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A proof theoretic study on intuitionistic tree sequent calculus

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A PROOF THEORETIC STUDY ON INTUITIONISTIC TREE SEQUENT CALCULUS

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Abstract. **TLJ** is a proof system for intuitionistic logic which has connections to many other areas of computer science and mathematics. In this paper, to make the base of those studies, we give a proof theoretic study on this system.

1. Introduction

The tree sequent calculus (or labelled sequent calculus) **TLJ** introduced by Kashima [2] is a proof system for intuitionistic logic which derives tree sequents. We can give a natural proof of the Kripke completeness theorem by use of this tool (see [1, 2]). Furthermore, this system has connections to many other areas of computer science and mathematics. For example, in [3], the author gives an intuitionistic fragment of the $\lambda\mu$-calculus by use of this proof system. In this paper, to make the base of those studies, we give a proof theoretic study on this system.

2. Preliminary

In this section, we prepare some notions we are going to use. To simplify the argument, we treat only the implicational formulas.

Suppose that a countable set P of atomic propositions is given. Then the set Fml of all formulas is defined as follows.

$$\alpha, \beta \in \mathrm{Fml} ::= \; p \mid (\alpha \supset \beta)$$
$$p \in \mathrm{P}$$

Parentheses are omitted in the usual manner. We use metavariables $\varphi, \psi, \alpha, \beta, \ldots$ to stand for arbitrary formulas and $p, q, \ldots$ for arbitrary atomic propositions. We write $\alpha \equiv \beta$ if α is syntactically equal to β.

Let $\mathbb{N}^{<\omega}$ be the set of all finite sequences of natural numbers and $*$ be the concatenation function on $\mathbb{N}^{<\omega}$, that is, $\langle n_1, \ldots, n_k\rangle * \langle m_1, \ldots, m_l\rangle = \langle n_1, \ldots, n_k, m_1, \ldots, m_l\rangle$. We write the empty sequence as ϵ. We use the abbreviation such as $\langle n\rangle = n$ if it causes no confusion. We define a partial order $\preceq$ on $\mathbb{N}^{<\omega}$ as follows.

$$\overline{n} \preceq \overline{m} \;\Leftrightarrow\; \exists \overline{k} \in \mathbb{N}^{<\omega} \text{ such that } \overline{m} = \overline{n} * \overline{k}$$

We write $\overline{n} \prec \overline{m}$ if both $\overline{n} \preceq \overline{m}$ and $\overline{n} \neq \overline{m}$ hold, and write $\overline{n} \prec_1 \overline{m}$ if there exists a natural number k such that $\overline{m} = \overline{n} * k$. A *tree* $\mathcal{T}$ is a finite subset of $\mathbb{N}^{<\omega}$ which satisfies:

- $\epsilon \in \mathcal{T}$.
- $\overline{n} \in \mathcal{T},\ \overline{m} \preceq \overline{n} \implies \overline{m} \in \mathcal{T}$.

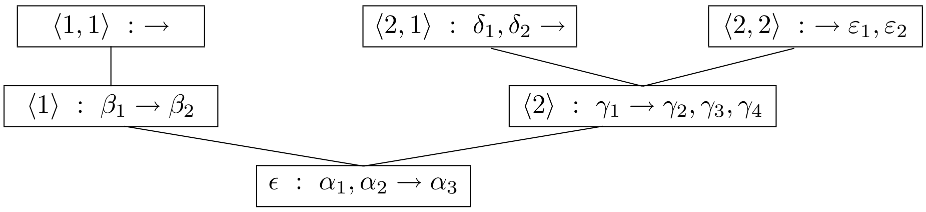

FIGURE 1

We say $\overline{n}$ is a node of $\mathcal{T}$ if $\overline{n} \in \mathcal{T}$. We also say $\overline{n}$ is a parent-node of $\overline{m}$ (or $\overline{m}$ is a child-node of $\overline{n}$) if $\overline{n} \prec_1 \overline{m}$, and say $\overline{n}$ is an ancestor of $\overline{m}$ (or $\overline{m}$ is a descendant of $\overline{n}$) if $\overline{n} \prec \overline{m}$.

Definition 2.1 (TLJ).

(1) A *tree sequent* is an expression of the form $\Gamma \xrightarrow{\mathcal{T}} \Delta$ where:
- $\mathcal{T}$ is a tree.
- Γ and Δ are sets of pairs of nodes of $\mathcal{T}$ and formulas written $\overline{n} : \alpha$.

We write $\Gamma(\overline{n}) = \{\alpha \mid \overline{n} : \alpha \in \Gamma\}$. We abbreviate $\emptyset \xrightarrow{\mathcal{T}} \Delta$ to $\xrightarrow{\mathcal{T}} \Delta$.

A tree sequent is viewed as a tree in which each node is labelled with a sequent. For example, the tree sequent

$$\epsilon : \alpha_1,\ \epsilon : \alpha_2,\ \langle 1\rangle : \beta_1,\ \langle 2\rangle : \gamma_1,\ \langle 2,1\rangle : \delta_1,\ \langle 2,1\rangle : \delta_2$$

$$\xrightarrow{\mathcal{T}} \epsilon : \alpha_3, \langle 1\rangle : \beta_2,\ \langle 2\rangle : \gamma_2,\ \langle 2\rangle : \gamma_3,\ \langle 2\rangle : \gamma_4,\ \langle 2,2\rangle : \varepsilon_1,\ \langle 2,2\rangle : \varepsilon_2$$

$$(\mathcal{T} = \{\epsilon, \langle 1\rangle, \langle 1,1\rangle, \langle 2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\})$$

can be viewed as the tree in figure 1.

(2) The tree sequent calculus **TLJ** is a proof system which derives tree sequents, and consists of the following rules.

[axiom] (Id) $\overline{n} : \alpha \xrightarrow{\mathcal{T}} \overline{n} * \overline{m} : \alpha$

[structural inference rule]

$$\frac{\Gamma_1 \xrightarrow{\mathcal{T}} \Gamma_2}{\Delta_1, \Gamma_1 \xrightarrow{\mathcal{T}} \Gamma_2, \Delta_2} \text{ (Weakening)}$$

[Logical inference rule]

$$\frac{\Gamma_1 \xrightarrow{\mathcal{T}} \Delta_1,\ \overline{n} : \alpha \qquad \overline{n} : \beta,\ \Gamma_2 \xrightarrow{\mathcal{T}} \Delta_2}{\overline{n} : \alpha \supset \beta,\ \Gamma_1, \Gamma_2 \xrightarrow{\mathcal{T}} \Delta_1, \Delta_2} \text{ } (\supset\rightarrow)$$

$$\frac{\overline{n} * k : \alpha,\ \Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} * k : \beta}{\Gamma \xrightarrow{\mathcal{T} \setminus \{n*k\}} \Delta,\ \overline{n} : \alpha \supset \beta} \text{ } (\rightarrow\supset)$$

In the last figure, because $\Gamma \xrightarrow{\mathcal{T} \setminus \{n*k\}} \Delta,\ \overline{n} : \alpha \supset \beta$ is also a tree sequent, the node $\overline{n} * k$ and its descendants do not occur in the lower sequent (see also figure 2). We say $\overline{n} * k$ is the *eigen-node* of this $(\rightarrow\supset)$-rule.

We write $\vdash_{\mathbf{TLJ}} \varphi$ (φ is provable in **TLJ**) if $\vdash_{\mathbf{TLJ}} \xrightarrow{\{\epsilon\}} \epsilon : \varphi$.

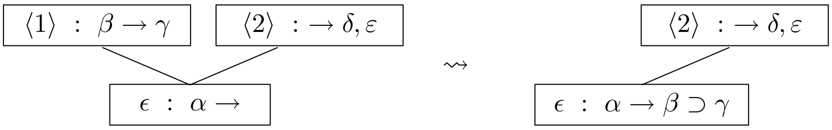

FIGURE 2. $(\to\supset)$

3. VARIANT SYSTEM: $\mathbf{TLJ}^\infty$

We introduce a new proof system $\mathbf{TLJ}^\infty$ as follows. This system is useful to analyze the **TLJ**-derivations proof theoretically (see the next section).

Definition 3.1.

(1) A *pseudo tree sequent* is an expression of the form $\Gamma \xrightarrow{\mathbb{N}^{<\omega}} \Delta$ where Γ and Δ are sets of of pairs of elements of $\mathbb{N}^{<\omega}$ and formulas written $\overline{n} : \alpha$. A pseudo tree sequent $\Gamma \xrightarrow{\mathbb{N}^{<\omega}} \Delta$ is simply written as $\Gamma \to \Delta$ if it causes no confusion.

(2) $\mathbf{TLJ}^\infty$ is a proof system which treats pseudo tree sequents, and obtained from **TLJ** by modifying $(\to\supset)$ as follows.

$$\frac{\overline{n} * k : \alpha,\ \Gamma \xrightarrow{\mathbb{N}^{<\omega}} \Delta,\ \overline{n} * k : \beta}{\Gamma \xrightarrow{\mathbb{N}^{<\omega}} \Delta,\ \overline{n} : \alpha \supset \beta}\ (\to\supset)$$

Here $\overline{n} * k$ and its descendants do not occur in the lower sequent. We call this condition the *label condition* of this $(\to\supset)$.

Theorem 3.2. There exists $\mathcal{T}$ such that $\Gamma \xrightarrow{\mathcal{T}} \Delta$ is provable in **TLJ**, if and only if $\Gamma \xrightarrow{\mathbb{N}^{<\omega}} \Delta$ is provable in $\mathbf{TLJ}^\infty$.

