A cut-free sequent calculus with ε-symbols

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Let $\mathcal{L}^0$ be a language of first-order predicate logic. The intermediate predicate logic **CD** (language $\mathcal{L}^0$) is obtained by adding the axiom $\forall x(A(x)\vee B)\to(\forall xA(x)\vee B)$ to the intuitionistic logic. **CD** is known to be complete with respect to Kripke-models with constant domains.

Problem 1 *Find a good (i.e., cut-free and simple) sequent calculus for* **CD**.

In [1], there are some solutions to this problem. Recently I try to give another solution which is a sequent calculus with single succedent ("LJ-style"), but this plan does not succeed yet.

$\mathcal{L}^\varepsilon$ is the extension of $\mathcal{L}^0$ with "ε-terms": $\varepsilon xA(x)$ and $\bar{\varepsilon}xA(x)$. The formulas $\exists xA(x)\to A(\varepsilon xA(x))$ and $A(\bar{\varepsilon}xA(x))\to\forall xA(x)$ are called "ε-axioms".

Cut-free sequent calculus LJ+ε (language $\mathcal{L}^\varepsilon$):

- Initial sequents. $\quad A^0 \Rightarrow A^0 \quad$ where A^0 is in $\mathcal{L}^0$.
- Inference rules.

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi}\ (\text{exchange}) \qquad \frac{A, A, \Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\ (\text{contraction})$$

$$\frac{\Gamma \Rightarrow \Pi}{A, \Gamma \Rightarrow \Pi}\ (\text{weakening}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A^0}\ (\text{weakening}) \ \text{ where } A^0 \text{ is in } \mathcal{L}^0.$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{A\to B, \Gamma, \Delta \Rightarrow \Pi}\ (\to \text{left}) \qquad \frac{\Gamma, A^0 \Rightarrow B}{\Gamma \Rightarrow A^0\to B}\ (\to \text{right}) \ \text{ where } A^0 \text{ is in } \mathcal{L}^0.$$

$$\frac{\Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow}\ (\neg \text{ left}) \qquad \frac{\Gamma, A^0 \Rightarrow}{\Gamma \Rightarrow \neg A^0}\ (\neg \text{ right}) \ \text{ where } A^0 \text{ is in } \mathcal{L}^0.$$

$$\frac{A, \Gamma \Rightarrow \Pi}{A\wedge B, \Gamma \Rightarrow \Pi}\ (\wedge \text{ left}) \qquad \frac{B, \Gamma \Rightarrow \Pi}{A\wedge B, \Gamma \Rightarrow \Pi}\ (\wedge \text{ left}) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\wedge B}\ (\wedge \text{ right})$$

$$\frac{A, \Gamma \Rightarrow \Pi \quad B, \Gamma \Rightarrow \Pi}{A\vee B, \Gamma \Rightarrow \Pi}\ (\vee \text{ left}) \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A\vee B}\ (\vee \text{ right}) \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A\vee B}\ (\vee \text{ right})$$

$$\frac{A(t),\ \Gamma \Rightarrow \Pi}{\forall xA(x),\ \Gamma \Rightarrow \Pi}\ (\forall \text{ left}) \qquad \frac{\Gamma(a) \Rightarrow A(a)}{\Gamma(\bar{\varepsilon}xA(x)) \Rightarrow \forall xA(x)}\ (\forall \text{ right})$$

$$\frac{A(\varepsilon xA(x)),\ \Gamma \Rightarrow \Pi}{\exists xA(x),\ \Gamma \Rightarrow \Pi}\ (\exists \text{ left}) \qquad \frac{\Gamma \Rightarrow A(t)}{\Gamma \Rightarrow \exists xA(x)}\ (\exists \text{ right})$$

$$\frac{\Gamma(a) \Rightarrow \Pi}{\Gamma(t) \Rightarrow \Pi}\ (\text{substitution}) \quad \text{where } a \text{ is not in } \Pi.$$

(a: free-variable. t: term. Π: empty or single formula.)

