Foundations of Palm Calculus: A Practical Guide for AoI Analysis

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Motivation: Why Palm Calculus for AoI Analysis?

Ubiquity of Stationarity: Standard AoI analysis typically assumes the underlying (e.g., queueing or communication) system has reached a stationary state

Fundamental Framework: Palm calculus provides the rigorous mathematical framework to analyze stationary systems associated with point processes

Model-Agnostic Nature: It does not rely on specific stochastic assumptions (e.g., Poisson arrivals, exponential services, or independence)

Power of Universality: By simply assuming stationarity, we can derive general structural identities that remain robust across diverse system architectures

ATTENTION: To evaluate the AoI of a specific system, some stochastic assumptions (e.g., Markov property or independence) may be required

Bibliography

Palm Calculus Approach to AoI Analysis

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- Y. Jiang & **NM**. Joint performance analysis of ages of information in a multi-source pushout server. *IEEE Trans. Inform. Theory*, **68**, 965–975, 2022
- G. Kesidis, T. Konstantopoulos & M. A. Zazanis. Age of information using Markov-renewal methods. *Queueing Syst.*, **103**, 95–130, 2023
- A. Rizk & J.-Y. Le Boudec. A Palm calculus approach to the distribution of the age of information. *IEEE Trans. Inform. Theory*, **69**, 8097–8110, 2023

To Lean More about Palm Calculus

- F. Baccelli & P. Brémaud. *Elements of Queueing Theory: Palm Martingale Calculus and Stochastic Recurrences, 2nd ed.*, Springer, 2003

1. Stationary Point Processes & Palm Calculus
   - Stationary Point Processes
   - Introduction to Palm Calculus
2. Application to AoI Analysis of Single-Source Systems
   - General Identities
   - Examples
     - Example 1: Single Pushout Server
     - Example 2: FCFS Single Server Queue
3. Application to AoI Analysis of Multi-Source Systems
   - General Identities
   - An Example
     - Example 3: Multi-source Single Pushout Server
4. Summary

1 Stationary Point Processes & Palm Calculus
  - Stationary Point Processes
  - Introduction to Palm Calculus

2 Application to AoI Analysis of Single-Source Systems
  - General Identities
  - Examples
    - Example 1: Single Pushout Server
    - Example 2: FCFS Single Server Queue

3 Application to AoI Analysis of Multi-Source Systems
  - General Identities
  - An Example
    - Example 3: Multi-source Single Pushout Server

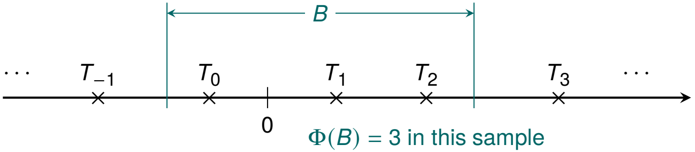
4 Summary

Point Process on $\mathbb{R}$

$\{T_n\}_{n\in\mathbb{Z}}$ $(T_n \le T_{n+1})$: Random sequence of times at which some events (e.g., packet arrivals, information updates) occur

$\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$: Point process (PP) on $\mathbb{R} = (-\infty, \infty)$

$\Phi(B)$: # of points falling in $B \in \mathcal{B}(\mathbb{R})$



Assumption

Simplicity (no multiple points): $\mathbb{P}\big(\Phi(\{t\}) \in \{0, 1\},\ ^{\forall}t \in \mathbb{R}\big) = 1$

$\Leftrightarrow\ \mathbb{P}\big(T_n < T_{n+1}\big) = 1,\ n \in \mathbb{Z}$

Conventionally, $T_0 \le 0 < T_1$

Stationary Point Process on $\mathbb{R}$

$\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$: A simple PP on $\mathbb{R}$

Assumption

Stationarity: $^\forall B_1 \dots, B_n \in \mathcal{B}(\mathbb{R}),\ k_1, \dots, k_n \in \underset{\{0,1,2,\dots\}}{\mathbb{N}_0},\ t \in \mathbb{R}$

$$\mathbb{P}\big(\Phi(B_1) = k_1, \dots, \Phi(B_n) = k_n\big) = \mathbb{P}\big(\Phi(B_1+t) = k_1, \dots, \Phi(B_n+t) = k_n\big)$$

where $B + t = \{s + t : s \in B\}$

Positive & Finite Intensity: $\lambda := \mathbb{E}\big[\Phi([0,1])\big] \in (0,\infty)$

