ISSN 1342-2804

Research Reports on Mathematical and Computing Sciences

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March 2010, B–461

Department of
Mathematical and
Computing Sciences
Tokyo Institute of Technology

SERIES B: Operations Research

Odds theorem with multiple selection chances

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Abstract

We study the multi-selection version of so-called *odds theorem* by Bruss (2000). We observe a finite number of independent 0/1 (failure/success) random variables sequentially and want to select the last success. We derive the optimal selection rule when we are given m (≥ 1) selection chances and find that the optimal rule has the form of combination of multiple odds-sums. We provide a formula for computing the maximum probability of selecting the last success when we have m selection chances and also give closed-form formulas for $m = 2$ and 3. For $m = 2$, we further give the bounds for the maximum probability of selecting the last success and derive its limit as the number of observations goes to infinity. An interesting implication of our result is that the limit of the maximum probability of selecting the last success for $m = 2$ is consistent to the corresponding limit for the classical secretary problem with two selection chances.

Keywords: Optimal stopping problem, selecting the last success, multiple selection chances.

1 Introduction

For a positive integer N, let $X_1, X_2, \ldots, X_N$ denote independent 0/1 random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We observe these X_i's sequentially and say that the ith trial is a success if $X_i = 1$. The problem is to find a rule $\tau \in \mathcal{T}$ to maximize the probability of selecting the last success, where $\mathcal{T}$ is the class of all selection rules such that $\{\tau = j\} \in \sigma(X_1, X_2, \cdots, X_j)$; that is, the decision to select the jth success or not depends on the information up to j. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ and let $p_i = \mathrm{P}(X_i = 1)$ and $q_i = 1 - p_i = \mathrm{P}(X_i = 0)$ for $i \in \mathcal{N}$. In addition, let r_i, $i \in \mathcal{N}$, denote the odds of the ith trial; that is, $r_i = p_i/q_i$, where we set $r_i = +\infty$ if $p_i = 1$. When exactly one selection chance is allowed, Bruss (2000) solved the problem with elegant simpleness as follows.

Proposition 1.1 (Theorem 1 in Bruss (2000)) *Suppose that exactly one selection chance is given in the problem above. Then, the optimal selection rule $\tau_*^{(1)}$ selects the first success after the sum of the future odds becomes less than one; that is,*

$$\tau_*^{(1)} = \min\bigl\{i \geq i_*^{(1)} : X_i = 1\bigr\}, \tag{1.1}$$

$$i_*^{(1)} = \min\Bigl\{i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1\Bigr\}, \tag{1.2}$$

where $\min(\emptyset) = +\infty$ *and* $\sum_{j=a}^{b} \cdot = 0$ *when $b < a$ conventionally. Furthermore, the maximum probability of "win" (selecting the last success) is given by*

$$P^{(1)}(\mathrm{win}) = P_N^{(1)}(p_1, \ldots, p_N) = \prod_{k=i_*^{(1)}}^{N} q_k \sum_{k=i_*^{(1)}}^{N} r_k. \tag{1.3}$$

This result, referred to as the sum-the-odds theorem or shortly odds theorem, is attractive since it can be applied to many basic optimal stopping problems such as betting, the classical secretary problem (CSP) and the group-interview secretary problem proposed by Hsiau and Yang (2000). Bruss (2000) also showed that $P^{(1)}(\mathrm{win})$ in (1.3) is bounded below by $R^{(1)}\, e^{-R^{(1)}}$ with $R^{(1)} = \sum_{j=i_*^{(1)}}^{N} r_j$, and remarkably, he found in (2003) that it is bounded below by e^{-1} when $\sum_{j=1}^{N} r_j \geq 1$. These results generalize the known lower bounds for the CSP, where each p_i has the specific value of $p_i = 1/i$ for $i \in \mathcal{N}$ (see, e.g., Hill and Krengel (1992)).

After Bruss (2000), where the case with the random number of observations was also considered, the odds theorem has been extended in several directions. Bruss and Paindaveine (2000) extended it to the problem of selecting the last ℓ (> 1) successes. Hsiau and Yang (2002) considered the problem with Markov-dependent trials. Recently, Ferguson (2008) extended the odds theorem in some ways, where the infinite number of trials is allowed, the payoff for not selecting to the end is different from the payoff for selecting a success that is not the last, and the trials are generally dependent. Furthermore, he applied his extension to the stopping game of Sakaguchi (1984).