Proof.

($\Longrightarrow$) Let Σ be a **TLJ**-derivation of $\Gamma \xrightarrow{\mathcal{T}} \Delta$. We can obtain a $\mathbf{TLJ}^\infty$-derivation of $\Gamma \xrightarrow{\mathbb{N}^{<\omega}} \Delta$ by replacing each tree sequent $\Pi \xrightarrow{\mathcal{T}} \Theta$ occurring in Σ by the pseudo tree sequent $\Pi \xrightarrow{\mathbb{N}^{<\omega}} \Theta$.

($\Longleftarrow$) Define a tree $T(\Gamma) = \{\overline{n} \mid \exists \overline{k} \in \mathbb{N}^{<\omega}, \exists \alpha \in \mathrm{Fml}$ such that $\overline{n} * \overline{k} : \alpha \in \Gamma\}$. We can show, by induction on the size of the $\mathbf{TLJ}^\infty$-derivation, if $\Gamma \to \Delta$ is provable in $\mathbf{TLJ}^\infty$ and $T(\Gamma \cup \Delta) \subseteq \mathcal{T}$ then $\Gamma \xrightarrow{\mathcal{T}} \Delta$ is provable in **TLJ**. $\square$

3.1. A proof theoretic study on $\mathbf{TLJ}^\infty$.

In this subsection, (Weakening) is restricted as the following form.

$$\frac{\Gamma \to \Delta}{\overline{n} : \alpha,\ \Gamma \to \Delta}\ (\text{Weakening}) \quad \frac{\Gamma \to \Delta}{\Gamma \to \Delta,\ \overline{n} : \alpha}\ (\text{Weakening})$$

In this subsection, we show that every $\mathbf{TLJ}^\infty$-derivation can be transformed into another derivation which satisfies some desirable conditions. This result is used in [4].

Definition 3.3 (Regular derivation)**.** A $\mathbf{TLJ}^\infty$-derivation Σ is *regular* if it satisfies the following conditions.

- All eigen-nodes in Σ are distinct from each other.
- If a node $\overline{n}$ is used as the eigen-node of an occurrence $\mathcal{I}$ of $(\to\supset)$, then $\overline{n}$ and its descendants occur only above $\mathcal{I}$.

Theorem 3.4. Each $\mathbf{TLJ}^\infty$-derivation Σ can be transformed into a regular derivation Σ' such that $|\Sigma| = |\Sigma'|$.

Lemma 3.5. Let

$$\Gamma_{\overline{n}}^{i\mapsto j} = \{\overline{m} : \alpha \in \Gamma \mid \overline{n} * i \not\preceq \overline{m}\} \cup \{\overline{n} * j * \overline{m} : \alpha \mid \overline{n} * i * \overline{m} : \alpha \in \Gamma\}.$$

Let $\Sigma_{\overline{n}}^{i\mapsto j}$ be a derivation obtained from a derivation Σ by replacing each node occurring in Σ of the form $\overline{n} * i * \overline{m}$ by $\overline{n} * j * \overline{m}$.

If Σ is a regular derivation of $\Gamma \to \Delta$ and $\overline{n} * j * \overline{m}$ and its descendants do not occur in Σ, then $\Sigma_{\overline{n}}^{i\mapsto j}$ is a regular derivation of $\Gamma_{\overline{n}}^{i\mapsto j} \to \Delta_{\overline{n}}^{i\mapsto j}$. Furthermore, if

$$\overline{n} * i * \overline{m}_1, \dots, \overline{n} * i * \overline{m}_k, \bar{l}_1, \dots, \bar{l}_p \quad (\overline{n} * i \not\preceq \bar{l}_1, \dots, \bar{l}_p)$$

occur as eigen-nodes in Σ, then

$$\overline{n} * j * \overline{m}_1, \dots, \overline{n} * j * \overline{m}_k, \bar{l}_1, \dots, \bar{l}_p \quad (\overline{n} * i \not\preceq \bar{l}_1, \dots, \bar{l}_p)$$

occur as eigen-nodes in $\Sigma_{\overline{n}}^{i\mapsto j}$ and the other nodes are not used as eigen-nodes. Furthermore $|\Sigma_{\overline{n}}^{i\mapsto j}| = |\Sigma|$

Proof. By induction on the size of Σ. □

Proof of theorem 3.4. By induction on the number N of eigen-nodes which violate the condition of the conditions written in definition 3.3.

Take a $(\to\supset)$-rule $\mathcal{I}$ whose eigen-node violates the conditions written in definition 3.3 above $\mathcal{I}$ all eigen-nodes satisfy the conditions. Suppose $\mathcal{I}$ has the following form.

$$\dfrac{\begin{matrix}\vdots\ \Pi \\ \overline{n} * i : \alpha,\ \Gamma \to \Delta,\ \overline{n} * i : \beta\end{matrix}}{\begin{matrix}\Gamma \to \Delta,\ \overline{n} : \alpha \supset \beta \\ \vdots\end{matrix}}\ \mathcal{I}$$

Let j be a natural number such that $\overline{n} * j$ and its descendant do not occur in Σ, and let Σ' be a derivation obtained from Σ by replacing Π by $\Pi_{\overline{n}}^{i\mapsto j}$.

$$\dfrac{\begin{matrix}\vdots\ \Pi_{\overline{n}}^{i\mapsto j} \\ \overline{n} * j : \alpha,\ \Gamma \to \Delta,\ \overline{n} * j : \beta\end{matrix}}{\begin{matrix}\Gamma \to \Delta,\ \overline{n} : \alpha \supset \beta \\ \vdots\end{matrix}}\ (\to\supset)$$

This transformation reduces N without changing the endsequent. By induction hypothesis, Σ' can be transformed into a regular derivation. □

Example 3.6.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle 1\rangle : \alpha \to \langle 1,1\rangle : \alpha}{\langle 1,1\rangle : \beta,\ \langle 1\rangle : \alpha \to \langle 1,1\rangle : \alpha}\,(\text{Weakening})}{\langle 1\rangle : \alpha \to \langle 1\rangle : \beta \supset \alpha}\,(\to\supset)}{\langle 1\rangle : \alpha \to \langle 1\rangle : \beta \supset \alpha,\ \langle 1,1\rangle : \alpha}\,(\text{Weakening})}{\langle 1,1\rangle : \beta,\ \langle 1\rangle : \alpha \to \langle 1\rangle : \beta \supset \alpha,\ \langle 1,1\rangle : \alpha}\,(\text{Weakening})}{\langle 1\rangle : \alpha \to \langle 1\rangle : \beta \supset \alpha}\,(\to\supset)}{\to \epsilon : \alpha \supset \beta \supset \alpha}\,(\to\supset)$$

$$\rhd \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle 1\rangle : \alpha \to \langle 1,2\rangle : \alpha}{\langle 1,2\rangle : \beta,\ \langle 1\rangle : \alpha \to \langle 1,2\rangle : \alpha}\,(\text{Weakening})}{\langle 1\rangle : \alpha \to \langle 1\rangle : \beta \supset \alpha}\,(\to\supset)}{\langle 1\rangle : \alpha \to \langle 1\rangle : \beta \supset \alpha,\ \langle 1,1\rangle : \alpha}\,(\text{Weakening})}{\langle 1,1\rangle : \beta,\ \langle 1\rangle : \alpha \to \langle 1\rangle : \beta \supset \alpha,\ \langle 1,1\rangle : \alpha}\,(\text{Weakening})}{\langle 1\rangle : \alpha \to \langle 1\rangle : \beta \supset \alpha}\,(\to\supset)}{\to \epsilon : \alpha \supset \beta \supset \alpha}\,(\to\supset) \quad : \text{ regular}$$

Definition 3.7 (Well-ordered derivation)**.** Let $\mathcal{I}$ be an inference rule occurring in a derivation Σ. We write $\mathcal{I}[\overline{n}]$ and say $\overline{n}$ is the main-node of $\mathcal{I}$, if $\mathcal{I}$ satisfies either of the following conditions.

- $\mathcal{I}$ is a (Weakening)-rule and it adds a formula labelled with $\overline{n}$.
- $\mathcal{I}$ is not a (Weakening)-rule and its side formulas are labelled with $\overline{n}$.

A $\mathbf{TLJ}^{\infty}$-derivation Σ is *well-ordered* if every pair $\langle \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{J}_1, \mathcal{J}_2\rangle$ of inference rules occurring in Σ satisfies the following condition (see also figure 3).

$$\left\{\begin{array}{l} \mathcal{I}_2 \text{ occurs immediately after } \mathcal{I}_1 \\ \mathcal{I}_3 \text{ occurs in the path from } \mathcal{I}_2 \text{ to the endsequent of } \Sigma \\ \mathcal{I}_4 \text{ occurs immediately after } \mathcal{I}_3 \\ \mathcal{I}_1[\overline{m}],\ \mathcal{I}_2[\overline{n}],\ \mathcal{I}_3[\overline{n}],\ \mathcal{I}_4[\overline{l}] \quad (\overline{n} \neq \overline{m},\ \overline{n} \neq \overline{l}) \end{array}\right.$$
$$\Longrightarrow \left\{\begin{array}{l} \overline{n} \text{ does not occur above } \mathcal{I}_1 \\ \overline{n} \text{ does not occur below } \mathcal{I}_4 \end{array}\right.$$

$$\begin{array}{l} \mathcal{J}_2 \text{ occurs in the path from } \mathcal{J}_1 \text{ to the endsequent of } \Sigma,\ \mathcal{J}_1[\overline{m}],\ \mathcal{J}_2[\overline{n}] \\ \Longrightarrow\ \overline{m} \not\preceq \overline{n} \end{array}$$

Theorem 3.8. Every $\mathbf{TLJ}^{\infty}$-derivation can be transformed into a well-ordered derivation with the same endsequent.

