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A dual form of Markov renewal equation and its application to asymptotic analysis of the single-server queue with a Markovian arrival stream

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Abstract

We consider a dual form of Markov renewal equation and derive limit results for the solution of it by analogy with the results for the standard form of Markov renewal equation. As an application of the derived results, we consider the tail asymptotics of the stationary workload distribution of the single-server queue with a Markovian arrival stream and show that our approach can give a simpler and more direct proof to the existing result.

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1 Introduction

The theory of Markov renewal equations is well known to be a powerful tool for studying many stochastic models (see, e.g., Çinlar [2] and Asmussen [1]). A standard form of Markov renewal equation is given by $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{R} * \boldsymbol{f}$ that is the matrix expression of

$$f_i(x) = g_i(x) + \sum_{j\in E} \int_0^x \mathrm{d}R_{i,j}(y)\, f_j(x-y), \quad x \ge 0,\ i \in E, \tag{1}$$

where E is a finite set called a state space, $\boldsymbol{f}$ is an unknown vector on E and $\boldsymbol{g}$ is a known vector on E such that their ith elements f_i and g_i, $i \in E$, are both functions on $[0,\infty)$, and $\boldsymbol{R}$ is a known matrix on $E \times E$ such that its (i,j)th element $R_{i,j}$, $i,j \in E$, is a nonnegative and nondecreasing function on $[0,\infty)$ with $\lim_{x\to\infty} R_{i,j}(x) < \infty$. In this note, instead of (1), we consider a dual form of Markov renewal equation $\boldsymbol{\phi} = \boldsymbol{\xi} + \boldsymbol{\phi} * \boldsymbol{R}$ that is the matrix expression of

$$\phi_j(x) = \xi_j(x) + \sum_{i\in E} \int_0^x \phi_i(x-y)\, \mathrm{d}R_{i,j}(y), \quad x \ge 0,\ j \in E, \tag{2}$$

where $\boldsymbol{\phi}$ is an unknown vector on E and $\boldsymbol{\xi}$ is a known vector on E such that their jth elements ϕ_j and ξ_j, $j \in E$, are both functions on $[0,\infty)$ (throughout this note, we use boldface Latin lowercase letters for column vectors, boldface Greek lowercase letters for row vectors and boldface uppercase letters for matrices). One may think that (2) is just a transposition of (1) and it is true in a sense. For example, it can be shown in a similar manner to the standard form that, under some weak conditions, (2) has the unique solution $\boldsymbol{\phi} = \sum_{n=0}^{\infty} \boldsymbol{\xi} * \boldsymbol{R}^{*n}$, where $\boldsymbol{R}^{*n}$ is defined inductively by $\boldsymbol{R}^{*0} = \boldsymbol{I}$ (identity matrix) and

$$R^{*n}_{i,j}(x) = \sum_{k\in E} \int_0^x \mathrm{d}R_{i,k}(y)\, R^{*(n-1)}_{k,j}(x-y), \quad x \ge 0,\ i,j \in E,\ n = 1,2,\dots.$$

This corresponds to the known result that the unique solution of (1) is given by $\boldsymbol{f} = \sum_{n=0}^{\infty} \boldsymbol{R}^{*n} * \boldsymbol{g}$ under the corresponding conditions (see, e.g., [1, 2]). However, (1) and (2) are different when, in particular, the limits of the solutions as $x \to \infty$ are concerned under the condition that $\boldsymbol{R}(\infty) = \lim_{x\to\infty} \boldsymbol{R}(x)$ is a stochastic matrix since, even if $\boldsymbol{R}(\infty)$ is stochastic, its transposition is not in general. In the former part of this note, we derive the limit results of the solution of (2) by analogy with the results for the standard form.

A motivation of this work comes from recent development of the study of stochastic models with Markovian environment, where a performance index is often provided as a row vector $\boldsymbol{\phi}(x)$, $x \ge 0$, such that its jth element $\phi_j(x)$ denotes the steady-state joint probability that some performance quantity is greater (or smaller) than x and the underlying Markov chain is in state $j \in E$. Thus, in the latter part of the note, we apply the limit results for the solution of (2) to the tail asymptotics of the stationary workload distribution for the single-server queue with a Markovian arrival stream. The asymptotic result we consider here was, in fact, proved by Takine [8] and Miyazawa [5]. In the proof by Takine [8], however, one has to assume the existence of the limits in advance because his proof relies on the Tauberian theorem (see, e.g., Feller [3, Chapter XIII]). On the other hand, Miyazawa [5] first derived the asymptotic result of the ruin probability for the corresponding risk process by applying a standard form of Markov renewal equation and then obtained the asymptotics for the queue by considering the time-reversed process. Here, we show that our approach due to the dual form of Markov renewal equation can give a simpler and more direct proof to the same result without, for example, considering the time-reversed process.

