+|(U \bigcap V_r)$ and $\rho_{K_r}^+$ is C^1 on U, it is readily inferred that $D_{\boldsymbol{x}}\rho_K^+(\bar{\boldsymbol{x}}) = D_{\boldsymbol{x}}\rho_{K_r}^+(\bar{\boldsymbol{x}})$ hold.
(ii): $D_{\boldsymbol{x}}\rho_{K_r}^+(\boldsymbol{x})$ is continuous on V_r. Therefore, (ii) follows immediately from (i).
(iii): $D_{\boldsymbol{x}}\rho_{K_r}^+(\bar{\boldsymbol{x}})$ is C^1 on a neighborhood of $\bar{\boldsymbol{x}}$. Consequently, it follows from Theorem 5.2 that $D_{\boldsymbol{x}}\rho_{K_r}^+(\bar{\boldsymbol{x}}) = P \begin{pmatrix} \boldsymbol{I}_r & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} P^T$ for $\bar{\boldsymbol{x}} \in K_r$. Part (iii) follows directly from part (i) and the fact that $int(V_r) \subset E_{\rho_K^+} \bigcap V_r$. ∎

Remark 5.5. It readily follows from (i) of Corollary 5.4 that the B-subderivative $\partial_B\rho_K^+$ of the Euclidean projector ρ_K^+ onto any closed convex set K with C^2 stratification 5.1 is "blind" to sets of Lebesgue measure zero as similarly as the generalized Jacobians, i.e., $\partial_B\rho_K^+(\bar{\boldsymbol{x}}) = \{\ \lim_{k\to\infty} D_{\boldsymbol{x}}f(\boldsymbol{x}_k) : \boldsymbol{x}_k \in E_f \setminus \mathcal{N}, (k = 1, 2, \cdots)$ and $\lim_{k\to\infty} \boldsymbol{x}_k = \bar{\boldsymbol{x}}\ \}$ holds for any set $\mathcal{N}$ of Lebesgue measure zero.

It is known that $E_{\rho^+_{S_+(n)}} = S^*(n)$ holds by Theorem 4.6 and Lemma 4.8 of [24], i.e., $E_{\rho^+_K} = \amalg_{r=0}^n int(V_r)$ holds in case of $K = S_+(n)$. Contrary to the case of $K = S_+(n)$, in the general setting of closed convex sets with C^2 stratification 5.1, we will show in Remark 5.6 an example in which $E_{\rho^+_K} \supseteq \amalg_{r=0}^n int(V_r)$ holds and $E_{\rho^+_K} \neq \amalg_{r=0}^n int(V_r)$.

Remark 5.6. We will make an example of $K \subset \boldsymbol{R}^2$ with C^2 stratification 5.1 that does not satisfy $E_{\rho^+_K} \neq \amalg_{r=0}^2 int(V_r)$.

Let $\alpha,\ \beta$ be $0 < \alpha,\ \beta < 1$ and $f(x) = x^{1+\alpha}$ and $g(x) = x^{\frac{1}{1+\beta}}$ be real valued functions of one variable. Consider $K = \{\ (x,y) \in \boldsymbol{R}^2 : 0 \le x \le 1 \text{ and } f(x) \le y \le g(x)\ \}$. Let

$$\left\{ \begin{array}{rcl} \boldsymbol{0} & = & (0,0) \\ \boldsymbol{a} & = & (1,1) \\ K_{11} & = & \{(x, y \in \boldsymbol{R}^2) : y = f(x) = x^{1+\alpha}\ (0 < x < 1)\ \} \\ K_{12} & = & \{(x, y \in \boldsymbol{R}^2) : y = g(x) = x^{\frac{1}{1+\beta}}\ (0 < x < 1)\ \} \end{array} \right. .$$

Then each stratum of K is given as

$$\left\{ \begin{array}{rcl} K_0 & = & \{\boldsymbol{0},\ \boldsymbol{a}\} \\ K_1 & = & K_{11} \amalg K_{12} \\ K_2 & = & \{(x,y) \in \boldsymbol{R}^2 : 0 < x < 1 \text{ and } f(x) < y < g(x)\ \} \end{array} \right.$$

From this, it is readily inferred that K satisfies C^2 stratification 5.1. Direct calculation obtains

$$\left\{ \begin{array}{rcl} \sigma_K(\boldsymbol{0}) & = & \{(x,y) \in \boldsymbol{R}^2 : x \le 0 \text{ and } y \le 0\} \\ \sigma_K(\boldsymbol{a}) & = & \left\{ (x,y) \in \boldsymbol{R}^2 : \begin{array}{c} (x \le 1 \text{ and } y \ge -(1+\beta)(x-1)+1\ ) \\ \text{or } (x \ge 1 \text{ and } y \le -\frac{1}{1+\alpha}(x-1)+1\ ) \end{array} \right\} \end{array} \right. .$$