This theorem follows from the following lemma.

$$
\begin{array}{l}
\quad\vdots \text{ no inference rules whose main nodes are } \overline{n} \text{ occur} \\
\dfrac{\dfrac{\Gamma_0 \to \Delta_0}{\Gamma_1 \to \Delta_1}\,\mathcal{I}_1[\overline{m}]}{\Gamma_2 \to \Delta_2}\,\mathcal{I}_2[\overline{n}] \\
\quad\vdots \text{ only inference rules whose main nodes are } \overline{n} \text{ occur} \\
\dfrac{\dfrac{\Gamma_2' \to \Delta_2'}{\Gamma_3 \to \Delta_3}\,\mathcal{I}_3[\overline{n}]}{\Gamma_4 \to \Delta_4}\,\mathcal{I}_4[\overline{l}] \\
\quad\vdots \text{ no inference rules whose main nodes are } \overline{n} \text{ occur}
\end{array}
$$

FIGURE 3. well-ordered derivation

Lemma 3.9. Suppose that Σ is not well-ordered and has the following form.

$$
\begin{array}{c}
\vdots \\
\dfrac{\dfrac{\Gamma'' \to \Delta''}{\Gamma' \to \Delta'}\,\mathcal{J}[\overline{m}]}{\Gamma \to \Delta}\,\mathcal{I}[\overline{n}]
\end{array}
$$

If the figure above $\Gamma' \to \Delta'$ be well-ordered, then Σ can be transformed into a well-ordered derivation in which the main-node of the lowest inference rule is $\overline{m}$.

Proof. In this proof we extend $(\to\supset)$ as follows.

$$
\dfrac{\Gamma \to \Delta}{\Gamma \setminus \{\overline{n} * k : \alpha\} \to \Delta \setminus \{\overline{n} * k : \beta\},\ \overline{n} : \alpha \supset \beta}\,(\to\supset)
$$

This extended rule can be derived in $\mathbf{TLJ}^{\infty}$ by the following figure.

$$
\dfrac{\dfrac{\Gamma \to \Delta}{\overline{n} * k : \alpha,\ \Gamma \setminus \{\overline{n} * k : \alpha\} \to \Delta \setminus \{\overline{n} * k : \beta\},\ \overline{n} * k : \beta}\,(\text{Weakening})}{\Gamma \setminus \{\overline{n} * k : \alpha\} \to \Delta \setminus \{\overline{n} * k : \beta\},\ \overline{n} : \alpha \supset \beta}\,(\to\supset)
$$

The proof of this lemma is given by induction on the size of Σ. We can assume that Σ is regular. We do not treat all cases, but the other cases can be showed in the same way.

(1) $\mathcal{I} : (\to\supset),\ \mathcal{J} : (\to\supset)$.

$$
\begin{array}{c}
\vdots\ \Omega \\
\dfrac{\dfrac{\Gamma \to \Delta}{\Gamma \setminus \{\overline{m} * k : \alpha\} \to \Delta \setminus \{\overline{m} * k : \beta\},\ \overline{m} : \alpha \supset \beta}\,\mathcal{J}}{\Gamma \setminus \{\overline{m} * k : \alpha,\ \overline{n} * l : \gamma\} \to \Delta \setminus \{\overline{m} * k : \beta,\ \overline{n} * l : \delta\},\ \overline{m} : \alpha \supset \beta,\ \overline{n} : \gamma \supset \delta}\,\mathcal{I}
\end{array}
$$

First, consider the following derivation.

$$
\begin{array}{c}
\vdots\ \Omega \\
\dfrac{\Gamma \to \Delta}{\Gamma \setminus \{\ \overline{n} * l : \gamma\} \to \Delta \setminus \{\overline{n} * l : \delta\},\ \overline{n} : \gamma \supset \delta}\,(\to\supset)
\end{array}
$$

Because the derivation above $\mathcal{J}$ is well-ordered and $\mathcal{I}$ violates the well-ordered condition, there is an inference rule whose main-node is $\overline{n} * l$ in Ω. We therefore obtain that $\overline{m} * k \not\preceq \overline{n} * l$, and can see above figure follows the label condition of $(\rightarrow\supset)$. By induction hypothesis, this derivation can be transformed into a well-ordered derivation Π. From the statement of the lemma, if the main-node of the lowest inference rule of Ω is $\overline{m}*k$, then the main-node of the lowest inference rule of Π is $\overline{m}*k$. Then the following derivation is well-ordered.

$$\dfrac{\Pi}{\Gamma\setminus\{\overline{m}*k:\alpha,\ \overline{n}*l:\gamma\}\rightarrow\Delta\setminus\{\overline{m}*k:\beta,\ \overline{n}*l:\delta\},\ \overline{m}:\alpha\supset\beta,\ \overline{n}:\gamma\supset\delta}\ (\rightarrow\supset)$$

(2) $\mathcal{I}:(\rightarrow\supset),\ \mathcal{J}:(\supset\rightarrow)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Omega_1\\ \Gamma_1\rightarrow\Delta_1,\ \overline{m}:\alpha\end{array}\qquad\begin{array}{c}\vdots\ \Omega_2\\ \overline{m}:\beta,\ \Gamma_2\rightarrow\Delta_2\end{array}}{\Gamma_1,\Gamma_2,\ \overline{m}:\alpha\supset\beta\rightarrow\Delta_1,\Delta_2}\ \mathcal{J}}{(\Gamma_1,\Gamma_2,\ \overline{m}:\alpha\supset\beta)\setminus\{\overline{n}*k:\gamma\}\rightarrow(\Delta_1,\Delta_2)\setminus\{\overline{n}*k:\delta\},\ \overline{n}:\gamma\supset\delta}\ \mathcal{I}$$

First, consider the following derivations.

$$\dfrac{\begin{array}{c}\vdots\ \Omega_1\\ \Gamma_1\rightarrow\Delta_1,\ \overline{m}:\alpha\end{array}}{\Gamma_1\setminus\{\overline{n}*k:\gamma\}\rightarrow(\Delta_1,\ \overline{m}:\alpha)\setminus\{\overline{n}*k:\delta\},\ \overline{n}:\gamma\supset\delta}\ (\rightarrow\supset)$$

$$\dfrac{\begin{array}{c}\vdots\ \Omega_2\\ \overline{m}:\beta,\ \Gamma_2\rightarrow\Delta_2\end{array}}{(\overline{m}:\beta,\ \Gamma_2)\setminus\{\overline{n}*k:\gamma\}\rightarrow\Delta_2\setminus\{\overline{n}*k:\delta\},\ \overline{n}:\gamma\supset\delta}\ (\rightarrow\supset)$$

By induction hypothesis, these derivations can be transformed into well-ordered derivations Π_1,Π_2. From the statement of the lemma, if the main-node of the lowest inference rule of Ω_i is $\overline{m}$, then the main-node of the lowest inference rule of Π_i is $\overline{m}$ $(i=1,2)$. Then the following derivation is well-ordered.

$$\dfrac{\Pi_1\quad\Pi_2}{(\Gamma_1,\Gamma_2,\ \overline{m}:\alpha\supset\beta)\setminus\{\overline{n}*k:\gamma\}\rightarrow(\Delta_1,\Delta_2)\setminus\{\overline{n}*k:\delta\},\ \overline{n}:\gamma\supset\delta}\ (\supset\rightarrow)$$

Note that the following conditions hold because $\overline{n}*k\not\preceq\overline{m}$.

$$\begin{aligned}&(\Gamma_1,\Gamma_2,\ \overline{m}:\alpha\supset\beta)\setminus\{\overline{n}*k:\gamma\}\\&=(\Gamma_1\setminus\{\overline{n}*k:\gamma\})\cup(\Gamma_2\setminus\{\overline{n}*k:\gamma\})\cup\{\overline{m}:\alpha\supset\beta\}.\\&(\Delta_1,\ \overline{m}:\alpha)\setminus\{\overline{n}*k:\delta\}=(\Delta_1\setminus\{\overline{n}*k:\delta\})\cup\{\overline{m}:\alpha\}.\\&(\overline{m}:\beta,\ \Gamma_2)\setminus\{\overline{n}*k:\gamma\}=(\Gamma_2\setminus\{\overline{n}*k:\gamma\})\cup\{\overline{m}:\beta\}.\end{aligned}$$

(3) $\mathcal{I}$: (Weakening), $\mathcal{J}:(\rightarrow\supset)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Omega\\ \Gamma\rightarrow\Delta\end{array}}{\Gamma\setminus\{\overline{m}*k:\alpha\}\rightarrow\Delta\setminus\{\overline{m}*k:\beta\},\ \overline{m}:\alpha\supset\beta}\ \mathcal{J}}{\Gamma\setminus\{\overline{m}*k:\alpha\}\rightarrow\Delta\setminus\{\overline{m}*k:\beta\},\ \overline{m}:\alpha\supset\beta,\ \overline{n}:\varphi}\ \mathcal{I}$$

Consider the following derivation.

$$\dfrac{\begin{array}{c}\vdots\ \Omega\\ \Gamma\rightarrow\Delta\end{array}}{\Gamma\rightarrow\Delta,\ \overline{n}:\varphi}\ (\text{Weakening})$$

By induction hypothesis, there is a well-ordered derivation Π of $\Gamma \to \Delta,\ \overline{n} : \varphi$. Because Σ is regular, $\overline{m} * k \not\preceq \overline{n}$. In addition, because Ω is well-ordered and Σ is not well-ordered, we obtain $\overline{m} * k \neq \overline{n}$. Then the following figure is well-ordered derivation.

$$\dfrac{\Pi}{\Gamma \setminus \{\overline{m} * k : \alpha\} \to \Delta \setminus \{\overline{m} * k : \beta\},\ \overline{m} : \alpha \supset \beta,\ \overline{n} : \varphi}\ \mathcal{I}$$

□

4. A new proof of the Kripke completeness theorem

When we prove the Kripke completeness theorem by use of the tree sequent method, the proof is given as the following steps.