Properties:

Locally Finite: $^\forall B \in \mathcal{B}(\mathbb{R})$ bounded, $\mathbb{P}\big(\Phi(B) < \infty\big) = 1$

$$\Leftrightarrow \lim_{n\to\pm\infty} T_n = \pm\infty$$

Mean # of Points Proportional to Lebesgue Measure:

$$\mathbb{E}[\Phi(B)] = \lambda\,|B|,\ ^\forall B \in \mathcal{B}(\mathbb{R})$$

Marked Point Processes

An arriving packet has some supplementary information
(e.g., content, source, required processing time, priority, etc.)

$\{Z_n\}_{n\in\mathbb{Z}}$: Sequence of random variables

$\Phi_Z = \sum_{n\in\mathbb{Z}} \delta_{(T_n,Z_n)}$: Marked point process

$\Phi_Z(B\times E)$: # of points in $B \in \mathcal{B}(\mathbb{R})$ with marks falling in E

Stationarity of Marked Point Process

A marked PP Φ_Z is stationary

$$\Leftrightarrow \; {}^\forall B_1,\dots,B_n \in \mathcal{B}(\mathbb{R}),\; E_1,\dots,E_n,\; k_1,\dots,k_n \in \mathbb{N}_0,\; t\in\mathbb{R}$$

$$\mathbb{P}\big(\Phi_Z(B_1\times E_1)=k_1,\dots,\Phi_Z(B_n\times E_n)=k_n\big)$$
$$=\mathbb{P}\big(\Phi_Z((B_1+t)\times E_1)=k_1,\dots,\Phi_Z((B_n+t)\times E_n)=k_n\big)$$

Marks follow the point shifts as remaining unchanged

1 Stationary Point Processes & Palm Calculus
Stationary Point Processes
Introduction to Palm Calculus

2 Application to AoI Analysis of Single-Source Systems
General Identities
Examples
Example 1: Single Pushout Server
Example 2: FCFS Single Server Queue

3 Application to AoI Analysis of Multi-Source Systems
General Identities
An Example
Example 3: Multi-source Single Pushout Server

4 Summary

Palm Probability

$\Phi := \sum_{n\in\mathbb{Z}} \delta_{T_n}$: Simple & stationary PP on $\mathbb{R}$ with intensity $\lambda \in (0,\infty)$

$\{X(t)\}_{t\in\mathbb{R}}$: A stochastic process jointly stationary with Φ;

$\forall s_1,\dots,s_n,\ t\in\mathbb{R},\ E_1,\dots,E_n$

$$\mathbb{P}\big(X(s_1)\in E_1,\dots,X(s_n)\in E_n\big) = \mathbb{P}\big(X(s_1+t)\in E_1,\dots,X(s_n+t)\in E_n\big)$$

Expectation w.r.t. Palm Probability

$^\forall f$ s.t. expectations below exist & $B\in\mathcal{B}(\mathbb{R})$ s.t. $|B|\in(0,\infty)$,

$$\mathbb{E}^0_\Phi[f(X(0))] = \frac{1}{\lambda\,|B|}\,\mathbb{E}\Big[\sum_{n\in\mathbb{Z}} f(X(T_n))\,\mathbf{1}_B(T_n)\Big] \qquad (\spadesuit)$$

$\lambda\,|B| = \mathbb{E}[\Phi(B)]$

$\Phi(B)$: # of Φ-points in B

Sum of $f(X)$ sampled at Φ-points in B

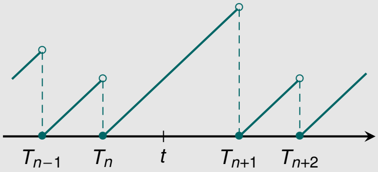
- Palm probability is defined for each given point process
- ($\spadesuit$) does not depend on $B\in\mathcal{B}(\mathbb{R})$ (as long as $|B|\in(0,\infty)$)
- $\mathbb{E}^0_\Phi[f(X(0))] \neq \mathbb{E}[f(X(0))]$ (Event-average $\neq$ Time-average)