In this paper, we consider yet another extension of the result by Bruss (2000); that is, we are interested in the problem with multiple selection chances. In our first main result, we derive the optimal rule for the problem of selecting the last success with m $(\in \mathcal{N})$ selection chances and find that the optimal rule is expressed as a combination of multiple odds-sums. Our extension is, of course, applied to the multi-selection versions of the problems to which the odds theorem can be applied (e.g., the CSP with multiple selection chances in Gilbert and Mosteller (1966) and Sakaguchi (1978)). In our second main result, we provide a formula that can be used for computing the probability of win for the problem with m $(\in \mathcal{N})$ selection chances and give the closed-form formulas for $m = 2$ and 3. Furthermore, we show the lower and upper bounds for the maximum probability of win for $m = 2$ and derive its limit as $N \to \infty$ under some condition of p_i, $i \in \mathcal{N}$. This limit of the maximum probability of win is consistent to the known limit $e^{-1} + e^{-3/2}$ for the CSP with two selection chances (see, e.g., Gilbert and Mosteller (1966), Ano and Ando (2000) or Bruss (1988)).

The paper is organized as follows. In Section 2, we consider the optimal rule for the problem of selecting the last success with m ($\in \mathcal{N}$) selection chances. Our approach is essentially based on the technique of Ano and Ando (2000), in which they studied the condition for the monotone (equivalently, one-step look-ahead) selection rule to be optimal in multiple selection problems. For greater details of the monotone selection problem, see Chow et al. (1971) or Ferguson (2006). In Section 3, we derive some formulas for the maximum probability of win. We give the bounds for the maximum probability of win for $m = 2$ and derive its limit as $N \to \infty$ under some condition of p_i, $i \in \mathcal{N}$. At the last, we conclude the paper making conjectures on the limits of the maximum probability of win for $m \geq 3$ and on the lower bound for $m \geq 2$.

2 Multiple sums-the-odds theorem

Suppose that we are given m ($\in \mathcal{N}$) selection chances in the problem described in the preceding section. Let $V_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and select this success when we have at most m selection chances left. Let $W_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and ignore this success when we have at most m selection chances left. Let, furthermore, $M_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and are faced with a decision to select or not when we have at most m selection chances left. The optimality equation is then given by

$$M_i^{(m)} = \max\{V_i^{(m)}, W_i^{(m)}\}, \quad i \in \mathcal{N}. \tag{2.1}$$

Clearly, if $m > N - i$ (the remaining selection chances are more than the remaining observations) and we observe $X_i = 1$, then the decision to select brings us win with probability 1, so that $M_i^{(m)} = V_i^{(m)} = 1$ for $i > N - m$. In particular, we have $M_N^{(m)} = V_N^{(m)} = 1$ and $W_N^{(m)} = 0$ for any $m \in \mathcal{N}$.

We can see that $V_i^{(m)}$ is represented as the sum of two conditional probabilities; one is that no success appears in $i+1, \ldots, N$ provided that $X_i = 1$ and the other is that we win finally in starting at $i+1$ with $m-1$ selection chances provided that $X_i = 1$. Since the latter conditional probability is just equal to $W_i^{(m-1)}$, we have

$$\begin{aligned} V_i^{(m)} &= \mathrm{P}(X_{i+1} = X_{i+2} = \cdots = X_N = 0 \mid X_i = 1) + W_i^{(m-1)} \\ &= \prod_{j=i+1}^{N} q_j + W_i^{(m-1)}, \quad i \in \mathcal{N}, \end{aligned} \tag{2.2}$$

where we set $W_i^{(0)} := 0$ for $i \in \mathcal{N}$ and $\prod_{j=a}^{b} \cdot = 1$ when $b < a$ conventionally. The second equality above follows from the independence of X_i's. On the other hand, $W_i^{(m)}$ is given as the conditional probability that we make the optimal decision at the first success after i and win finally provided that $X_i = 1$, so that,

$$\begin{aligned} W_i^{(m)} &= \sum_{j=i+1}^{N} \mathrm{P}(X_{i+1} = \cdots = X_{j-1} = 0, X_j = 1 \mid X_i = 1)\, M_j^{(m)} \\ &= \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j\, M_j^{(m)}, \quad i \in \mathcal{N}. \end{aligned} \tag{2.3}$$

As a preparation to studying the problem with multiple selection chances, we here give another proof of the odds theorem (Proposition 1.1) by using the notion of monotone stopping rule in Chow et al. (1971).