φ is valid in every Kripke models

$\Nearrow$ $\qquad\qquad$ $\Searrow$

φ is provable in **LJ** $\quad \underset{[\sharp]}{\Longleftarrow} \quad$ φ is provable in **TLJ**

In the standard method, the relation $[\sharp]$ is proved by use of a translation called *formulaic translation* (see [1, Definition 3.10] or [2, Section 2]). In this section, we give a proof of $[\sharp]$ without this translation.

4.1. Proof of $[\sharp]$.

Our proof is given by only one procedure to remove all redundant weakening rules. First, we define the notion "redundant weakening".

Definition 4.1. A (Weakening)-rule $\mathcal{W}$ in a $\mathbf{TLJ}^{\infty}$-derivation Σ is *necessary* if it satisfies the following conditions.

- A $(\to\supset)$-rule appears just after $\mathcal{W}$, and $\mathcal{W}$ add only one side formula of this $(\to\supset)$-rule.
- $\mathcal{W}$ is the lowest inference rule in Σ.

A (Weakening)-rule is *redundant* if it is not necessary.

We call a $\mathbf{TLJ}^{\infty}$-derivation in which no redundant (Weakening)-rules occur an *essential derivation*.

Example 4.2. In the following figures, the rules $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$ are all necessary. The rules $\mathcal{W}_4, \mathcal{W}_5$ and $\mathcal{W}_6$ are all redundant.

$$\dfrac{\dfrac{\vdots}{\Gamma \to \Delta,\ \overline{n} * k : \beta}\ \mathcal{W}_1}{\dfrac{\overline{n} * k : \alpha,\ \Gamma \to \Delta,\ \overline{n} * k : \beta}{\Gamma \to \Delta,\ \overline{n} : \alpha \supset \beta}\ (\to\supset)}\ \vdots \qquad \dfrac{\dfrac{\vdots}{\overline{n} * k : \alpha,\ \Gamma \to \Delta}\ \mathcal{W}_2}{\dfrac{\overline{n} * k : \alpha,\ \Gamma \to \Delta,\ \overline{n} * k : \beta}{\Gamma \to \Delta,\ \overline{n} : \alpha \supset \beta}\ (\to\supset)}\ \vdots$$

$$\dfrac{\vdots}{\dfrac{\Gamma \to \Delta}{\Gamma', \Gamma \to \Delta, \Delta'}\ \mathcal{W}_3}$$

(endsequent)

$$\dfrac{\dfrac{\vdots \atop \Gamma \to \Delta}{\overline{n}*k:\alpha,\ \Gamma \to \Delta,\ \overline{n}*k:\beta}\ \mathcal{W}_4}{\Gamma \to \Delta,\ \overline{n}:\alpha \supset \beta \atop \vdots}\ (\to\supset) \qquad \dfrac{\dfrac{\vdots \atop \Gamma \to \Delta,\ \overline{n}*k:\beta}{\overline{n}*k:\alpha,\ \Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}*k:\beta}\ \mathcal{W}_5}{\Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}:\alpha \supset \beta \atop \vdots}\ (\to\supset)$$

$$\dfrac{\dfrac{\vdots \atop \Gamma_1 \to \Delta_1}{\Gamma_1 \to \Delta_1,\ \overline{n}:\alpha}\ \mathcal{W}_6 \qquad \overline{n}:\beta,\ \Gamma_2 \to \Delta_2}{\overline{n}:\alpha \supset \beta,\ \Gamma_1,\Gamma_2 \to \Delta_1,\Delta_2 \atop \vdots}\ (\supset\to)$$

Theorem 4.3. Every $\mathbf{TLJ}^\infty$-derivation can be transformed into an essential $\mathbf{TLJ}^\infty$-derivation with the same endsequent.

Proof. The proof is given by double induction on:

- The number w of occurrence of redundant (Weakening)-rule in Σ.
- The height[1] h of the lowest redundant (Weakening)-rule in Σ.

Take one of the lowest redundant (Weakening)-rules $\mathcal{W}$, and do the following operations. In each case, we can easily check that the operation reduces h without changing w or reduces w.

(1) Suppose that $\mathcal{W}$ adds two side formulas of $(\to\supset)$-rule occurring immediately after $\mathcal{W}$:

$$\dfrac{\dfrac{\vdots\ (a) \atop \Gamma \to \Delta}{\overline{n}*k:\alpha,\ \Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}*k:\beta}\ \mathcal{W}}{\Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}:\alpha \supset \beta \atop \vdots\ (b)}\ (\to\supset)$$

In this case, we transform this figure into the following figure.

$$\dfrac{\vdots\ (a) \atop \Gamma \to \Delta}{\Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}:\alpha \supset \beta \atop \vdots\ (b)}\ (\text{Weakening})$$

(2) Suppose that $\mathcal{W}$ adds one side formula of $(\to\supset)$-rule occurring immediately after $\mathcal{W}$ but adds also some other formulas:

$$\dfrac{\dfrac{\vdots\ (a) \atop \overline{n}*k:\alpha,\ \Gamma \to \Delta}{\overline{n}*k:\alpha,\ \Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}*k:\beta}\ \mathcal{W}}{\Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}:\alpha \supset \beta \atop \vdots\ (b)}\ (\to\supset) \qquad \dfrac{\dfrac{\vdots\ (a) \atop \Gamma \to \Delta,\ \overline{n}*k:\beta}{\overline{n}*k:\alpha,\ \Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}*k:\beta}\ \mathcal{W}}{\Gamma',\Gamma \to \Delta,\Delta',\ \overline{n}:\alpha \supset \beta \atop \vdots\ (b)}\ (\to\supset)$$

[1] The height of a occurrence $\mathcal{I}$ of an inference rule in Σ is the number of tree sequents occurring in the path from $\mathcal{I}$ to the endsequent of Σ. An occurrence $\mathcal{W}$ of redundant (Weakening)-rule is the lowest in Σ if its height is smallest in all redundant (Weakening)-rules in Σ.

In this case, we transform this figure as follows.

$$\begin{array}{c}\vdots\,(a)\\ \dfrac{\dfrac{\dfrac{\overline{n}*k:\alpha,\ \Gamma\to\Delta}{\overline{n}*k:\alpha,\ \Gamma\to\Delta,\ \overline{n}*k:\beta}\,(\text{Weakening})}{\Gamma\to\Delta,\ \overline{n}:\alpha\supset\beta}\,(\to\supset)}{\Gamma',\Gamma\to\Delta,\Delta',\ \overline{n}:\alpha\supset\beta}\,(\text{Weakening})\\ \vdots\,(b)\end{array}\qquad \begin{array}{c}\vdots\,(a)\\ \dfrac{\dfrac{\dfrac{\Gamma\to\Delta,\ \overline{n}*k:\beta}{\overline{n}*k:\alpha,\ \Gamma\to\Delta,\ \overline{n}*k:\beta}\,(\text{Weakening})}{\Gamma\to\Delta,\ \overline{n}:\alpha\supset\beta}\,(\to\supset)}{\Gamma',\Gamma\to\Delta,\Delta',\ \overline{n}:\alpha\supset\beta}\,(\text{Weakening})\\ \vdots\,(b)\end{array}$$

(3) Suppose that $\mathcal{W}$ adds a side formula of ($\supset\to$)-rule occurring immediately after $\mathcal{W}$:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\,(a)\\ \Gamma_1\to\Delta_1\end{array}}{\Gamma_1,\Gamma_2\to\Delta_1,\Delta_2\ \overline{n}:\alpha}\,\mathcal{W}\qquad \begin{array}{c}\vdots\,(b)\\ \overline{n}:\beta,\ \Gamma_3\to\Delta_3\end{array}}{\overline{n}:\alpha\supset\beta,\ \Gamma_1,\Gamma_2,\Gamma_3\to\Delta_1,\Delta_2,\Delta_3}\,(\supset\to)\\ \vdots\,(c)\end{array}$$

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\,(a)\\ \Gamma_1\to\Delta_1,\ \overline{n}:\alpha\end{array}\qquad \dfrac{\begin{array}{c}\vdots\,(b)\\ \Gamma_2\to\Delta_2\end{array}}{\overline{n}:\beta,\ \Gamma_2,\Gamma_3\to\Delta_2,\Delta_3}\,\mathcal{W}}{\overline{n}:\alpha\supset\beta,\ \Gamma_1,\Gamma_2,\Gamma_3\to\Delta_1,\Delta_2,\Delta_3}\,(\supset\to)\\ \vdots\,(c)\end{array}$$