This note is organized as follows. In the next section, we derive the limit results for the solution of (2) by analogy with those for the standard form. Then, in Section 3, we apply the derivation in Section 2 to the tail asymptotics for the single-server queue with a Markovian arrival stream, where we will see that the existing result is easily proved by using the limit result for the dual form of Markov renewal equation and some matrix calculations.

2 Limit results for dual form of Markov renewal equations

Throughout this section, we impose the following assumption on the matrix function $\boldsymbol{R}$ in (2).

Assumption 2.1 (i) $\boldsymbol{R}(\infty)$ is irreducible and aperiodic; that is, there exists a positive integer n such that $\boldsymbol{R}(\infty)^n$ is strictly positive.

(ii) $\boldsymbol{R}$ is nonlattice in the sense that each $R_{i,j}$, $i,j \in E$, is *not* a step function such that it jumps only on the set $\{\delta_{i,j}, \delta_{i,j}+\delta, \delta_{i,j}+2\,\delta, \ldots\}$ for some $\delta_{i,j} \geq 0$ and $\delta > 0$.

For convenience, we further suppose that $R_{i,j}(0) = 0$ for each pair $i, j \in E$. If $\boldsymbol{R}(\infty)$ is a stochastic matrix, we say that the Markov renewal equation is proper. We consider the proper case first and then extend the result to the case where (2) is not necessarily proper.

In the proper Markov renewal equation, $\boldsymbol{R}$ is interpreted as the semi-Markov kernel of a Markov renewal point process $\{(T_n, M_n)\}_{n\in\mathbb{Z}}$, where $\{M_n\}_{n\in\mathbb{Z}}$ is a discrete-time Markov chain on E driven by transition matrix $\boldsymbol{R}(\infty)$ and $\mathrm{P}(T_{n+1} - T_n \leq x \mid M_n = i,\ M_{n+1} = j) = R_{i,j}(x)/R_{i,j}(\infty)$ for $x \geq 0$, $n \in \mathbb{Z}$ and $i, j \in E$ such that $R_{i,j}(\infty) > 0$. Under Assumption 2.1(i), the Markov chain $\{M_n\}_{n\in\mathbb{Z}}$ has a unique (up to a multiplicative factor) invariant measure, that is a positive solution $\boldsymbol{\nu}$ of $\boldsymbol{\nu} = \boldsymbol{\nu}\,\boldsymbol{R}(\infty)$. In the Markov renewal process $\{(T_n, M_n)\}_{n\in\mathbb{Z}}$ satisfying Assumption 2.1(i), the return times to each state in E form a nonterminating renewal process. Furthermore, with Assumption 2.1(ii), the distribution of the inter-return times to each state is nonlattice (see Proposition 2.27 in [2, Chapter 10]). Let $\boldsymbol{m} = \int_0^\infty x\,\mathrm{d}\boldsymbol{R}(x)\,\boldsymbol{e}$, where $\boldsymbol{e}$ denotes the column vector on E such that each element is equal to unity; that is, let m_i denote the mean sojourn time of each visit to state $i \in E$. Then, the mean inter-return time to state $i \in E$ is given by $\boldsymbol{\nu}\,\boldsymbol{m}/\nu_i$ (see, e.g., [1, 2]) and the following limit result holds.