Properties of Palm Probability (1/2)

$\mathbb{E}[\Phi(\{0\})] = \lambda|\{0\}|$

- $\mathbb{P}^0_\Phi(T_0 = 0) = 1$ (cf. $\mathbb{P}(T_0 = 0) = 0$)

∵) $T_-(t) := \sup\{T_n \le t\}$, $X(t) := t - T_-(t)$, $t \in \mathbb{R}$ (← stationary)



$$\mathbb{P}^0_\Phi(T_0 = 0) = \mathbb{E}^0_\Phi[\mathbf{1}_{\{X(0)=0\}}] = \frac{1}{\lambda}\mathbb{E}\Bigg[\sum_{n\in\mathbb{Z}} \mathbf{1}_{\{X(T_n)=0\}}\, \mathbf{1}_{[0,1]}(T_n)\Bigg] = \frac{\mathbb{E}[\Phi([0,1])]}{\lambda}$$

$T_0 = T_-(0)$ $B = [0,1]$ 1 $(\because\ T_-(T_n) = T_n)$

- $\{X(t)\}_{t\in\mathbb{R}}$ is stationary in $\mathbb{P}$ ➜ $\{X(T_n)\}_{n\in\mathbb{Z}}$ is stationary in $\mathbb{P}^0_\Phi$

  ⟺ ${}^\forall m \in \mathbb{N}_0$, $n \in \mathbb{Z}$,

$$\mathbb{P}^0_\Phi\big(X(0) \in E_0, \ldots, X(T_m) \in E_m\big) = \mathbb{P}^0_\Phi\big(X(T_n) \in E_0, \ldots, X(T_{m+n}) \in E_m\big)$$

$X(0)$: $0 = T_0$

Properties of Palm Probability (2/2)

$\Phi_Z = \sum_{n\in\mathbb{Z}} \delta_{(T_n,Z_n)}$: A simple & stationary marked PP with intensity $\lambda \in (0,\infty)$

${}^{\forall} f$ s.t. expectations below exist & $B \in \mathcal{B}(\mathbb{R})$ s.t. $|B| \in (0,\infty)$

$$\mathbb{E}_{\Phi}^{0}[f(Z_0)] = \frac{1}{\lambda\,|B|}\,\mathbb{E}\Biggl[\sum_{n\in\mathbb{Z}} f(Z_n)\,\mathbf{1}_B(T_n)\Biggr] \qquad (\clubsuit)$$

($\because$ Substitute $X(t) := Z_n,\ t \in [T_n, T_{n+1})$ into ($\spadesuit$))

- $\{Z_n\}_{n\in\mathbb{Z}}$ is stationary in $\mathbb{P}_{\Phi}^{0}$ $\Leftrightarrow$ $\forall m \in \mathbb{N}_0,\ n \in \mathbb{Z}$,
  $\mathbb{P}_{\Phi}^{0}\bigl(Z_0 \in E_0, \ldots, Z_m \in E_m\bigr) = \mathbb{P}_{\Phi}^{0}\bigl(Z_n \in E_0, \ldots, Z_{m+n} \in E_m\bigr)$,
- ${}^{\forall} n \in \mathbb{Z},\ B \in \mathcal{B}(\mathbb{R}),\ \mathbb{P}_{\Phi}^{0}(T_{n+1} - T_n \in B) = \mathbb{P}_{\Phi}^{0}(T_1 \in B)$
  ($\because$ Substitute $Z_n := T_{n+1} - T_n$)

Basic Formula 1: Campbell's Formula

$\Phi_Z = \sum_{n\in\mathbb{Z}} \delta_{(T_n,Z_n)}$: A simple & stationary marked PP with intensity $\lambda \in (0,\infty)$

$^\forall g$ s.t. expectations below exist

$$\mathbb{E}\left[\sum_{n\in\mathbb{Z}} g(T_n, Z_n)\right] = \lambda \int_{\mathbb{R}} \mathbb{E}^0_\Phi[g(t, Z_0)]\, \mathrm{d}t$$