Another Proof of Proposition 1.1: We prove the first part of Proposition 1.1 only. The monotone selection region for the single selection problem is given by $B^{(1)} := \{i \in \mathcal{N} : G_i^{(1)} > 0\}$, where

$$G_i^{(1)} := V_i^{(1)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(1)}, \quad i \in \mathcal{N}. \tag{2.4}$$

Note that $B^{(1)}$ is the region of $i \in \mathcal{N}$ such that the probability of win by selecting $X_i = 1$ is greater than that by ignoring $X_i = 1$ and then selecting the first success after X_i. From (2.2), we have $V_i^{(1)} = \prod_{j=i+1}^{N} q_j$ and, if there exists a $j \in \{i+1, \ldots, N\}$ such that $q_j = 0$, then (2.4) leads to $G_i^{(1)} \leq 0$. If $q_j > 0$ for all $j = i+1, \ldots, N$, on the other hand, (2.4) is written as

$$\begin{aligned} G_i^{(1)} &= \prod_{j=i+1}^{N} q_j - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \left(\prod_{k=j+1}^{N} q_k \right) \\ &= \prod_{j=i+1}^{N} q_j \left(1 - \sum_{j=i+1}^{N} r_j \right). \end{aligned} \tag{2.5}$$

Therefore, if $G_i^{(1)} > 0$ for some $i \in \mathcal{N}$, then $q_j > 0$ for all $j = i+1, \ldots, N$ and (2.5) gives $\sum_{j=i+1}^{N} r_j < 1$. Conversely, if $\sum_{j=i+1}^{N} r_j < 1$ for some $i \in \mathcal{N}$, then $q_j > 0$ for all $j = i+1, \ldots, N$ and (2.5) gives $G_i^{(1)} > 0$. Namely, $G_i^{(1)} > 0$ is equivalent to $\sum_{j=i+1}^{N} r_j < 1$ and $B^{(1)}$ is given by

$$B^{(1)} = \left\{ i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1 \right\}.$$

Since $\sum_{j=i+1}^{N} r_j$ is clearly nonincreasing in i, $B^{(1)}$ is "closed" in the sense of the monotone problem in Chow et al (1971); that is, $i \in B^{(1)}$ implies that $j \in B^{(1)}$ for all $j = i, i+1, \ldots, N$. Hence, the optimal rule for the single selection problem is given by (1.1) and (1.2). □

We are now at the position to give the optimal rules for the multiple selection problem. For each $i \in \mathcal{N}$, we define $H_i^{(m)}$, $m \in \mathcal{N}$, recursively by

$$H_i^{(1)} := 1 - \sum_{j=i+1}^{N} r_j, \tag{2.6}$$

$$H_i^{(m)} := H_i^{(1)} + \sum_{j=(i+1) \vee i_*^{(m-1)}}^{N} r_j \, H_j^{(m-1)}, \tag{2.7}$$

where $a \vee b = \max\{a, b\}$ for a, $b \in \mathbb{R}$. In (2.7), if there exists a $j \in \{i+1, \ldots, N\}$ such that $p_j = 1$ (that is, $r_j = +\infty$), then we set $H_i^{(m)} := -\infty$.

Theorem 2.1 *Suppose that we have at most m ($\in \mathcal{N}$) selection chances. Then, the optimal selection rule $\tau_*^{(m)}$ is given by*

$$\tau_*^{(m)} = \min\{i \geq i_*^{(m)} : X_i = 1\}, \tag{2.8}$$

$$i_*^{(m)} = \min\{i \in \mathcal{N} : H_i^{(m)} > 0\}, \tag{2.9}$$

where $\min(\emptyset) = +\infty$. *Furthermore, we have*

$$1 \leq i_*^{(m)} \leq i_*^{(m-1)} \leq \cdots \leq i_*^{(1)} \leq N. \tag{2.10}$$

Proof: The monotone selection region for the problem with m ($\in \mathcal{N}$) selection chances is defined by $B^{(m)} := \{i \in \mathcal{N} : G_i^{(m)} > 0\}$, where

$$G_i^{(m)} := V_i^{(m)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.11}$$

To derive (2.8) and (2.9), it suffices to show that $B^{(m)}$ is closed and satisfies $B^{(m)} = \{i \in \mathcal{N} : H_i^{(m)} > 0\}$, which is also deduced by checking that $G_i^{(m)} > 0$ is equivalent to $H_i^{(m)} > 0$ for each $i \in \mathcal{N}$ and that $i \mapsto H_i^{(m)}$ changes the sign at most once from nonpositive to positive. To have (2.10), on the other hand, it suffices to show that $H_i^{(m)} \geq H_i^{(m-1)}$ for $i \in \mathcal{N}$ such that $H_i^{(m-1)} > -\infty$. We verify them by the induction on m.

We have already seen in the proof of Proposition 1.1 that $G_i^{(1)} > 0$ is equivalent to $H_i^{(1)} > 0$ for $i \in \mathcal{N}$. In particular, if $q_j = 0$ for some $j \in \{i+1, \ldots, N\}$, then $G_i^{(1)} \leq 0$, while if $q_j > 0$ for all $j = i+1, \ldots, N$, then it holds that $G_i^{(1)} = (\prod_{j=i+1}^{N} q_j) \, H_i^{(1)}$ (see (2.5) and (2.6)). We have also seen that $i \mapsto H_i^{(1)}$ changes the sign at most once from nonpositive to positive. The inequality $H_i^{(2)} \geq H_i^{(1)}$ for $i \in \mathcal{N}$ such that $H_i^{(1)} > -\infty$ is immediate from (2.7); that is,

$$H_i^{(2)} - H_i^{(1)} = \sum_{j=(i+1) \vee i_*^{(1)}}^{N} r_j \, H_j^{(1)} \geq 0,$$

where the last inequality follows from $H_j^{(1)} > 0$ for $j \geq i_*^{(1)}$.