Lemma 1 *Suppose that each ξ_i, $i \in E$, is nonnegative and directly Riemann integrable (see, e.g., [1, 2] for its definition). Then, under Assumption 2.1,*

$$\lim_{x\to\infty} \phi_j(x) = \frac{\nu_j}{\boldsymbol{\nu}\,\boldsymbol{m}} \int_0^\infty \boldsymbol{\xi}(x)\,\boldsymbol{e}\,\mathrm{d}x, \quad j \in E. \tag{3}$$

Proof: The proof follows by analogy with that in the standard case (see, e.g., Proposition 4.9 in [2, Chapter 10]) and is omitted. □

Remark 1 In the standard Markov renewal equation (1), the limit of the solution $\boldsymbol{f}$ is given by

$$\lim_{x\to\infty} f_i(x) = \frac{1}{\boldsymbol{\nu}\,\boldsymbol{m}} \int_0^\infty \boldsymbol{\nu}\,\boldsymbol{g}(x)\,\mathrm{d}x, \quad i \in E,$$

provided that each g_i, $i \in E$, is nonnegative and directly Riemann integrable, where we can see that the limit of $f_i(x)$ as $x \to \infty$ is invariant of $i \in E$. In contrast, we can note that the limit of ϕ_j in (3) is proportional to the steady-state probability that the underlying Markov chain $\{M_n\}_{n\in\mathbb{Z}}$ is in state $j \in E$.

Now, we do not necessarily suppose that the Markov renewal equation (2) is proper; that is, we admit that $\boldsymbol{R}(\infty)$ is not stochastic. Let $\widehat{\boldsymbol{R}}(\theta)$ denote the moment-generating function of $\boldsymbol{R}$ for a real number θ; that is,

$$\widehat{\boldsymbol{R}}(\theta) = \int_0^\infty e^{\theta x}\,\mathrm{d}\boldsymbol{R}(x).$$

Clearly, $\widehat{\boldsymbol{R}}(\theta)$ always exists if θ is not positive. Also, we can consider the case where $\boldsymbol{R}(x)$ has light tails; that is, there is a $\theta_0 > 0$ such that $\widehat{\boldsymbol{R}}(\theta)$ exists for all $\theta < \theta_0$, where θ_0 is possibly infinity. Under Assumptions 2.1(i), $\widehat{\boldsymbol{R}}(\theta)$ is, if exists, also nonnegative and irreducible, and therefore, by the Perron-Frobenius theory (see, e.g., Seneta [7]), it has a positive eigenvalue $\delta(\theta)$ that dominates the real parts of all other eigenvalues and the associated left and right eigenvectors are positive. Denote these eigenvectors by $\boldsymbol{\eta}(\theta)$ and $\boldsymbol{h}(\theta)$ respectively. We here assume the following.

Assumption 2.2 There exists an $\alpha \in \mathbb{R}$ such that $\delta(\alpha) = 1$.

A condition under which the α in Assumption 2.2 exists is found in Problem 4.3 of [1, Chapter VII] and, in the case where $\boldsymbol{R}(\infty)$ is substochastic and $\boldsymbol{R}$ has light tails, the condition under which the positive α exits is discussed in [5]. Using the α in Assumption 2.2, define an $E \times E$-matrix function $\boldsymbol{R}^\dagger$ by

$$R^\dagger_{i,j}(x) = \frac{h_j(\alpha)}{h_i(\alpha)} \int_0^x e^{\alpha y}\, \mathrm{d}R_{i,j}(y), \quad x \ge 0,\ i,j \in E.$$

Then, we can see that $\boldsymbol{R}^\dagger(\infty) = \lim_{x\to\infty} \boldsymbol{R}^\dagger(x)$ is stochastic and, under Assumption 2.1(ii), $\boldsymbol{R}^\dagger$ is also nonlattice. An invariant measure $\boldsymbol{\nu}^\dagger$ of $\boldsymbol{R}^\dagger(\infty)$ is given by $\nu^\dagger_i = \eta_i(\alpha)\, h_i(\alpha)$, $i \in E$, and the vector of the mean sojourn times $\boldsymbol{m}^\dagger$ is given by

$$m^\dagger_i = \frac{1}{h_i(\alpha)} \sum_{j\in E} h_j(\alpha) \int_0^\infty x\, e^{\alpha x}\, \mathrm{d}R_{i,j}(x), \quad i \in E.$$

Define $\boldsymbol{\phi}^\dagger$ and $\boldsymbol{\xi}^\dagger$ respectively by $\phi^\dagger_j(x) = h_j(\alpha)\, e^{\alpha x}\, \phi_j(x)$ and $\xi^\dagger_j(x) = h_j(\alpha)\, e^{\alpha x}\, \xi_j(x)$, $x \ge 0$, $j \in E$. Then, from (2), we have the proper Markov renewal equation $\boldsymbol{\phi}^\dagger = \boldsymbol{\xi}^\dagger + \boldsymbol{\phi}^\dagger * \boldsymbol{R}^\dagger$ in the dual form, and hence, by applying Lemma 1, we readily obtain the following result which is also a dual of Theorem 4.6 in [1, Chapter VII].