A generalization of definition formula

Substitute $g(t,z) := f(z)\, \mathbf{1}_B(t)$

$$\to \begin{cases} (\text{LHS}) = \mathbb{E}\left[\sum_{n\in\mathbb{Z}} f(Z_n)\, \mathbf{1}_B(T_n)\right] \\ (\text{RHS}) = \lambda \int_{\mathbb{R}} \mathbb{E}^0_\Phi\left[f(Z_0)\, \mathbf{1}_B(t)\right] \mathrm{d}t = \lambda |B|\, \mathbb{E}^0_\Phi[f(Z_0)] \end{cases}$$

→ Reduced to (♣)

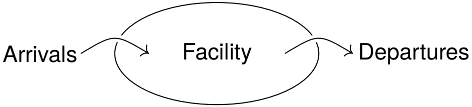
Example: Stationary Version of Little's Law

Customers arrive at a facility, stay for a while, and then leave

$\{T_n\}_{n\in\mathbb{Z}}$: Sequence of arrival times of customers

W_n, $n \in \mathbb{Z}$: Sojourn time of the customer arriving at T_n

$L(t)$, $t \in \mathbb{R}$: # of customers in the facility at time t



Little's Law (Stationary Version)

Marked PP $\Phi_W = \sum_{n\in\mathbb{Z}} \delta_{(T_n, W_n)}$ is simple & stationary with intensity $\lambda \in (0, \infty)$ ➜ $\mathbb{E}[L(0)] = \lambda\, \mathbb{E}^0_\Phi[W_0]$

$(\because)$

Campbell's Formula w/ $g(t,z) = \mathbf{1}_{[t,t+z)}(0)$

$$\mathbb{E}[L(0)] = \mathbb{E}\Bigg[\underbrace{\sum_{n\in\mathbb{Z}} \mathbf{1}_{[T_n, T_n+W_n)}(0)}_{\text{\# of customers at time 0}}\Bigg] = \lambda \int_{\mathbb{R}} \mathbb{E}^0_\Phi[\underbrace{\mathbf{1}_{[t,t+W_0)}(0)}_{=\,\mathbf{1}_{(-W_0,0]}(t)}]\,\mathrm{d}t = \lambda\, \mathbb{E}^0_\Phi[W_0]$$

# of customers at time 0

Basic Formula 2: Inversion Formula

Expresses time-stationary average in terms of Palm-average
(Most useful for AoI analysis)

$\Phi := \sum_{n \in \mathbb{Z}} \delta_{T_n}$: Simple & stationary PP on $\mathbb{R}$ with intensity $\lambda \in (0, \infty)$

$\{X(t)\}_{t \in \mathbb{R}}$: A stochastic process jointly stationary with Φ

${}^\forall f$ s.t. expectations below exist,

$$\mathbb{E}[f(X(0))] = \underbrace{\lambda}_{= \frac{1}{\mathbb{E}^0_\Phi[T_1]}} \mathbb{E}^0_\Phi\left[\int_0^{T_1} f(X(t))\,\mathrm{d}t\right]$$

Integral of $f(X)$ over $(0, T_1]$

Note:

- $\mathbb{E}^0_\Phi[T_1] = \dfrac{1}{\lambda}$ ($\because$ Substitute $f(\cdot) \equiv 1$)
- Corresponds to Renewal Reward Theorem
  (Regenerative structure is NOT needed)

Example: Waiting-time Paradox

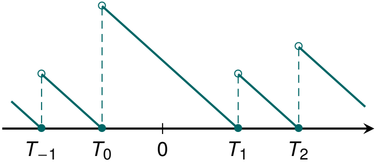
Waiting-time Paradox (Stationary Version)

$\{T_n\}_{n\in\mathbb{Z}}$: Sequence of bus arrival times at a certain bus stop

PP $\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$ is simple & stationary ➜ $\mathbb{E}[T_1] \geq \dfrac{\mathbb{E}^0_\Phi[T_1]}{2}$

Mean waiting time (unaware of timetable) is always longer than half of mean bus interarrival time (Equality holds iff $T_{n+1} - T_n = \text{const.}$)

$\because$) $T_+(t) := \inf\{T_n > t\}$, $X(t) := T_+(t) - t$, $t \in \mathbb{R}$ ($\leftarrow$ stationary)



$$\begin{cases} \mathrm{Var}^0_\Phi[X] = \mathbb{E}^0_\Phi[X^2] - (\mathbb{E}^0_\Phi[X])^2 \\ \lambda = 1/\mathbb{E}^0_\Phi[T_1] \end{cases}$$