As the induction hypotheses, we now assume the following for $m' = 1, 2, \ldots, m$ with some fixed $m \in \{1, 2, \ldots, N-1\}$.

(i) $G_i^{(m')} > 0$ is equivalent to $H_i^{(m')} > 0$ for each $i \in \mathcal{N}$. In particular, if $q_j = 0$ for some $j \in \{i+1, \ldots, N\}$, then $G_i^{(m')} \leq 0$, and if $q_j > 0$ for all $j = i+1, \ldots, N$, then it holds that $G_i^{(m')} = (\prod_{j=i+1}^{N} q_j) \, H_i^{(m')}$.

(ii) $i \mapsto H_i^{(m')}$ changes the sign at most once from nonpositive to positive.

(iii) $H_i^{(m'+1)} - H_i^{(m')} \geq 0$ for $i \in \mathcal{N}$ such that $H_i^{(m')} > -\infty$.

Note that $H_i^{(m)} > 0$ and equivalently $G_i^{(m)} > 0$ for $i \geq i_*^{(m)}$ by the induction hypothesis. Thus, by (i) above, $q_j > 0$ for all $j = i_*^{(m)} + 1, \ldots, N$. Let us show (i)–(iii) above for $m' = m+1$. We first examine (i). From (2.11), the monotone selection region in the case with $m+1$ selection chances is given by $B^{(m+1)} = \{i \in \mathcal{N} : G_i^{(m+1)} > 0\}$, where

$$G_i^{(m+1)} = V_i^{(m+1)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(m+1)}, \quad i \in \mathcal{N}. \tag{2.12}$$

Since $V_j^{(m+1)} = V_j^{(1)} + W_j^{(m)}$ from (2.2), substituting this into (2.12), we have

$$\begin{aligned} G_i^{(m+1)} &= V_i^{(1)} + W_i^{(m)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, (V_j^{(1)} + W_j^{(m)}) \\ &= G_i^{(1)} + \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, (M_j^{(m)} - W_j^{(m)}), \end{aligned} \tag{2.13}$$

where the first term on the right-hand side comes from (2.4) and the second term does from (2.3). By the induction hypothesis, we have $M_j^{(m)} = V_j^{(m)}$ for $j \geq i_*^{(m)}$ and $M_j^{(m)} = W_j^{(m)}$ for $j < i_*^{(m)}$ in (2.1);

that is,

$$M_j^{(m)} - W_j^{(m)} = \begin{cases} V_j^{(m)} - W_j^{(m)} & \text{for } j \ge i_*^{(m)}, \\ 0 & \text{for } j < i_*^{(m)}. \end{cases}$$

Furthermore, the induction hypothesis reads (2.3) as

$$W_j^{(m)} = \sum_{\ell=j+1}^{N} \left(\prod_{k=j+1}^{\ell-1} q_k \right) p_\ell \, V_\ell^{(m)} \quad \text{for } j \ge i_*^{(m)}.$$

Therefore, we have from (2.11) that

$$M_j^{(m)} - W_j^{(m)} = G_j^{(m)} \quad \text{for } j \ge i_*^{(m)},$$

and substituting this into (2.13), we obtain

$$G_i^{(m+1)} = G_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, G_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.14}$$

Here, if there exists a $j \in \{i+1, \ldots, N\}$ such that $q_j = 0$, this j is less than or equal to $i_*^{(m)}$ since $q_j > 0$ for all $j = i_*^{(m)} + 1, \ldots, N$. Namely, this occurs only in the case of $i < i_*^{(m)}$. In this case, the first term on the right-hand side of (2.14) is not greater than zero and the second term is equal to zero; that is, $G_i^{(m+1)} \le 0$. Conversely, suppose that $q_j > 0$ for all $j = i+1, \ldots, N$ with some $i \in \mathcal{N}$. Then, by the induction hypothesis, applying $G_i^{(m')} = (\prod_{j=i+1}^{N} q_j) \, H_i^{(m')}$ for $m' = 1$ and $m' = m$ into (2.14), we have

$$\begin{aligned} G_i^{(m+1)} &= \left(\prod_{j=i+1}^{N} q_j \right) H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \left(\prod_{\ell=j+1}^{N} q_\ell \right) H_j^{(m)} \\ &= \prod_{j=i+1}^{N} q_j \left(H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, H_j^{(m)} \right), \end{aligned}$$