Lemma 2 *Under Assumptions 2.1 and 2.2, if each ξ_i, $i \in E$, is nonnegative and $e^{\alpha x}\, \xi_i(x)$ is directly Riemann integrable, then*

$$\lim_{x\to\infty} e^{\alpha x}\, \phi_j(x) = \frac{\eta_j(\alpha)}{\boldsymbol{\eta}(\alpha)\, \widehat{\boldsymbol{R}}^{(1)}(\alpha)\, \boldsymbol{h}(\alpha)} \int_0^\infty e^{\alpha x}\, \boldsymbol{\xi}(x)\, \mathrm{d}x\, \boldsymbol{h}(\alpha), \quad j \in E, \tag{4}$$

where $\widehat{\boldsymbol{R}}^{(1)}(\alpha) = (\mathrm{d}/\mathrm{d}\theta)\widehat{\boldsymbol{R}}(\theta)\big|_{\theta=\alpha}$.

Remark 2 The α in Assumption 2.2 is, of course, equal to zero in the proper case and that (4) in Lemma 2 then reduces to (3) in Lemma 1.

3 Application to asymptotic analysis of a single-server queue

In this section, we apply Lemma 2 in the preceding section to the tail asymptotics of the stationary workload distribution for the single-server queue with a Markovian arrival stream. We will see that the existing result is proved very easily with some matrix calculations.

Consider a work-conserving single-server queue with an infinite buffer. Customer arrivals and their service times are supposed to follow a Markovian arrival stream with representation $(\boldsymbol{C}, \boldsymbol{D})$, where $\boldsymbol{C}$ denotes an $E \times E$-matrix with negative diagonal elements and nonnegative off-diagonal elements, and $\boldsymbol{D}$ denotes an $E \times E$-matrix function such that its (i,j)th elements $D_{i,j}$, $i,j \in E$, are nonnegative and nondecreasing function on $[0,\infty)$ with $D_{i,j}(\infty) = \lim_{x\to\infty} D_{i,j}(x) < \infty$. The matrix $\boldsymbol{C}+\boldsymbol{D}(\infty)$ is a rate matrix; that is, $(\boldsymbol{C} + \boldsymbol{D}(\infty))\, \boldsymbol{e} = \boldsymbol{0}$, where $\boldsymbol{0}$ denotes the column vector on E such that each element is equal to zero. We suppose that the rate matrix $\boldsymbol{C} + \boldsymbol{D}(\infty)$ is irreducible. Since the state space E is finite, the continuous-time Markov chain driven by the rate matrix $\boldsymbol{C}+\boldsymbol{D}(\infty)$ always has the stationary distribution, which is denoted by $\boldsymbol{\pi}$; that is, $\boldsymbol{\pi}\,(\boldsymbol{C}+\boldsymbol{D}(\infty)) = \boldsymbol{0}$ and $\boldsymbol{\pi}\, \boldsymbol{e} = 1$. We refer to this stationary Markov chain as the underlying Markov chain. When a state transition driven by $\boldsymbol{D}(\infty)$ (including one not changing the current state driven by $D_{i,i}(\infty)$, $i \in E$) occurs, a customer arrives to the queue, where we suppose that there exists at least one pair $(i,j) \in E \times E$ such that $D_{i,j}(\infty) > 0$, and thus the arrivals are certain. The service times of customers whose arrivals are driven by $D_{i,j}(\infty) > 0$ are i.i.d. and according to distribution $D_{i,j}(x)/D_{i,j}(\infty)$. For convenience, we suppose that $D_{i,j}(0) = 0$ for each pair $i, j \in E$. The traffic intensity ρ of the queue is given by $\rho = \boldsymbol{\pi} \int_0^\infty x \, \mathrm{d}\boldsymbol{D}(x) \, \boldsymbol{e}$, which is assumed to be less than unity, and thus the queue is stable (see, e.g., Loynes [4]).