Inversion Formula ($T_1 = T_+(0)$)

$$\mathbb{E}[T_1] = \lambda\,\mathbb{E}^0_\Phi\left[\int_{T_0=0}^{T_1} (\underbrace{T_+(t)}_{=T_1} - t)\,\mathrm{d}t\right] = \frac{\lambda\,\mathbb{E}^0_\Phi[{T_1}^2]}{2} = \frac{\mathbb{E}^0_\Phi[T_1]}{2} + \frac{\mathrm{Var}^o_\Phi[T_1]}{2\,\mathbb{E}[T_1]}$$

Basic Formula 3: Neveu's Exchange Formula

Connects two Palm distributions

$\Phi := \sum_{n\in\mathbb{Z}} \delta_{T_n}$
$\Xi := \sum_{n\in\mathbb{Z}} \delta_{R_n}$: $\begin{cases}\text{Both simple \& jointly stationary PPs on } \mathbb{R} \text{ with} \\ \text{respective intensities } \lambda_\Phi \in (0,\infty) \text{ \& } \lambda_\Xi \in (0,\infty)\end{cases}$

$\{X(t)\}_{t\in\mathbb{R}}$: A stochastic process jointly stationary with Φ & Ξ

$^\forall f$ s.t. expectations below exist,

$$\lambda_\Phi \, \mathbb{E}^0_\Phi[f(X(0))] = \lambda_\Xi \, \mathbb{E}^0_\Xi\left[\sum_{n\in\mathbb{Z}} f(X(T_n)) \, \mathbf{1}_{[0,R_1)}(T_n)\right] \qquad (\blacklozenge)$$

- $\mathbb{E}^0_\Xi\big[\Phi([0,R_1))\big] = \dfrac{\lambda_\Phi}{\lambda_\Xi}$ ($\because$ Substitute $f(\cdot) \equiv 1$)

  Mean # of Φ-points in a Ξ-cycle

Example: Cycle Formula for Queues

$\{T_n\}_{n\in\mathbb{Z}}$: Arrival times of customers at a queue; $\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$

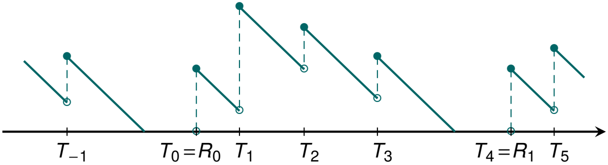
$\{R_n\}_{n\in\mathbb{Z}}$: Arrival times of customers finding the queue empty;

$\Xi = \sum_{n\in\mathbb{Z}} \delta_{R_n}$

$$\lambda_\Xi = \mathbb{E}[\Xi([0,1])] = \mathbb{E}\left[\sum_{n\in\mathbb{Z}} \mathbf{1}_{[0,1]}(T_n)\, \mathbf{1}_{\{L(T_n-)=0\}}\right] = \lambda_\Phi \mathbb{P}^0_\Phi\big(L(0-)=0\big)$$

# of customers

Definition formula



Sum of $f(X)$ sampled for customers in a busy cycle

$$(\blacklozenge) \leftrightarrow \mathbb{E}^0_\Phi[f(X(0))] = \frac{\mathbb{E}^0_\Xi\left[\sum_{n\in\mathbb{Z}} f(X(T_n))\, \mathbf{1}_{[0,R_1)}(T_n)\right]}{\mathbb{E}^0_\Xi\big[\Phi([0,R_1))\big]}$$

# of customers in a busy cycle

1 Stationary Point Processes & Palm Calculus
Stationary Point Processes
Introduction to Palm Calculus

2 Application to AoI Analysis of Single-Source Systems
General Identities
Examples
Example 1: Single Pushout Server
Example 2: FCFS Single Server Queue

3 Application to AoI Analysis of Multi-Source Systems
General Identities
An Example
Example 3: Multi-source Single Pushout Server

4 Summary

General Network Model (Single-Source)

Input to Network:

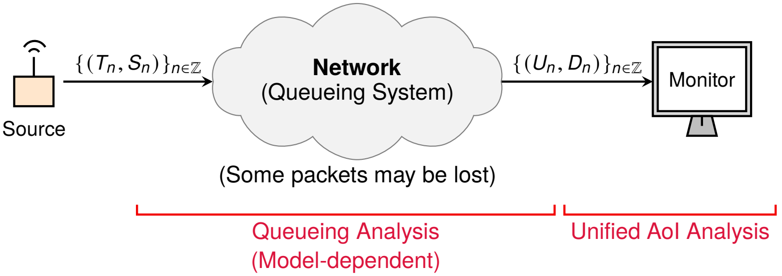
$\{T_n\}_{n\in\mathbb{Z}}$: Times at which packets are generated & timestamped

$S_n, n \in \mathbb{Z}$: Required service time of the packet generated at T_n

Output from Network:

$\{U_n\}_{n\in\mathbb{Z}}$: Service completion times of informative packets (= Times of information updates)

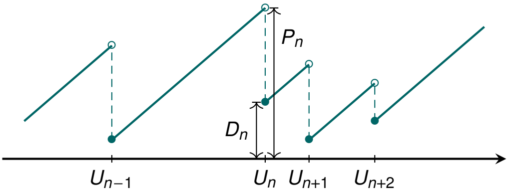
$D_n, n \in \mathbb{Z}$: Delay of the informative packet whose service is completed at U_n (➜ generated & timestamped at $U_n - D_n$)



AoI Processes

$\{A(t)\}_{t\in\mathbb{R}}$: AoI process;

$$A(t) := D_n + t - U_n, \quad t \in [U_n, U_{n+1}),\ n \in \mathbb{Z}$$



$\{P_n\}_{n\in\mathbb{Z}}$: Sequence of peak AoI (PAoI) (AoI just before updates);

$$P_{n+1} := D_n + (U_{n+1} - U_n), \quad n \in \mathbb{Z}$$

Lemma [Jiang & M '22]

Marked PP $\Psi_D = \sum_{n\in\mathbb{Z}} \delta_{(U_n,D_n)}$ is simple & stationary with

$$\begin{cases} \lambda_\Psi \in (0,\infty) \\ \mathbb{P}^0_\Psi(D_0 < \infty) = 1 \end{cases} \rightarrow \begin{cases} \{A(t)\}_{t\in\mathbb{R}} \text{ is jointly stationary with } \Psi_D \\ \mathbb{P}\big(A(0) < \infty\big) = 1 \end{cases}$$

Stationary AoI Distribution

Proposition [Jiang & M '22]

Marked PP $\Psi_D = \sum_{n\in\mathbb{Z}} \delta_{(U_n,D_n)}$ is simple & stationary with

$\begin{cases} \lambda_\Psi \in (0,\infty) \\ \mathbb{P}^0_\Psi(D_0 < \infty) = 1 \end{cases}$ ➜

$$\mathbb{P}\big(A(0) \le x\big) = \lambda_\Psi \int_0^x \big(\mathbb{P}^0_\Psi(P_0 > y) - \mathbb{P}^0_\Psi(D_0 > y)\big)\,\mathrm{d}y, \quad x \ge 0$$

Note:

- Stationary version of the Theorem in [Inoue et al. '19]
- The stationary AoI distribution is characterized by the distributions of delays & PAoIs
- The stationary AoI distribution has density

$$\lambda_\Psi\big(\mathbb{P}^0_\Psi(P_0 > x) - \mathbb{P}^0_\Psi(D_0 > x)\big), \quad x \ge 0$$

Y. Inoue, H. Masuyama, T. Takine & T. Tanaka. A general formula for the stationary distribution of the age of information and its application to single-server queues. *IEEE Trans. Inform. Theory*, **65**, 8305–8324, 2019

Proof via Inversion Formula

$$\mathbb{E}[f(A(0))] = \lambda_\Psi \, \mathbb{E}^0_\Psi\left[\int_{U_0=0}^{U_1} f(A(t))\, dt\right]$$