so that (2.7) leads to

$$G_i^{(m+1)} = \left(\prod_{j=i+1}^{N} q_j \right) H_i^{(m+1)}. \tag{2.15}$$

From the observation above, if $G_i^{(m+1)} > 0$, then $q_j > 0$ for all $j = i+1, \ldots, N$ and (2.15) leads to $H_i^{(m+1)} > 0$. Conversely, if $H_i^{(m+1)} > 0$, then (2.7) states that $H_i^{(1)} > -\infty$; that is, $q_j > 0$ for all $j = i+1, \ldots, N$. Thus, (2.15) also leads to $G_i^{(m+1)} > 0$. Hence, we have (i) for $m' = m+1$.

We next show (ii). By the induction hypothesis, $H_i^{(m+1)} \ge H_i^{(m)}$ for $i \in \mathcal{N}$ such that $H_i^{(m)} > -\infty$ and $H_i^{(m)} > 0$ for $i \ge i_*^{(m)}$; that is, $H_i^{(m+1)} > 0$ for $i \ge i_*^{(m)}$. For $i < i_*^{(m)}$, we have $\sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, H_j^{(m)} = \sum_{j=i_*^{(m)}}^{N} r_j \, H_j^{(m)}$, which is invariant to i. Thus, (2.7) states that $H_i^{(m+1)}$ ($= H_i^{(1)} + \text{Constant}$) is nondecreasing in i ($< i_*^{(m)}$). Hence, $i \mapsto H_i^{(m+1)}$ changes the sign at most once from nonpositive to positive and (ii) holds for $m' = m+1$.

Finally, to show (iii) for $m' = m+1$, we use (2.7) and take the difference between $H_i^{(m+2)}$ and $H_i^{(m+1)}$; that is,

$$\begin{aligned} H_i^{(m+2)} - H_i^{(m+1)} &= \sum_{j=(i+1)\vee i_*^{(m+1)}}^{N} r_j \, H_j^{(m+1)} - \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, H_j^{(m)} \\ &\ge \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, (H_j^{(m+1)} - H_j^{(m)}) \ge 0, \end{aligned}$$

where the first inequality follows from $H_j^{(m+1)} > 0$ for $j \geq i_*^{(m+1)}$ and $i_*^{(m+1)} \leq i_*^{(m)}$ by the induction hypothesis. The second inequality also follows from the induction hypothesis. Hence, the induction is completed and so is the proof. $\square$

Let $h_i^{(m)} := 1 - H_i^{(m)}$ for i and $m \in \mathcal{N}$. From (2.7), $h_i^{(m)}$ for $m \in \mathcal{N}$ are then given by

$$h_i^{(1)} = \sum_{j=i+1}^{N} r_j,$$
$$h_i^{(m)} = \sum_{j=i+1}^{i_*^{(m-1)}-1} r_j + \sum_{j=(i+1)\vee i_*^{(m-1)}}^{N} r_j\, h_j^{(m-1)}, \quad m = 2, 3, \ldots.$$

We can see from the above that each $h_i^{(m)}$ is expressed as a combination of multiple odds-sums. For instance, $h_i^{(2)}$ and $h_i^{(3)}$ are calculated as

$$h_i^{(2)} = \sum_{j=i+1}^{i_*^{(1)}-1} r_j + \sum_{j=(i+1)\vee i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k, \tag{2.16}$$
$$h_i^{(3)} = \sum_{j=i+1}^{i_*^{(2)}-1} r_j + \sum_{j=(i+1)\vee i_*^{(2)}}^{N} r_j \left\{ \sum_{k=j+1}^{i_*^{(1)}-1} r_k + \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k \sum_{\ell=k+1}^{N} r_\ell \right\}.$$

The optimal rule for the problem with m ($\in \mathcal{N}$) selection chances then reduces to $\tau_*^{(m)} = \min\{i \in \mathcal{N} : h_i < 1 \ \&\ X_i = 1\}$. That is why we call Theorem 2.1 "multiple sums-the-odds theorem" or shortly "multiple odds theorem."

3 Maximum probability of win

In this section, we first derive a formula for computing the maximum probability of win under the optimal rule with m ($\in \mathcal{N}$) selection chances and then provide closed-form formulas for $m = 2$ and 3. Second, we give its lower and upper bounds and the limit as $N \to \infty$ for $m = 2$.