Let V denote the workload in the steady state and let M denote the associated state of the underlying Markov chain. We consider vector distribution $\boldsymbol{\phi}(x)$, $x \geq 0$, whose jth element represents the stationary joint probability $\phi_j(x) = \mathrm{P}(V > x, M = j)$. Then, using the results in Takine [9], we can show the following.

Lemma 3 *$\boldsymbol{\phi}$ satisfies the dual form of Markov renewal equation;*

$$\boldsymbol{\phi}(x) = \boldsymbol{\pi} \, \overline{\boldsymbol{R}}(x) + \int_0^x \boldsymbol{\phi}(x-y) \, \mathrm{d}\boldsymbol{R}(y), \tag{5}$$

where

$$\boldsymbol{R}(x) = \int_0^x \mathrm{d}y \int_y^\infty \mathrm{d}\boldsymbol{D}(w) \, e^{\boldsymbol{Q}(w-y)}, \quad x \geq 0, \tag{6}$$

with $\boldsymbol{R}(\infty) = \lim_{x \to \infty} \boldsymbol{R}(x) < \infty$. Also, $\overline{\boldsymbol{R}}(x) = \boldsymbol{R}(\infty) - \boldsymbol{R}(x)$, $x \geq 0$, and $\boldsymbol{Q}$ is given as a matrix satisfying

$$\boldsymbol{Q} = \boldsymbol{C} + \int_0^\infty \mathrm{d}\boldsymbol{D}(x) \, e^{\boldsymbol{Q}x}. \tag{7}$$

We can easily see that each element of $\boldsymbol{R}$ in (6) is a nonnegative and nondecreasing function on $[0, \infty)$ and it is shown that the matrix $\boldsymbol{Q}$ satisfying (7) is an irreducible rate matrix (see, e.g., [9]). In the proof of Lemma 3 below and thereafter, $\boldsymbol{\kappa}$ denotes the stationary distribution of $\boldsymbol{Q}$; that is, $\boldsymbol{\kappa} \boldsymbol{Q} = \boldsymbol{0}$ and $\boldsymbol{\kappa} \boldsymbol{e} = 1$.

Proof: Let Y denote the random variable representing the queue length under the preemptive last-come, first-served discipline in the steady state. Define the vector distributions $\boldsymbol{\psi}(x)$ and $\boldsymbol{\psi}^{(n)}(x)$, $x \geq 0$, such that their jth elements represent $\psi_j(x) = \pi_j - \phi_j(x) = \mathrm{P}(V \leq x, M = j)$ and $\psi_j^{(n)}(x) = \mathrm{P}(V \leq x, M = j, Y = n)$ respectively. Then, Takine [9] derived that

$$\begin{aligned}
\boldsymbol{\psi}^{(0)}(x) &= (1-\rho) \, \boldsymbol{\kappa}, \quad x \geq 0, \\
\boldsymbol{\psi}^{(n)}(x) &= \int_0^x \boldsymbol{\psi}^{(n-1)}(x-y) \, \mathrm{d}\boldsymbol{R}(y) \quad x \geq 0, \ n = 1, 2, \ldots.
\end{aligned}$$

Thus, summing up over $n = 0, 1, 2, \ldots$, we have

$$\boldsymbol{\psi}(x) = (1-\rho) \, \boldsymbol{\kappa} + \int_0^x \boldsymbol{\psi}(x-y) \, \mathrm{d}\boldsymbol{R}(y), \quad x \geq 0. \tag{8}$$

Hence, we have $\boldsymbol{\pi} = (1-\rho) \, \boldsymbol{\kappa} + \boldsymbol{\pi} \, \boldsymbol{R}(\infty)$ by taking $x \to \infty$ in (8) and obtain (5) by $\boldsymbol{\phi}(x) = \boldsymbol{\pi} - \boldsymbol{\psi}(x)$. $\square$

Now, we assume the following.

Assumption 3.1 $\boldsymbol{D}$ has light tails; that is, there is a $\theta_0 > 0$ such that $\widehat{\boldsymbol{D}}(\theta) = \int_0^\infty e^{\theta x} \, \mathrm{d}\boldsymbol{D}(x)$ exists for all $\theta < \theta_0$, where θ_0 is possibly infinity.