In this case, we transform this figure as follows.

$$\begin{array}{c}\vdots\,(a)\\ \dfrac{\Gamma_1\to\Delta_1}{\overline{n}:\alpha\supset\beta,\ \Gamma_1,\Gamma_2,\Gamma_3\to\Delta_1,\Delta_2,\Delta_3}\,(\text{Weakening})\\ \vdots\,(c)\end{array}$$

$$\begin{array}{c}\vdots\,(b)\\ \dfrac{\Gamma_2\to\Delta_2}{\overline{n}:\alpha\supset\beta,\ \Gamma_1,\Gamma_2,\Gamma_3\to\Delta_1,\Delta_2,\Delta_3}\,(\text{Weakening})\\ \vdots\,(c)\end{array}$$

(4) Suppose that (Weakening)-rule occurs immediately after $\mathcal{W}$:

$$\begin{array}{c}\vdots\,(a)\\ \dfrac{\dfrac{\Gamma_1\to\Delta_2}{\Gamma_1,\Gamma_2\to\Delta_1,\Delta_2}\,\mathcal{W}}{\Gamma_1,\Gamma_2,\Gamma_3\to\Delta_1,\Delta_2,\Delta_3}\,(\text{Weakening})\\ \vdots\,(b)\end{array}$$

In this case, we transform this figure into the following figure.

$$\dfrac{\begin{matrix}\vdots\ (a)\\ \Gamma_1 \to \Delta_2\end{matrix}}{\begin{matrix}\Gamma_1, \Gamma_2, \Gamma_3 \to \Delta_1, \Delta_2, \Delta_3\\ \vdots\ (b)\end{matrix}}\ \text{(Weakening)}$$

(5) Suppose that $\mathcal{W}$ adds no formulas:

$$\dfrac{\begin{matrix}\vdots\ (a)\\ \Gamma \to \Delta\end{matrix}}{\begin{matrix}\emptyset, \Gamma \to \Delta, \emptyset\\ \vdots\ (b)\end{matrix}}\ \mathcal{W}$$

In this case, we transform this figure into the following figure.

$$\begin{matrix}\vdots\ (a)\\ \Gamma \to \Delta\\ \vdots\ (b)\end{matrix}$$

(6) Suppose that $\mathcal{W}$ adds only formulas which are not side formulas of the inference rule occurring immediately after $\mathcal{W}$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ (a)\\ \Gamma \to \Delta\end{matrix}}{\Pi, \Gamma \to \Delta, \Theta}\ \mathcal{W}}{\begin{matrix}\Pi, \Gamma' \to \Delta', \Theta\\ \vdots\ (b)\end{matrix}}\ \mathcal{I}$$

In this case, we transform this derivation into the following figure.

$$\dfrac{\dfrac{\begin{matrix}\vdots\ (a)\\ \Gamma \to \Delta\end{matrix}}{\Gamma' \to \Delta'}\ \mathcal{I}}{\begin{matrix}\Pi, \Gamma' \to \Delta', \Theta\\ \vdots\ (b)\end{matrix}}\ \mathcal{W}$$

□

Lemma 4.4. If Σ is an essential $\mathbf{TLJ}^\infty$-derivation of $\Pi \to \Theta$, then Σ has the following form for some $\Pi' \subseteq \Pi$ and $\Theta' \subseteq \Theta$.

$$\begin{matrix}\vdots\\ \Pi' \to \Theta'\\ \hline\hline \Pi \to \Theta\end{matrix}\ \text{(Weakening)}$$

Then, if $\Gamma \to \Delta$ occurs above the sequent $\Pi' \to \Theta'$, then the following conditions hold.

- Δ is a singleton set.
- Suppose that $\Delta = \{\overline{n} : \beta\}$ and $\overline{m} : \alpha \in \Gamma$, then $\overline{m} \preceq \overline{n}$.

Proof. By induction on the size of the derivation of $\Gamma \to \Delta$.

(0) Suppose that $\Gamma \to \Delta$ is an axiom. Then this sequent has the form $\overline{n} : \alpha \to \overline{n} * \overline{k} : \alpha$ and satisfies the required conditions obviously.

(1) Suppose that $\Gamma \to \Delta$ is derived by the $(\to\supset)$-rule.

$$\dfrac{\begin{array}{c}\vdots\\ \overline{n} * k : \alpha,\ \Gamma \to \Delta',\ \overline{n} * k : \beta\end{array}}{\Gamma \to \Delta',\ \overline{n} : \alpha \supset \beta}\ (\to\supset) \quad (\Delta = \Delta' \cup \{\overline{n} : \alpha \supset \beta\})$$

By induction hypothesis, we have the following conditions.
- $\Delta' = \emptyset$.
- If $\overline{m} : \gamma \in \Gamma$, then $\overline{m} \preceq \overline{n} * k$.

By label condition of $(\to\supset)$, $\overline{n} * k$ and its descendants do not occur in the lower sequent. Therefore we obtain that if $\overline{m} : \gamma \in \Gamma$ then $\overline{m} \preceq \overline{n}$.

(2) Suppose that $\Gamma \to \Delta$ is derived by the $(\supset\to)$-rule.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \to \Delta_1,\ \overline{k} : \alpha\end{array} \quad \begin{array}{c}\vdots\\ \overline{k} : \beta,\ \Gamma_2 \to \Delta_2\end{array}}{\overline{k} : \alpha \supset \beta,\ \Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}\ (\supset\to)$$

$$(\Gamma = \{\overline{k} : \alpha \supset \beta\} \cup \Gamma_1 \cup \Gamma_2,\ \Delta = \Delta_1 \cup \Delta_2)$$

By induction hypothesis, we have the following conditions.
- $\Delta_1 = \emptyset$, and $\Delta_2 = \{\overline{n} : \gamma\}$ for some $\overline{n} \in \mathbb{N}^{<\omega}$ and $\gamma \in \mathrm{Fml}$.
- If $\overline{m} : \delta \in \Gamma_1$ then $\overline{m} \preceq \overline{k}$.
- If $\overline{l} : \theta \in \{\overline{k} : \beta\} \cup \Gamma_2$ then $\overline{l} \preceq \overline{n}$.

From the first condition, we obtain $\Delta = \{\overline{n} : \gamma\}$. In addition, from the third condition, we obtain $\overline{k} \preceq \overline{n}$. Therefore, with the second condition, we obtain that if $\overline{m} : \delta \in \Gamma$ then $\overline{m} \prec \overline{n}$.

(3) Suppose that $\Gamma \to \Delta$ is not the endsequent of Σ, and is derived by the (Weakening)-rule. Because Σ is essential, the form of this figure is either of the following figures.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma' \to \Delta',\ \overline{n} : \beta\end{array}}{\overline{n} * k : \alpha,\ \Gamma' \to \Delta',\ \overline{n} : \beta}\ (\text{Weakening})}{\Gamma' \to \Delta',\ \overline{n} : \alpha \supset \beta}\ (\to\supset) \qquad (\Gamma = \{\overline{n} * k : \alpha\} \cup \Gamma',\ \Delta = \Delta' \cup \{\overline{n} : \beta\})$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \overline{n} * k : \alpha,\ \Gamma' \to \Delta'\end{array}}{\overline{n} * k : \alpha,\ \Gamma' \to \Delta',\ \overline{n} : \beta}\ (\text{Weakening})}{\Gamma' \to \Delta',\ \overline{n} : \alpha \supset \beta}\ (\to\supset) \qquad (\Gamma = \{\overline{n} * k : \alpha\} \cup \Gamma',\ \Delta = \Delta' \cup \{\overline{n} : \beta\})$$

First, we show that the later figure can not occur. Suppose that this figure occurs in Σ. By induction hypothesis, we have the following conditions.
- There are $\overline{l} \in \mathbb{N}^{<\omega}$ and $\gamma \in \mathrm{Fml}$ such that $\Delta' = \{\overline{m} : \gamma\}$.
- If $\overline{l} : \delta \in \{\overline{n} * k : \alpha\} \cup \Gamma'$, then $\overline{l} \preceq \overline{m}$.

From the later condition, we have $\overline{n} * k \preceq \overline{l}$. This violates the label condition of the $(\to\supset)$-rule.

Then we consider the former figure. By induction hypothesis, we have the following conditions.
- $\Delta' = \emptyset$.

- If $\bar{l} : \delta \in \Gamma'$, then $\bar{l} \preceq \overline{n} * k$.

These are just as the required conditions.

□

Theorem 4.5. ([♯]) If φ is provable in **TLJ**, then φ is provable in Gentzen's **LJ** [2]
.