Theorem 3.1 *For the problem with at most m ($\in \mathcal{N}$) selection chances, the maximum probability of win under the optimal rule, $P^{(m)}(\text{win}) = P_N^{(m)}(p_1, \ldots, p_N)$, is given by*

$$P^{(m)}(\text{win}) = \prod_{j=i_*^{(m)}}^{N} q_j \sum_{j=i_*^{(m)}}^{N} r_j + \sum_{j=i_*^{(m)}}^{N} \left(\prod_{k=i_*^{(m)}}^{j} q_k \right) r_j\, W_j^{(m-1)}, \tag{3.1}$$

where if $p_{i_*^{(m)}} = 1$, then $P^{(m)}(\text{win}) = \prod_{k=i_*^{(m)}+1}^{N} q_k + W_{i_*^{(m)}}^{(m-1)}$ (note that $p_j < 1$ for all $j = i_*^{(m)}+1, \ldots, N$). Especially, for $m = 2$ and 3,

$$P^{(2)}(\text{win}) = \prod_{j=i_*^{(2)}}^{N} q_j \sum_{j=i_*^{(2)}}^{N} r_j \left(1 + \prod_{k=j+1}^{i_*^{(1)}-1} (1 + r_k) \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k \right), \tag{3.2}$$

$$P^{(3)}(\text{win}) = \prod_{j=i_*^{(3)}}^{N} q_j \sum_{j=i_*^{(3)}}^{N} r_j \left[1 + \prod_{k=j+1}^{i_*^{(2)}-1} (1 + r_k) \right.$$
$$\left. \times \sum_{k=(j+1)\vee i_*^{(2)}}^{N} r_k \left(1 + \prod_{\ell=k+1}^{i_*^{(1)}-1} (1 + r_\ell) \sum_{\ell=(k+1)\vee i_*^{(1)}}^{N} r_\ell \right) \right]. \tag{3.3}$$

Proof: Note that the independence of X_i's leads to $P^{(m)}(\text{win}) = W^{(m)}_{i_*^{(m)}-1}$ under the optimal selection rule. Thus, we have from (2.2) and (2.3) that

$$\begin{aligned} P^{(m)}(\text{win}) &= \sum_{j=i_*^{(m)}}^{N} \left(\prod_{k=i_*^{(m)}}^{j-1} q_k \right) p_j \, M_j^{(m)} \\ &= \sum_{j=i_*^{(m)}}^{N} \left(\prod_{k=i_*^{(m)}}^{j-1} q_k \right) p_j \left(\prod_{\ell=j+1}^{N} q_\ell + W_j^{(m-1)} \right), \end{aligned}$$

where the second equality follows from $M_j^{(m)} = V_j^{(m)}$ for $j \geq i_*^{(m)}$. Hence, (3.1) is readily obtained.

$P^{(2)}(\text{win})$ and $P^{(3)}(\text{win})$ are derived from straightforward calculation. Since the optimal rule requires to select the first success after $i_*^{(1)}$, we have $M_k^{(1)} = V_k^{(1)} = \prod_{\ell=k+1}^{N} q_k$ for $k \geq i_*^{(1)}$. It then follows from (2.3) that

$$W_j^{(1)} = \sum_{k=j+1}^{N} \left(\prod_{\ell=j+1}^{k-1} q_\ell \right) p_k \, M_k^{(1)} = \prod_{\ell=j+1}^{N} q_\ell \sum_{k=j+1}^{N} r_k \quad \text{for } j \geq i_*^{(1)} - 1.$$

For $j < i_*^{(1)} - 1$, on the other hand, we have $W_j^{(1)} = W^{(1)}_{i_*^{(1)}-1} = \prod_{\ell=i_*^{(1)}}^{N} q_\ell \sum_{j=i_*^{(1)}}^{N} r_j$. Therefore, for each $j \in \mathcal{N}$,

$$W_j^{(1)} = \prod_{\ell=(j+1)\vee i_*^{(1)}}^{N} q_\ell \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k.$$

Substituting this into (3.1) with $m = 2$ and using $1/q_k = 1 + r_k$, we obtain (3.2).

By the similar approach to the above, we have

$$W_j^{(2)} = \prod_{\ell=(j+1)\vee i_*^{(2)}}^{N} q_\ell \sum_{k=(j+1)\vee i_*^{(2)}}^{N} r_k \left(1 + \prod_{\ell=k+1}^{i_*^{(1)}-1} (1 + r_\ell) \sum_{\ell=(k+1)\vee i_*^{(1)}}^{N} r_\ell \right).$$