Noting that $(\theta \boldsymbol{I} - \boldsymbol{Q})$, $\theta > 0$, is nonsingular since $\boldsymbol{Q}$ is a rate matrix, we have from (6) that the moment-generating function of $\boldsymbol{R}$ is given by

$$\begin{aligned}
\widehat{\boldsymbol{R}}(\theta) &= \int_0^\infty e^{\theta x} \int_x^\infty \mathrm{d}\boldsymbol{D}(w) \, e^{\boldsymbol{Q}(w-x)} \, \mathrm{d}x \\
&= \int_0^\infty \mathrm{d}\boldsymbol{D}(w) \, e^{\boldsymbol{Q}w} \int_0^w e^{(\theta \boldsymbol{I} - \boldsymbol{Q})x} \, \mathrm{d}x \\
&= \int_0^\infty \mathrm{d}\boldsymbol{D}(w) \, (e^{\theta \boldsymbol{I} w} - e^{\boldsymbol{Q}w}) \, (\theta \boldsymbol{I} - \boldsymbol{Q})^{-1}
\end{aligned}$$

$$= (\widehat{\boldsymbol{D}}(\theta) + \boldsymbol{C} - \boldsymbol{Q})\,(\theta \boldsymbol{I} - \boldsymbol{Q})^{-1}, \quad \theta \in (0, \theta_0), \tag{9}$$

where we use Fubini's theorem in the second equality and (7) in the last equality. Here, for $\theta < \theta_0$, $\boldsymbol{C} + \widehat{\boldsymbol{D}}(\theta)$ is irreducible and Metzler-Leontief (ML); that is, its off-diagonal elements are nonnegative. Thus, by the Perron-Frobenius theory (see, e.g., [7]), it has a real eigenvalue $\delta(\theta)$ that dominates the real parts of all other eigenvalues, and the associated right and left eigenvectors are positive. Let $\boldsymbol{\eta}(\theta)$ and $\boldsymbol{r}(\theta)$, $\theta < \theta_0$, denote the left and right eigenvectors associated with $\delta(\theta)$. It is shown that the equation $\delta(\theta) = \theta$ has at most one solution (see [8]) and we here assume the following.

Assumption 3.2 There exists an $\alpha \in (0, \theta_0)$ such that $\delta(\alpha) = \alpha$.

Then, from (9), we can easily check that $\boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{R}}(\alpha) = \boldsymbol{\eta}(\alpha)$ and $\widehat{\boldsymbol{R}}(\alpha)\,\boldsymbol{h}(\alpha) = \boldsymbol{h}(\alpha)$ with $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$; that is, $\boldsymbol{\eta}(\alpha)$ and $\boldsymbol{h}(\alpha)$ are respectively the left and right eigenvectors of $\widehat{\boldsymbol{R}}(\alpha)$ associated with the eigenvalue equal to unity. We here normalize $\boldsymbol{\eta}(\alpha)$ and $\boldsymbol{r}(\alpha)$ such that $\boldsymbol{\eta}(\alpha)\,\boldsymbol{e} = \boldsymbol{\eta}(\alpha)\,\boldsymbol{r}(\alpha) = 1$. With this set-up, we can prove the following by applying Lemma 2.

Proposition 1 (Theorem 4.6 of [8], Theorem 4.1 of [5]) *Under Assumptions 3.1 and 3.2, we have*

$$\lim_{x\to\infty} e^{\alpha x}\,\phi_j(x) = \frac{(1-\rho)\,\boldsymbol{\kappa}\,\boldsymbol{r}(\alpha)}{\boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{D}}^{(1)}(\alpha)\,\boldsymbol{r}(\alpha) - 1}\,\eta_j(\alpha). \tag{10}$$