Proof. Let Σ_0 be a **TLJ**-derivation of $\rightarrow \epsilon : \varphi$. By theorems 3.2 and 4.3, we can transform Σ_0 into an essential $\mathbf{TLJ}^{\infty}$-derivation Σ_1 of $\rightarrow \epsilon : \varphi$. By lemma 4.4, we can check that if $\Gamma \rightarrow \Delta$ occurs in Σ_1, then Δ is a singleton set. Then, we can obtain a **LJ**-derivation Σ_2 of φ from Σ_1 by replacing each tree sequent occurring in Σ_1 of the form $\overline{n_1} : \alpha_1, \ldots, \overline{n_k} : \alpha_k \xrightarrow{\mathcal{T}} \overline{m} : \beta$ by a sequent $\alpha_1, \ldots, \alpha_k \rightarrow \beta$. □

Example 4.6. Suppose that the following **TLJ**-derivation Σ_2 is given.

$$\dfrac{\dfrac{\langle 1\rangle : p \xrightarrow{\{\epsilon,\langle 1\rangle,\langle 1,1\rangle\}} \langle 1,1\rangle : p \qquad \dfrac{\langle 1\rangle : p \xrightarrow{\{\epsilon,\langle 1\rangle,\langle 1,1\rangle\}} \langle 1,1\rangle : p}{\langle 1,1\rangle : q,\ \langle 1\rangle : p \xrightarrow{\{\epsilon,\langle 1\rangle,\langle 1,1\rangle\}} \langle 1,1\rangle : p}\text{(Weakening)}}{\dfrac{\langle 1,1\rangle : p \supset q,\ \langle 1\rangle : p \xrightarrow{\{\epsilon,\langle 1\rangle,\langle 1,1\rangle\}} \langle 1,1\rangle : p}{\langle 1\rangle : p \xrightarrow{\{\epsilon,\langle 1\rangle\}} \langle 1\rangle : (p \supset q) \supset p}(\rightarrow\supset)}(\supset\rightarrow)}{\xrightarrow{\{\epsilon\}} \epsilon : p \supset (p \supset q) \supset p}(\rightarrow\supset)$$

Then we extract an **LJ**-derivation from Σ_2 as follows.

$$\Sigma_2$$

$\bigtriangledown$ Theorem 3.2

$$\dfrac{\dfrac{\dfrac{\langle 1\rangle : p \rightarrow \langle 1,1\rangle : p \qquad \dfrac{\langle 1\rangle : p \rightarrow \langle 1,1\rangle : p}{\langle 1,1\rangle : q,\ \langle 1\rangle : p \rightarrow \langle 1,1\rangle : p}\text{(Weakening)}}{\langle 1,1\rangle : p \supset q,\ \langle 1\rangle : p \rightarrow \langle 1,1\rangle : p}(\supset\rightarrow)}{\langle 1\rangle : p \rightarrow \langle 1\rangle : (p \supset q) \supset p}(\rightarrow\supset)}{\rightarrow \epsilon : p \supset (p \supset q) \supset p}(\rightarrow\supset)$$

$\bigtriangledown$ Theorem 4.3

$$\dfrac{\dfrac{\dfrac{\langle 1\rangle : p \xrightarrow{\mathcal{T}} \langle 1,1\rangle : p}{\langle 1,1\rangle : p \supset q,\ \langle 1\rangle : p \rightarrow \langle 1,1\rangle : p}\text{(Weakening)}}{\langle 1\rangle : p \rightarrow \langle 1\rangle : (p \supset q) \supset p}(\rightarrow\supset)}{\rightarrow \epsilon : p \supset (p \supset q) \supset p}(\rightarrow\supset)$$

$\bigtriangledown$ Theorem 4.5

$$\dfrac{\dfrac{\dfrac{p \rightarrow p}{p \supset q, p \rightarrow p}\text{(Weakening)}}{p \rightarrow (p \supset q) \supset p}(\rightarrow\supset)}{\rightarrow p \supset (p \supset q) \supset p}(\rightarrow\supset)$$

[2]**LJ** is a proof system for intuitionistic logic introduced by Gentzen. See [5] in detail.

5. Admissible rules

In this section, we show proof theoretically that some useful rules are admissible in **TLJ**.

5.1. (Cut), $(h \to)$, $(\to h)$.

Definition 5.1. The system $\mathbf{TLJ}^{ch}$ is obtained from **TLJ** by adding (Cut), $(h \to)$ and $(\to h)$ written below.

$$\frac{\Gamma_1 \xrightarrow{\mathcal{T}} \Delta_1,\ \overline{n} : \alpha \quad \overline{n} : \alpha,\ \Gamma_2 \xrightarrow{\mathcal{T}} \Delta_2}{\Gamma_1, \Gamma_2 \xrightarrow{\mathcal{T}} \Delta_1, \Delta_2}\ (\text{Cut})$$

$$\frac{\overline{n} * k : \alpha,\ \Gamma \xrightarrow{\mathcal{T}} \Delta}{\overline{n} : \alpha,\ \Gamma \xrightarrow{\mathcal{T}} \Delta}\ (h \to) \qquad \frac{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} : \alpha}{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} * k : \alpha}\ (\to h)$$

In addition, we define the system $\mathbf{TLJ}^{c}$ from **TLJ** by adding (Cut), and define the system $\mathbf{TLJ}^{h}$ from **TLJ** by adding $(h \to)$ and $(\to h)$.

Theorem 5.2. The following conditions are all equivalent.

- φ is provable in **TLJ**.
- φ is provable in $\mathbf{TLJ}^{c}$.
- φ is provable in $\mathbf{TLJ}^{h}$.
- φ is provable in $\mathbf{TLJ}^{ch}$.

Proof. Obviously, the following relation holds.

$$\begin{array}{ccccc} & & \varphi \text{ is provable in } \mathbf{TLJ}^{c} & & \\ & \nearrow & & \searrow & \\ \varphi \text{ is provable in } \mathbf{TLJ} & & & & \varphi \text{ is provable in } \mathbf{TLJ}^{ch} \\ & \searrow & & \nearrow & \\ & & \varphi \text{ is provable in } \mathbf{TLJ}^{h} & & \end{array}$$

Furthermore, we can show that each $\mathbf{TLJ}^{ch}$-derivation can be transformed into an **LJ**-derivation in the same way as subsection 4.1. This implies that if φ is provable in $\mathbf{TLJ}^{ch}$ then φ is provable in **TLJ**. □

In the following argument, we write $\mathbf{TLJ}^{ch}$ as **TLJ** simply.

5.2. Other admissible rules.

We show that some additional rules are admissible in **TLJ**. These rules are very useful to analyze **TLJ**-derivations proof theoretically (see [3, 4]).

Theorem 5.3.

(1) Suppose $\overline{n} * j \notin \mathcal{T}$. Let

$$\mathcal{T}_{\overline{n}}^{i \mapsto j} = \{\overline{m} \in \mathcal{T} \mid \overline{n} * i \not\preceq \overline{m}\} \cup \{\overline{n} * j * \overline{m} \mid \overline{n} * i * \overline{m} \in \mathcal{T}\},$$
$$\Gamma_{\overline{n}}^{i \mapsto j} = \{\overline{m} : \alpha \in \Gamma \mid \overline{n} * i \not\preceq \overline{m}\} \cup \{\overline{n} * j * \overline{m} : \alpha \mid \overline{n} * i * \overline{m} : \alpha \in \Gamma\}.$$

Then the following rule is admissible in **TLJ** (see also figure 4).

$$\frac{\Gamma \xrightarrow{\mathcal{T}} \Delta}{\Gamma^{i \mapsto j}_{\overline{n}} \xrightarrow{\mathcal{T}^{i \mapsto j}_{\overline{n}}} \Delta^{i \mapsto j}_{\overline{n}}} \text{ (Transplant)}$$

Furthermore, if there is a **TLJ**-derivation Σ of $\Gamma \xrightarrow{\mathcal{T}} \Delta$, then there is a **TLJ**-derivation Σ' of $\Gamma^{i \mapsto j}_{\overline{n}} \xrightarrow{\mathcal{T}^{i \mapsto j}_{\overline{n}}} \Delta^{i \mapsto j}_{\overline{n}}$ such that $|\Sigma'| = |\Sigma|$.

(2) Suppose $\overline{n} \in \mathcal{T}$ and $\overline{n} * i \notin \mathcal{T}$. The following inference rule is admissible in **TLJ** (see also figure 5).

$$\frac{\Gamma \xrightarrow{\mathcal{T}} \Delta}{\Gamma \xrightarrow{\mathcal{T} \cup \{\overline{n} * i\}} \Delta} \text{ (Grow)}$$

(3) The following rule is admissible in **TLJ**.

$$\frac{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} : \top \supset \alpha}{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} : \alpha} \text{ (Remove } \top)$$

Here $\top \equiv p \supset p$.

(4) Let $\mathcal{T}^{\not\succeq \overline{n}} = \{\overline{m} \in \mathcal{T} \mid \overline{n} \not\preceq \overline{m}\}$. Then the following rule is admissible in **TLJ** (see also figure 6).

$$\frac{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} : \alpha}{\Gamma \xrightarrow{\mathcal{T}^{\not\succeq \overline{n} * i}} \Delta,\ \overline{n} : \alpha} \text{ (Trim)}$$

Here, $n * i$ and its descendant do not occur in $\Gamma \cup \Delta$.

(5) The following rule is admissible in **TLJ** (see also figure 7).

$$\frac{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} * i * \overline{m} : \alpha}{\Gamma \xrightarrow{\mathcal{T}^{\not\succeq \overline{n} * i}} \Delta,\ \overline{n} : \alpha} \text{ (Drop)}$$

Here, $n * i$ and its descendant do not occur in $\Gamma \cup \Delta$.

(6) The following rule is admissible in **TLJ** (see also figure 8).

$$\frac{\overline{n} * i * \overline{m} : \alpha,\ \Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} * i * \overline{m} : \beta}{\Gamma \xrightarrow{\mathcal{T}^{\not\succeq \overline{n} * i}} \Delta,\ \overline{n} : \alpha \supset \beta} \ (\rightarrow\supset)^*$$

Here, $n * i$ and its descendant do not occur in $\Gamma \cup \Delta$.