The characteristic formulas of **CD** are provable:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Rightarrow P(a)}{P(\bar{\varepsilon} x P(x)) \Rightarrow \forall x P(x)}\,(\forall\text{ right}) \qquad Q \Rightarrow Q}{P(\bar{\varepsilon} x P(x)) \vee Q \Rightarrow \forall x P(x) \vee Q}}{\forall x (P(x) \vee Q) \Rightarrow \forall x P(x) \vee Q}\,(\forall\text{ left})}{\Rightarrow \forall x (P(x) \vee Q) \to (\forall x P(x) \vee Q)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Rightarrow P(a)}{\neg P(a),\ P(a) \Rightarrow}}{\neg P(\varepsilon x P(x)),\ P(\varepsilon x P(x)) \Rightarrow}\,(\text{substitution})}{\neg P(\varepsilon x P(x)),\ \exists x P(x) \Rightarrow}\,(\exists\text{ left})}{\neg P(\varepsilon x P(x)) \Rightarrow \neg \exists x P(x)} \qquad Q \Rightarrow Q}{\neg P(\varepsilon x P(x)) \vee Q \Rightarrow \neg \exists x P(x) \vee Q}}{\forall x (\neg P(x) \vee Q) \Rightarrow \neg \exists x P(x) \vee Q}\,(\forall\text{ left})}{\Rightarrow \forall x (\neg P(x) \vee Q) \to (\neg \exists x P(x) \vee Q)}$$

Theorem 1 **CD** $\vdash A \quad \Longrightarrow \quad$ cut-free **LJ**$+\varepsilon \vdash \Rightarrow A$, *where A is in $\mathcal{L}^0$.*

Proof (sketch)

$$\begin{array}{rll} \mathbf{CD} \vdash A & \Longrightarrow & \text{cut-free sequent calculus with “connection”} \vdash \Rightarrow A \qquad ([1]) \\ & \Longrightarrow & \text{cut-free } \mathbf{LJ}+\varepsilon \vdash \Rightarrow A \qquad (\text{similarly to [2]: “}\mathbf{LK}^{\cup}\mathbf{R} \Longrightarrow \mathbf{LJ}^{\cup}\mathbf{R}\text{”}). \end{array}$$

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Theorem 2 (by Izumi Takeuti) *The converse of Theorem 1 does not hold.*

Proof Counterexample:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \Rightarrow P(x)}{P(\bar{\varepsilon} x P(x)) \Rightarrow \forall x P(x)}\,(\forall\text{ right})}{Q \to P(\bar{\varepsilon} x P(x)) \Rightarrow Q \to \forall x P(x)} \qquad \dfrac{\dfrac{P(x) \Rightarrow P(x)}{P(\bar{\varepsilon} x P(x)) \Rightarrow \forall x P(x)}\,(\forall\text{ right})}{R \to P(\bar{\varepsilon} x P(x)) \Rightarrow R \to \forall x P(x)}}{(Q \to P(\bar{\varepsilon} x P(x))) \vee (R \to P(\bar{\varepsilon} x P(x))) \Rightarrow (Q \to \forall x P(x)) \vee (R \to \forall x P(x))}}{\forall x \Big((Q \to P(x)) \vee (R \to P(x)) \Big) \Rightarrow (Q \to \forall x P(x)) \vee (R \to \forall x P(x))}\,(\forall\text{ left})}{\Rightarrow \forall x \Big((Q \to P(x)) \vee (R \to P(x)) \Big) \to \Big((Q \to \forall x P(x)) \vee (R \to \forall x P(x)) \Big)}$$

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References

[1] R. Kashima and T. Shimura: *Cut-elimination theorem for the logic of constant domains*, **MLQ 40**, 153-172 (1994).

[2] R. Kashima: *On semilattice relevant logics*, **MLQ** (to appear).