Let

$$\left\{ \begin{array}{rcl} V_{11} & = & \left\{ (x,y) \in \boldsymbol{R}^2 : \begin{array}{l} (0 < x \le 1 \text{ and } y \le f(x)\ ) \\ \text{ or } (x \ge 1 \text{ and } y \le -\frac{1}{1+\alpha}(x-1)+1\ ) \end{array} \right\} \\ V_{12} & = & \left\{ (x,y) \in \boldsymbol{R}^2 : \begin{array}{l} (x \le 0 \text{ and } 0 \le y \le -(1+\beta)(x-1)+1\ ) \\ \text{ or } (0 \le x \le 1 \text{ and } g(x) \le y \le -(1+\beta)(x-1)+1\ ) \end{array} \right\} \end{array} \right. .$$

Then the following holds:

$$\left\{ \begin{array}{rcl} V_0 & = & \sigma_K(\boldsymbol{0}) \amalg \sigma_K(\boldsymbol{a}) \\ V_1 & = & V_{11} \amalg V_{12} \\ V_2 & = & K_2 \end{array} \right. .$$

First, consider the case of $\boldsymbol{x}^+ = (x,y) = (x, f(x)) \in K_{11}$. The radius $r(\boldsymbol{x}^+)$ of the principal curvature of K_{11} at $\boldsymbol{x}^+$ is $r(\boldsymbol{x}^+) = \frac{(1+(f'(x))^2)^{\frac{3}{2}}}{f''(x)} = \frac{(1+((1+\alpha)x^\alpha)^2)^{\frac{3}{2}}}{(1+\alpha)\alpha x^{\alpha-1}}$. From this formula, it is readily inferred that

$$\lim_{\substack{\boldsymbol{x} \in K_{11} \\ \boldsymbol{x} \to \boldsymbol{0}}} r(\boldsymbol{x}^+) \ = \ 0. \tag{21}$$

Secondly, consider the case of $\boldsymbol{x}^+ = (x, y) = (x, g(x)) \in K_{12}$. In this case, we use the inverse function h of $g(x)$ restricted on $0 \le x \le 1$, i.e., $h(y) = y^{1+\beta}$. Then the radius $r(\boldsymbol{x}^+)$ of principal curvature of K_{12} at $\boldsymbol{x}^+ = (h(y), y)$ is $r(\boldsymbol{x}^+) = \frac{(1+(h'(y))^2)^{\frac{3}{2}}}{h''(y)} = \frac{(1+((1+\beta)y^{\beta})^2)^{\frac{3}{2}}}{(1+\beta)\beta y^{\beta-1}}$ and

$$\lim_{\substack{\boldsymbol{x}\in K_{12}\\ \boldsymbol{x}\to\boldsymbol{0}}} r(\boldsymbol{x}^+) \;=\; 0. \tag{22}$$

From Theorem 5.2 and Remark 5.3 and the results of (21) and (22), it follows that $\partial_B \rho_K^+(\boldsymbol{x}) = \{\boldsymbol{O}\}$ holds for $\boldsymbol{x} \in Z = \{(x, 0) \in \boldsymbol{R}^2 : x < 0\} \amalg \{(0, y) \in \boldsymbol{R}^2 : y < 0\}$. Therefore, from Proposition 3.11 it is readily inferred that ρ_K^+ is differentiable at $\boldsymbol{x} \in Z$, i.e., $E_{\rho_K^+} \ne \amalg_{r=0}^2 int(V_r)$. In fact, through a tedious calculation, one can deduce that $E_{\rho_K^+} = (\amalg_{r=0}^2 int(V_r)) \amalg Z$.

Before ending this section we would like to make the following conjecture.

Conjecture 5.7. *If a closed convex set K has C^2 stratification 5.1, then the Euclidean projector ρ_K^+ might be directionally differentiable and the regular boundary condition 4.4 might hold.*

If the above conjecture 5.7 is proved, then Corollary 4.9 will hold for any closed convex cone pointed at $\boldsymbol{0}$ having C^2 stratification 5.1.

6 Conclusions.

We have investigated the relation between strong stability of a stationary solution $\bar{\boldsymbol{x}}^+$ of $\mathbf{Pro}(\bar{f}, \bar{h}; K)$ with its associate stationary point $(\bar{\boldsymbol{x}}, \bar{\lambda})$ and the B-subderivative $\partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$ under the LICQ condition 2.3. First, we introduced a simple condition about the B-subderivative $\partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$ such that the Euclidean projector ρ_K^+ is semismooth. Consequently, it follows that the Euclidean projector onto any closed convex cone pointed at $\boldsymbol{0}$ is semismooth. Secondly, under the condition that the Euclidean projector ρ_K^+ is semismooth and with the additional assumption of the regular boundary condition 4.4 for K, we proved that a locally isolated stationary solution $\bar{\boldsymbol{x}}^+$ of program $\mathbf{Pro}(\bar{f}, \bar{h}; K)$ is strongly stable if and only if the coherent orientation property holds for $\partial_B\psi(\bar{\boldsymbol{x}}, \bar{\lambda}; \bar{f}, \bar{h})$. This characterization is considered as a complete generalization of Kojima's theory stated in Theorem 3.3 and Corollary 4.3 of [14]. Thirdly, we treated a closed convex set $K \subset \boldsymbol{R}^n$ satisfying C^2 stratification 5.1. Furthermore, we interpreted the regular boundary condition 4.4 geometrically in terms of principal curvatures of the stratum.

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