Proof.

(1) By induction on $|\Sigma|$. We give a proof only the case when $\Gamma \xrightarrow{\mathcal{T}} \Delta$ is derived by $(\rightarrow\supset)$-rule.

$$\frac{\begin{array}{c}\vdots\\ \overline{m} * k : \alpha,\ \Gamma \xrightarrow{\mathcal{T} \cup \{\overline{m} * k\}} \Delta',\ \overline{m} * k : \beta\end{array}}{\Gamma \xrightarrow{\mathcal{T}} \Delta',\ \overline{m} : \alpha \supset \beta} \ (\rightarrow\supset)$$

If $\overline{m} * k \neq \overline{n} * j$, then we obtain a derivation of

$$\overline{m} * k : \alpha,\ \Gamma \xrightarrow{\mathcal{T}^{i \mapsto j}_{\overline{n}} \cup \{\overline{m} * k\}} \Delta',\ \overline{m} * k : \beta$$

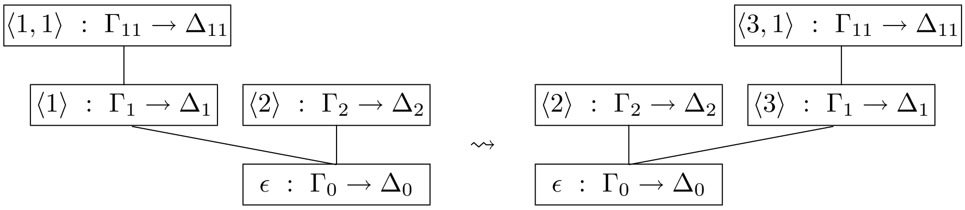

Figure 4. (Transplant)

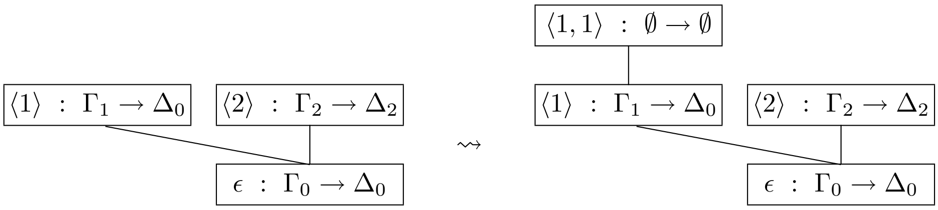

Figure 5. (Grow)

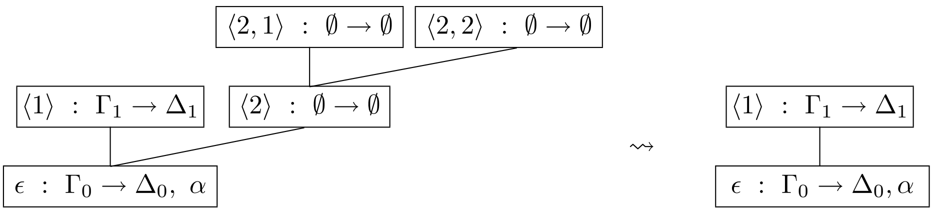

Figure 6. (Trim)

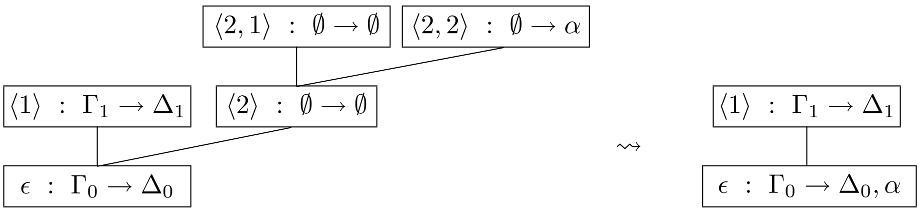

Figure 7. (Drop)

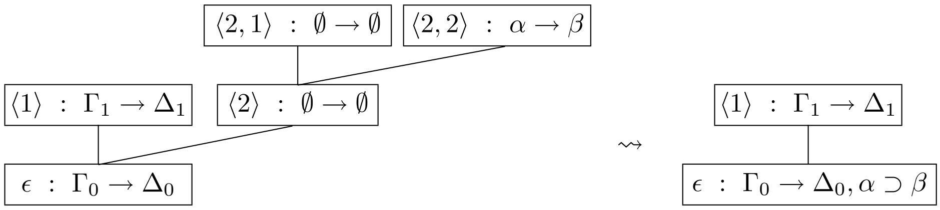

FIGURE 8. $(\to\supset)^*$

by induction hypothesis, and can derive $\Gamma \xrightarrow{\mathcal{T}^{i\mapsto j}_{\overline{n}}} \Delta',\ \overline{m} : \alpha \supset \beta$. If $\overline{m} * k = \overline{n} * j$ then, by induction hypothesis, we obtain a derivation Σ'' of

$$\overline{n} * l : \alpha,\ \Gamma \xrightarrow{\mathcal{T}\cup\{\overline{n}*l\}} \Delta,\ \overline{n} * l : \beta\ (l \neq j)$$

such that $|\Sigma''| = |\Sigma| - 1$. Then we also obtain a derivation of $\overline{n} * l : \alpha,\ \Gamma \xrightarrow{\mathcal{T}\cup\{\overline{n}*l\}} \Delta,\ \overline{n} * l : \beta$ by use of induction hypothesis again, and can derive $\Gamma \xrightarrow{\mathcal{T}^{i\mapsto j}_{\overline{n}}} \Delta',\ \overline{m} : \alpha \supset \beta$.

(2) By induction on the size of the derivation of $\Gamma \xrightarrow{\mathcal{T}} \Delta$. The only nontrivial case is the case when $\Gamma \xrightarrow{\mathcal{T}} \Delta$ is derived as follows.

$$\dfrac{\begin{matrix}\vdots\\ \overline{n} * i : \alpha,\ \Gamma \xrightarrow{\mathcal{T}} \Delta',\ \overline{n} * i : \beta\end{matrix}}{\Gamma \xrightarrow{\mathcal{T}} \Delta',\ \overline{n} : \alpha \supset \beta}\ (\to\supset)$$

In this case, we can prove in the same way as the proof of (1) by use of (Transplant)-rule.

(3) (Remove $\top$) can be supplemented by the following figure.

$$\dfrac{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} : \top \supset \alpha \qquad \dfrac{\dfrac{\overline{n} * 1 : p \xrightarrow{\mathcal{T}\cup\{\overline{n}*1\}} \overline{n} * 1 : p}{\xrightarrow{\mathcal{T}} \overline{n} : \top}\ (\to\supset) \qquad \overline{n} : \alpha \xrightarrow{\mathcal{T}} \overline{n} : \alpha}{\overline{n} : \top \supset \alpha \xrightarrow{\mathcal{T}} \overline{n} : \alpha}\ (\supset\to)}{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} : \alpha}\ (\mathrm{Cut})$$

(4) As an example, we give a derivation which supplements figure 6 as follows. The other cases are supplemented in the same way.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \xrightarrow{\mathcal{T}_3} \Delta,\ \epsilon : \alpha}{\langle 2,2\rangle : \top,\ \Gamma \xrightarrow{\mathcal{T}_3} \Delta,\ \epsilon : \alpha,\ \langle 2,2\rangle : \alpha}\text{(Weakening)}}{\Gamma \xrightarrow{\mathcal{T}_2} \Delta,\ \epsilon : \alpha,\ \langle 2\rangle : \top \supset \alpha}(\rightarrow\supset)}{\langle 2,1\rangle : \top,\ \Gamma \xrightarrow{\mathcal{T}_2} \Delta,\ \epsilon : \alpha,\ \langle 2\rangle : \top \supset \alpha,\ \langle 2,1\rangle : \alpha}\text{(Weakening)}}{\Gamma \xrightarrow{\mathcal{T}_1} \Delta,\ \epsilon : \alpha,\ \langle 2\rangle : \top \supset \alpha}(\rightarrow\supset)}{\langle 2\rangle : \top,\ \Gamma \xrightarrow{\mathcal{T}_1} \Delta,\ \epsilon : \alpha,\ \langle 2\rangle : \top \supset \alpha}\text{(Weakening)}}{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \epsilon : \alpha,\ \epsilon : \top \supset \top \supset \alpha}(\rightarrow\supset)}{\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \epsilon : \alpha}\text{(Remove }\top)$$

Here,

$$\mathcal{T} = \{\epsilon, \langle 1\rangle\},\quad \mathcal{T}_1 = \{\epsilon, \langle 1\rangle, \langle 2\rangle\},\quad \mathcal{T}_2 = \{\epsilon, \langle 1\rangle, \langle 2\rangle, \langle 2,1\rangle\},$$
$$\mathcal{T}_3 = \{\epsilon, \langle 1\rangle, \langle 2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}.$$

(5) By induction on $|\overline{m}|$. Suppose that $\Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} * i * \overline{m}$ is provable in **TLJ**. Suppose that $\overline{n}$ and its descendants do not occur in $\Gamma \cup \Delta$. Let $\overline{m} = \overline{k} * j$. The following figure shows that $\Gamma \xrightarrow{\mathcal{T}\backslash\{\overline{n}*i*\overline{m}\}} \Delta,\ \overline{n} * i * \overline{k} : \alpha$ is provable in **TLJ**.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} * i * \overline{m} : \alpha\end{matrix}}{\overline{n} * i * \overline{m} : \top,\ \Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} * i * \overline{m} : \alpha}\text{(Weakening)}}{\Gamma \xrightarrow{\mathcal{T}\backslash\{\overline{n}*i*\overline{m}\}} \Delta,\ \overline{n} * i * \overline{k} : \top \supset \alpha}(\rightarrow\supset)}{\Gamma \xrightarrow{\mathcal{T}\backslash\{\overline{n}*i*\overline{m}\}} \Delta,\ \overline{n} * i * \overline{k} : \alpha}\text{(Remove }\top)$$

Then, by induction hypothesis, $\Gamma \xrightarrow{\mathcal{T}^{\not\succeq \overline{n}*i}} \Delta,\ \overline{n} : \alpha$ is provable in **TLJ**.