Proof: First, it is easy to see from (6) that the matrix function $\boldsymbol{R}$ satisfies Assumption 2.1 in the preceding section. By (5), $\boldsymbol{\xi}$ in Lemma 2 is now given by $\boldsymbol{\xi}(x) = \boldsymbol{\pi}\,\overline{\boldsymbol{R}}(x)$, $x \ge 0$. In order to apply Lemma 2, we have to check that $e^{\alpha x}\,\boldsymbol{\xi}(x)$, $x \ge 0$, is directly Riemann integrable. To this end, it is sufficient to show that $e^{\alpha x}\,\boldsymbol{\xi}(x)$ is integrable since $\boldsymbol{\xi}$ is nonincreasing on $[0,\infty)$ and $e^{\alpha x}$ is nondecreasing with $e^{\alpha x} \to 1$ as $x \downarrow 0$ (see Rolski *et al.* [6, Lemma 6.1.4]). By (6) and applying Fubini's theorem,

$$\begin{aligned}
\int_0^\infty e^{\alpha x}\,\boldsymbol{\xi}(x)\,\mathrm{d}x &= \boldsymbol{\pi}\int_0^\infty \mathrm{d}x\, e^{\alpha x}\int_x^\infty \mathrm{d}\boldsymbol{D}(w)\int_0^{w-x} e^{\boldsymbol{Q}y}\,\mathrm{d}y \\
&= \boldsymbol{\pi}\int_0^\infty \mathrm{d}\boldsymbol{D}(w)\int_0^w \mathrm{d}y\, e^{\boldsymbol{Q}y}\int_0^{w-y} e^{\alpha x}\,\mathrm{d}x \\
&= \frac{\boldsymbol{\pi}}{\alpha}\Big[\int_0^\infty \mathrm{d}\boldsymbol{D}(w)\,e^{\alpha w}\int_0^w e^{(\boldsymbol{Q}-\alpha\boldsymbol{I})y}\,\mathrm{d}y - \int_0^\infty \mathrm{d}\boldsymbol{D}(w)\int_0^w e^{\boldsymbol{Q}y}\,\mathrm{d}y\Big].
\end{aligned} \tag{11}$$

Here, the first term in the brackets of (11) reduces to

$$\begin{aligned}
\int_0^\infty \mathrm{d}\boldsymbol{D}(w)\,e^{\alpha w}\,(e^{(\boldsymbol{Q}-\alpha\boldsymbol{I})w} - \boldsymbol{I})\,(\boldsymbol{Q} - \alpha\,\boldsymbol{I})^{-1} &= \big(\boldsymbol{Q} - \boldsymbol{C} - \widehat{\boldsymbol{D}}(\alpha)\big)\,(\boldsymbol{Q} - \alpha\,\boldsymbol{I})^{-1} \\
&= (\alpha\,\boldsymbol{I} - \boldsymbol{C} - \widehat{\boldsymbol{D}}(\alpha))\,(\boldsymbol{Q} - \alpha\,\boldsymbol{I})^{-1} + \boldsymbol{I},
\end{aligned} \tag{12}$$

where (7) is used in the first equality. On the other hand, for the second term in the brackets of (11), by using the relation $\boldsymbol{\kappa} = \boldsymbol{\kappa}\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1}$ and using the fact that $e^{\boldsymbol{Q}y}$ is a stochastic matrix, we have

$$\begin{aligned}
\int_0^\infty \mathrm{d}\boldsymbol{D}(w)\int_0^w e^{\boldsymbol{Q}y}\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})\,\mathrm{d}y\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1} &= \int_0^\infty \mathrm{d}\boldsymbol{D}(w)\,(\boldsymbol{e}\,\boldsymbol{\kappa}\,w + \boldsymbol{I} - e^{\boldsymbol{Q}w})\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1} \\
&= \big(\widehat{\boldsymbol{D}}^{(1)}(0)\,\boldsymbol{e}\,\boldsymbol{\kappa} + \boldsymbol{D}(\infty) + \boldsymbol{C} - \boldsymbol{Q}\big)\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1} \\
&= \big(\widehat{\boldsymbol{D}}^{(1)}(0) - \boldsymbol{I}\big)\,\boldsymbol{e}\,\boldsymbol{\kappa} + \big(\boldsymbol{C} + \boldsymbol{D}(\infty)\big)\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1} + \boldsymbol{I},
\end{aligned} \tag{13}$$

where we use (7) again in the second equality. Therefore, by (11)–(13), the direct Riemann integrability of $e^{\alpha x}\,\boldsymbol{\xi}(x)$ is verified. Now, substituting (12) and (13) into (11) and then post-multiplying by $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$, we have