Substituting this into (3.1) with $m = 3$, we obtain (3.3). $\square$

Next, we consider the lower and upper bounds for the maximum probability of win for $m = 2$ and its limit as $N \to \infty$. In the following, we put subscript "N" and write $P_N^{(m)}(\text{win})$ and $i_{*,N}^{(m)}$, on occasions, to emphasize the dependence on N. Let $R_N^{(m)} = \sum_{j=i_{*,N}^{(m)}}^{N} r_j$ and $R_N^{(m,2)} = \sum_{j=i_{*,N}^{(m)}}^{N} r_j^2$ for $m \in \mathcal{N}$. For the single selection problem, Bruss (2000) finds that

$$R_N^{(1)} \, e^{-R_N^{(1)}} < P_N^{(1)}(\text{win}) \leq R_N^{(1)} \, e^{-R_N^{(1)} + R_N^{(1,2)}},$$

and further shows that, if $R_N^{(1)} \to 1$ and $R_N^{(1,2)} \to 0$ as $N \to \infty$, then

$$P_N^{(1)}(\text{win}) \to 1/e \quad \text{as } N \to \infty.$$

For the double selection problem, we give below the bounds and the limit as $N \to \infty$ for the maximum probability of win. We find that the same limit $e^{-1} + e^{-3/2}$ as that for the CSP with two selection chances is obtained under a reasonable condition for $R_N^{(m)}$ and $R_N^{(m,2)}$ as $N \to \infty$ (see, e.g., Gilbert and Mosteller (1966), Ano and Ando (2000) or Bruss (1988)).

Theorem 3.2 *For the maximum probability of win with $m = 2$, we have*

$$P_N^{(2)}(\text{win}) > R_N^{(1)} \, e^{-R_N^{(1)}} + e^{-R_N^{(2)}}, \tag{3.4}$$

$$P^{(2)}(\text{win}) < R^{(1)} \, e^{-R^{(1)} + R^{(1,2)}}$$

$$+\,(1 + r_{i_*^{(1)}}\, R^{(1)} + r_{i_*^{(2)}})\, e^{-R^{(2)}+R^{(2,2)}}. \tag{3.5}$$

Furthermore, if $R_N^{(1)} \to 1$, $R_N^{(2)} \to 3/2$, $R_N^{(1,2)} \to 0$ *and* $R_N^{(2,2)} \to 0$ *as* $N \to \infty$, *then*

$$P_N^{(2)}(\mathrm{win}) \to e^{-1} + e^{-3/2} \quad \text{as } N \to \infty. \tag{3.6}$$

Proof: We first derive the lower bound (3.4). A simple expansion of (3.2) in Theorem 3.1 yields

$$\begin{aligned} P^{(2)}(\mathrm{win}) &= R^{(2)} \prod_{j=i_*^{(2)}}^{N} q_j + R^{(1)} \sum_{j=i_*^{(2)}}^{i_*^{(1)}-1} \Bigl(\prod_{k=i_*^{(2)}}^{j-1} q_k\Bigr) p_j \Bigl(\prod_{k=i_*^{(1)}}^{N} q_k\Bigr) \\ &\quad + \prod_{j=i_*^{(2)}}^{N} q_j \sum_{j=i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k, \end{aligned} \tag{3.7}$$

where the subscript "N" is omitted for simplicity of the notation. In the second term on the right-hand side above, we note that $\sum_{j=i_*^{(2)}}^{i_*^{(1)}-1} \bigl(\prod_{k=i_*^{(2)}}^{j-1} q_k\bigr) p_j = 1 - \prod_{j=i_*^{(2)}}^{i_*^{(1)}-1} q_j$ since it represents the probability that at least one success appears from $i_*^{(2)}$ to $i_*^{(1)} - 1$. Thus, we have

$$\begin{aligned} (\text{2nd term on RHS of (3.7)}) &= R^{(1)} \Bigl(1 - \prod_{j=i_*^{(2)}}^{i_*^{(1)}-1} q_j\Bigr) \prod_{k=i_*^{(1)}}^{N} q_k \\ &= R^{(1)} \Bigl(\prod_{j=i_*^{(1)}}^{N} q_j - \prod_{j=i_*^{(2)}}^{N} q_j\Bigr). \end{aligned} \tag{3.8}$$

Consider the third term on the right-hand side in (3.7). Since $h_i^{(2)} = 1 - H_i^{(2)} \geq 1$ for $i < i_*^{(2)}$, putting $i = i_*^{(2)} - 1$ in (2.16), we have $\sum_{j=i_*^{(2)}}^{i_*^{(1)}-1} r_j + \sum_{j=i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k \geq 1$, which is equivalent to

$$\sum_{j=i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k \geq 1 + R^{(1)} - R^{(2)}.$$

Therefore, we have

$$(\text{3rd term on RHS of (3.7)}) \geq (1 + R^{(1)} - R^{(2)}) \prod_{j=i_*^{(2)}}^{N} q_j. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7) yields

$$P^{(2)}(\mathrm{win}) \geq R^{(1)} \prod_{j=i_*^{(1)}}^{N} q_j + \prod_{j=i_*^{(2)}}^{N} q_j. \tag{3.10}$$

Here, noting $1/q_j = 1 + r_j$ and taking logarithm, we have for any $s \in \mathcal{N}$,

$$\log \prod_{j=s}^{N} q_j = -\sum_{j=s}^{N} \log(1 + r_j) > -\sum_{j=s}^{N} r_j,$$

where the inequality follows since $\log(1+x) \leq x$ for $x \in \mathbb{R}$ with the equality only when $x = 0$. Hence, we have $\prod_{j=s}^{N} q_j > e^{-R}$ with $R = \sum_{j=s}^{N} r_j$. Applying this into (3.10) with $s = i_*^{(1)}$ and $s = i_*^{(2)}$, we obtain (3.4).