(6) Let $\overline{m} = \overline{k} * j$. Then the following figure shows that the $(\rightarrow\supset)^*$-rule is admissible in **TLJ**.

$$\dfrac{\dfrac{\overline{n} * i * \overline{m} : \alpha,\ \Gamma \xrightarrow{\mathcal{T}} \Delta,\ \overline{n} * i * \overline{m} : \beta}{\Gamma \xrightarrow{\backslash\{\overline{n}*i*\overline{m}\}} \Delta,\ \overline{n} * i * \overline{k} : \alpha \supset \beta}(\rightarrow\supset)}{\Gamma \xrightarrow{\mathcal{T}^{\not\succeq \overline{n}*i}} \Delta,\ \overline{n} : \alpha \supset \beta}\text{(Drop)}$$

□

6. Variant system: TNJ

In this section, we introduce a natural deduction stile proof system **TNJ**. In [3], we give an intuitionistic fragment of $\lambda\mu$-calculus by use of this system.

Definition 6.1. The system **TNJ** is defined from **TLJ** by

- removing $(\supset\rightarrow)$-rule,

- adding the following rule.

$$\dfrac{\Gamma_1 \xrightarrow{\mathcal{I}} \Delta_1,\ \overline{n} : \alpha \supset \beta \qquad \Gamma_2 \xrightarrow{\mathcal{I}} \Delta_2,\ \overline{n} : \alpha}{\Gamma_1, \Gamma_2 \xrightarrow{\mathcal{I}} \Delta_1, \Delta_2,\ \overline{n} : \beta}\ (\mathbf{MP})$$

Theorem 6.2. φ is provable in **TNJ** if and only if φ is intuitionistically valid.

Proof.

($\Longrightarrow$) We can show that if $\Gamma \xrightarrow{\mathcal{I}} \Delta$ is provable in **TNJ** then $\Gamma \xrightarrow{\mathcal{I}} \Delta$ is provable in $\mathbf{TLJ}^{ch}$ by induction on the size of the **TNJ**-derivation of $\Gamma \xrightarrow{\mathcal{I}} \Delta$. It suffices to show the case when $\Gamma \xrightarrow{\mathcal{I}} \Delta$ is derived by (**MP**).

$$\dfrac{\Gamma_1 \xrightarrow{\mathcal{I}} \Delta_1,\ \overline{n} : \alpha \supset \beta \qquad \Gamma_2 \xrightarrow{\mathcal{I}} \Delta_2,\ \overline{n} : \alpha}{\Gamma_1, \Gamma_2 \xrightarrow{\mathcal{I}} \Delta_1, \Delta_2,\ \overline{n} : \beta}\ (\mathbf{MP})$$

$$(\Gamma = \Gamma_1 \cup \Gamma_2,\ \Delta = \Delta_1 \cup \Delta_2 \cup \{\overline{n} : \beta\})$$

By induction hypothesis, $\Gamma_1 \xrightarrow{\mathcal{I}} \Delta_1,\ \overline{n} : \alpha \supset \beta$ and $\Gamma_2 \xrightarrow{\mathcal{I}} \Delta_2,\ \overline{n} : \alpha$ are both provable in $\mathbf{TLJ}^{ch}$. Then, we can construct a $\mathbf{TLJ}^{ch}$-derivation of $\Gamma \xrightarrow{\mathcal{I}} \Delta$ as follows.

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \xrightarrow{\mathcal{I}} \Delta_1,\ \overline{n} : \alpha \supset \beta\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Gamma_2 \xrightarrow{\mathcal{I}} \Delta_2,\ \overline{n} : \alpha\end{array} \qquad \overline{n} : \beta \xrightarrow{\mathcal{I}} \overline{n} : \beta}{\overline{n} : \alpha \supset \beta,\ \Gamma_2 \xrightarrow{\mathcal{I}} \Delta_2,\ \overline{n} : \beta}\ (\supset\rightarrow)}{\Gamma_1, \Gamma_2 \xrightarrow{\mathcal{I}} \Delta_1, \Delta_2,\ \overline{n} : \beta}\ (\mathrm{Cut})$$

($\Longleftarrow$) The set $\{\alpha \mid \vdash_{\mathbf{TNJ}} \alpha\}$ is obviously closed under modus ponens. Furthermore, we can see that $\alpha \supset \beta \supset \alpha$ and $(\alpha \supset \beta \supset \gamma) \supset (\alpha \supset \beta) \supset \alpha \supset \gamma$ are both provable in **TNJ** by the following figures.

$$\dfrac{\dfrac{\dfrac{\langle 1 \rangle : \alpha \xrightarrow{\mathcal{I}_1} \langle 1 \rangle : \alpha}{\langle 1 \rangle : \alpha,\ \langle 1, 1 \rangle : \beta \xrightarrow{\mathcal{I}_1} \langle 1, 1 \rangle : \alpha}\ (\mathrm{Weakening})}{\langle 1 \rangle : \alpha \xrightarrow{\mathcal{I}_1} \langle 1 \rangle : \beta \supset \alpha}\ (\rightarrow\supset)}{\xrightarrow{\{\epsilon\}} \epsilon : \alpha \supset \beta \supset \alpha}\ (\rightarrow\supset)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Sigma_1 \qquad \Sigma_2}{\langle 1,1,1 \rangle : \alpha, \langle 1,1 \rangle : \alpha \supset \beta,\ \langle 1 \rangle : \alpha \supset \beta \supset \gamma \xrightarrow{\mathcal{I}_3} \langle 1,1,1 \rangle : \gamma}\ (\mathbf{MP})}{\langle 1,1 \rangle : \alpha \supset \beta,\ \langle 1 \rangle : \alpha \supset \beta \supset \gamma \xrightarrow{\mathcal{I}_2} \langle 1,1 \rangle : \alpha \supset \gamma}\ (\rightarrow\supset)}{\langle 1 \rangle : \alpha \supset \beta \supset \gamma \xrightarrow{\mathcal{I}_1} \langle 1 \rangle : (\alpha \supset \beta) \supset \alpha \supset \gamma}\ (\rightarrow\supset)}{\xrightarrow{\{\epsilon\}} \epsilon : (\alpha \supset \beta \supset \gamma) \supset (\alpha \supset \beta) \supset \alpha \supset \gamma}\ (\rightarrow\supset)$$

$$\Sigma_1\ :\ \dfrac{\langle 1 \rangle : \alpha \supset \beta \supset \gamma \xrightarrow{\mathcal{I}_3} \langle 1,1,1 \rangle : \alpha \supset \beta \supset \gamma \quad \langle 1,1,1 \rangle : \alpha \xrightarrow{\mathcal{I}_3} \langle 1,1,1 \rangle : \alpha}{\langle 1,1,1 \rangle : \alpha,\ \langle 1 \rangle : \alpha \supset \beta \supset \gamma \xrightarrow{\mathcal{I}_3} \langle 1,1,1 \rangle : \beta \supset \gamma}\ (\mathbf{MP})$$

$$\Sigma_2 \ : \ \dfrac{\langle 1,1\rangle : \alpha \supset \beta \xrightarrow{\mathcal{T}_3} \langle 1,1,1\rangle : \alpha \supset \beta \quad \langle 1,1,1\rangle : \alpha \xrightarrow{\mathcal{T}_3} \langle 1,1,1\rangle : \alpha}{\langle 1,1,1\rangle : \alpha,\ \langle 1,1\rangle : \alpha \supset \beta \xrightarrow{\mathcal{T}_3} \langle 1,1,1\rangle : \beta} \ (\mathbf{MP})$$

Here $\mathcal{T}_1 = \{\epsilon, \langle 1\rangle\}$, $\mathcal{T}_2 = \{\epsilon, \langle 1\rangle, \langle 1,1\rangle\}$ and $\mathcal{T}_3 = \{\epsilon, \langle 1\rangle, \langle 1,1\rangle, \langle 1,1,1\rangle\}$.

□

References

[1] I. Hasuo and R. Kashima. Kripke completeness of first-order constructive logics with strong negation. *Logic Journal of IGPL*, 11-6:615–646, 2003.

[2] R. Kashima. Sequent calculus of non-classical logics - proofs of completeness theorems by sequent calculus (in japanese)). In *Mathematical Society of Japan Annual Colloquium*, 1999.

[3] N. Matsuda. Intuitionistic fragment of the $\lambda\mu$-calculus. to appear.

[4] N. Matsuda. $\lambda\rho$-calculus and intuitionistic tree sequent calculus. to appear.

[5] M. E. Szabo. *THE COLLECTED PAPERS OF GERHARD GENTZEN*. NORTH-HOLLAND AMSTERDAM, 1969.

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