$$\int_0^\infty e^{\alpha x}\,\boldsymbol{\xi}(x)\,\mathrm{d}x\,\boldsymbol{h}(\alpha) = \frac{\boldsymbol{\pi}}{\alpha}\Big[(\boldsymbol{I} - \widehat{\boldsymbol{D}}^{(1)}(0))\,\boldsymbol{e}\,\boldsymbol{\kappa} - (\boldsymbol{C} + \boldsymbol{D}(\infty))\,(\boldsymbol{e}\,\boldsymbol{\kappa} - \boldsymbol{Q})^{-1}\Big]\,(\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$$

$$= (1-\rho)\,\boldsymbol{\kappa}\,\boldsymbol{r}(\alpha), \tag{14}$$

where we use $(\alpha\,\boldsymbol{I} - \boldsymbol{C} - \widehat{\boldsymbol{D}}(\alpha))\,\boldsymbol{r}(\alpha) = \boldsymbol{0}$ in the first equality, and $\boldsymbol{\kappa}\,\boldsymbol{Q} = \boldsymbol{\pi}\,(\boldsymbol{C} + \boldsymbol{D}(\infty)) = \boldsymbol{0}$, $\boldsymbol{\pi}\,\boldsymbol{e} = 1$ and $\boldsymbol{\pi}\,\widehat{\boldsymbol{D}}^{(1)}(0)\,\boldsymbol{e} = \rho$ in the second equality. Next, consider the term corresponding to denominator of (4). We have from (6) that

$$\begin{aligned}
\widehat{\boldsymbol{R}}^{(1)}(\alpha) &= \int_0^\infty x\, e^{\alpha x} \int_x^\infty \mathrm{d}\boldsymbol{D}(w)\, e^{\boldsymbol{Q}(w-x)}\, \mathrm{d}x \\
&= \int_0^\infty \mathrm{d}\boldsymbol{D}(w)\, e^{\boldsymbol{Q}w} \int_0^w x\, e^{(\alpha\boldsymbol{I}-\boldsymbol{Q})x}\, \mathrm{d}x \\
&= \int_0^\infty \mathrm{d}\boldsymbol{D}(w)\, e^{\boldsymbol{Q}w} \left[w\, e^{(\alpha\boldsymbol{I}-\boldsymbol{Q})w} - (e^{(\alpha\boldsymbol{I}-\boldsymbol{Q})w} - \boldsymbol{I})\,(\alpha\boldsymbol{I}-\boldsymbol{Q})^{-1} \right] (\alpha\boldsymbol{I}-\boldsymbol{Q})^{-1} \\
&= \left[\widehat{\boldsymbol{D}}^{(1)}(\alpha) - (\widehat{\boldsymbol{D}}(\alpha) + \boldsymbol{C} - \boldsymbol{Q})\,(\alpha\boldsymbol{I}-\boldsymbol{Q})^{-1} \right] (\alpha\boldsymbol{I}-\boldsymbol{Q})^{-1},
\end{aligned}$$

where we use (7) in the last equality. Hence, multiplying by $\boldsymbol{\eta}(\alpha)$ and $\boldsymbol{h}(\alpha) = (\alpha\,\boldsymbol{I} - \boldsymbol{Q})\,\boldsymbol{r}(\alpha)$ from both the sides, we have

$$\begin{aligned}
\boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{R}}^{(1)}(\alpha)\,\boldsymbol{h}(\alpha) &= \boldsymbol{\eta}(\alpha) \left[\widehat{\boldsymbol{D}}^{(1)}(\alpha) - \boldsymbol{I} \right] \boldsymbol{r}(\alpha) \\
&= \boldsymbol{\eta}(\alpha)\,\widehat{\boldsymbol{D}}^{(1)}(\alpha)\,\boldsymbol{r}(\alpha) - 1, \qquad (15)
\end{aligned}$$

where we use $\boldsymbol{\eta}(\alpha)\,(\widehat{\boldsymbol{D}}(\alpha) + \boldsymbol{C}) = \alpha\,\boldsymbol{\eta}(\alpha)$ in the first equality. Finally, substituting (14) and (15) into (4), we obtain (10). □

4 Concluding remark

In this note, we have derived the limit results for the solution of a dual form of Markov renewal equation and then applied this derivation to the tail asymptotics for the stationary workload distribution of the single-server queue with a Markovian arrival stream. We have seen that our approach can give a simpler and more direct proof than the existing proofs. It would further be expected that our approach could help the analysis of other stochastic models with Markovian environments.

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