We next derive the upper bound (3.5). To this end, we examine the third term on the right-hand side in (3.7). Since $h_i^{(2)} < 1$ for $i \geq i_*^{(2)}$, putting $i = i_*^{(2)}$ in (2.16), we have $\sum_{j=i_*^{(2)}+1}^{i_*^{(1)}-1} r_j +$

$\sum_{j=(i_*^{(2)}+1)\vee i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k < 1$, so that,

$$\sum_{j=i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k < 1 + (1 + r_{i_*^{(1)}})\, R^{(1)} - (R^{(2)} - r_{i_*^{(2)}}).$$

Therefore, we have

$$\text{(3rd term on RHS of (3.7))} < \big(1 + (1 + r_{i_*^{(1)}})\, R^{(1)} - R^{(2)} + r_{i_*^{(2)}}\big) \prod_{j=i_*^{(2)}}^{N} q_j. \tag{3.11}$$

Applying (3.8) and (3.11) into (3.7), we have

$$P^{(2)}(\text{win}) < R^{(1)} \prod_{j=i_*^{(1)}}^{N} q_j + (1 + r_{i_*^{(1)}}\, R^{(1)} + r_{i_*^{(2)}}) \prod_{j=i_*^{(1)}}^{N} q_j. \tag{3.12}$$

Here, since $1/q_j = 1 + r_j$, using $\log(1+x) \geq x - x^2$ for $x \in \mathbb{R}$, we obtain for any $s \in \mathcal{N}$,

$$\log \prod_{j=s}^{N} q_j \leq -\sum_{j=s}^{N} r_j + \sum_{j=s}^{N} r_j^2.$$

Hence, letting $\sum_{j=s}^{N} r_j = R$ and $\sum_{j=s}^{N} r_j^2 = R'$, we have $\prod_{j=s}^{N} q_j \leq e^{-R+R'}$. Applying this into (3.12) with $s = i_*^{(1)}$ and $s = i_*^{(2)}$, we have (3.5).

Finally, we have $r_{i_{*,N}^{(1)}} \to 0$ and $r_{i_{*,N}^{(2)}} \to 0$ as $N \to \infty$ since $R_N^{(1,2)} \to 0$ and $R_N^{(2,2)} \to 0$ as $N \to \infty$, respectively. Therefore, (3.4) and (3.5) yield (3.6) as $N \to \infty$. $\square$

As a final remark, we make two conjectures on the limits and lower bounds for the maximum probability of win in the multiple selection problem. First, we guess that, if $R_N^{(m)}$ and $R_N^{(m,2)}$, $m = 1, 2, \ldots$, have the same limits as those for the CSP with multiple selection chances, then the limit of the maximum probability of win is also consistent to that for the CSP; that is,

$$\lim_{N\to\infty} P_N^{(m)}(\text{win}) = \lim_{N\to\infty} \sum_{j=1}^{m} \frac{i_*^{(j)}}{N} \quad \text{for } m = 1, 2, \ldots.$$

The case of $m = 1$ was solved by Bruss (2000) and the case of $m = 2$ is done above. For the triple selection problem, for instance, our conjecture says that, if $R_N^{(1)} \to 1$, $R_N^{(2)} \to 3/2$ and $R_N^{(3)} \to 47/24$ with $R_N^{(m,2)} \to 0$, $m = 1, 2, 3$ as $N \to \infty$, then

$$\lim_{N\to\infty} P_N^{(3)}(\text{win}) = e^{-1} + e^{-3/2} + e^{-47/24}.$$

This triple selection case will be able to be confirmed by the similar approach to the one for $P_N^{(2)}(\text{win})$, with some more delicate and complicated calculations. The case of general m is more challenging.

Second, for the lower bounds for the maximum probability of win, our conjecture is stated as that, for some reasonable condition of p_i, $i \in \mathcal{N}$,

$$P^{(m)}(\text{win}) > \lim_{N\to\infty} \sum_{j=1}^{m} \frac{i_*^{(j)}}{N} \quad \text{for } m = 1, 2, \ldots.$$

For this problem, the case of $m = 1$ was shown by Bruss (2003). However, even the case of $m = 2$ is open.

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