$f(\cdot) = \mathbf{1}_{[0,x]}(\cdot)$

$$\mathbb{P}\big(A(0) \le x\big) = \lambda_\Psi \, \mathbb{E}^0_\Psi\left[\int_{U_0=0}^{U_1} \mathbf{1}_{[0,x]}\big(A(t)\big)\, dt\right]$$

$A(t) = D_0 + t, \ \ t \in [0, U_1)$, $P_1 = D_0 + U_1$

$$= \lambda_\Psi \, \mathbb{E}^0_\Psi\left[\int_{D_0}^{P_1} \mathbf{1}_{[0,x]}(u)\, du\right]$$

$\int_{D_0}^{P_1} = \int_0^{P_1} - \int_0^{D_0}$, $P_1 =_d P_0$ in $\mathbb{P}^0_\Psi$

$$= \lambda_\Psi \left(\mathbb{E}^0_\Psi[P_0 \wedge x] - \mathbb{E}^0_\Psi[D_0 \wedge x]\right)$$

$\mathbb{E}[X \wedge x] = \int_0^x \mathbb{P}(X > y)\, dy$

$$= \lambda_\Psi \int_0^x \left(\mathbb{P}^0_\Psi(P_0 > y) - \mathbb{P}^0_\Psi(D_0 > y)\right) dy$$

where $a \wedge b = \min\{a, b\}$

Moments & Laplace Transform of Stationary AoI

Corollary

1. Moment (if exist):

$$\mathbb{E}[A(0)^k] = \frac{\lambda_\Psi}{k+1}\left(\mathbb{E}_\Psi^0[{P_0}^{k+1}] + \mathbb{E}_\Psi^0[{D_0}^{k+1}]\right), \quad k = 1, 2, \dots$$

2. Laplace Transform:

$$\mathcal{L}_A(s) := \mathbb{E}[e^{-sA(0)}] = \frac{\lambda_\Psi}{s}\left(\mathcal{L}_D(s) - \mathcal{L}_P(s)\right), \quad s > 0$$

where $\mathcal{L}_D(s) := \mathbb{E}_\Psi^0[e^{-sD_0}]$, $\mathcal{L}_P(s) := \mathbb{E}_\Psi^0[e^{-sP_0}]$

$\because$) 1. $\mathbb{E}[A(0)^k] = \lambda_\Psi \int_0^\infty x^k \left(\mathbb{P}_\Psi^0(P_0 > x) - \mathbb{P}_\Psi^0(D_0 > x)\right) \mathrm{d}x$

$$\int_0^\infty x^k\, \mathbb{P}(X > x)\, \mathrm{d}x = \frac{\mathbb{E}[X^{k+1}]}{k+1}$$

2. $\mathcal{L}_A(s) = \lambda_\Psi \int_0^\infty e^{-sx} \left(\mathbb{P}_\Psi^0(P_0 > x) - \mathbb{P}_\Psi^0(D_0 > x)\right) \mathrm{d}x$

$$\int_0^\infty e^{-sx}\, \mathbb{P}(X > x)\, \mathrm{d}x = \frac{1 - \mathbb{E}[e^{-sX}]}{s}$$

1 Stationary Point Processes & Palm Calculus
  - Stationary Point Processes
  - Introduction to Palm Calculus

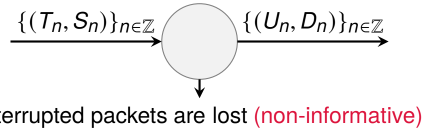
2 Application to AoI Analysis of Single-Source Systems
  - General Identities
  - Examples
    - Example 1: Single Pushout Server
    - Example 2: FCFS Single Server Queue

3 Application to AoI Analysis of Multi-Source Systems
  - General Identities
  - An Example
    - Example 3: Multi-source Single Pushout Server

4 Summary

Example 1: Single Pushout Server (1/6: Model)

- A generated packet is immediately served without waiting
- At the generation time of a packet, another packet in service (if any) is interrupted & lost



Input to the Server

$\{T_n\}_{n\in\mathbb{Z}}$: Times at which packets are generated & timestamped,
$\tau_n := T_{n+1} - T_n,\ n \in \mathbb{Z}$ (interarrival times)

$S_n,\ n \in \mathbb{Z}$: Required service time of the packet generated at T_n

Output from the Server

$\{U_n\}_{n\in\mathbb{Z}}$: Times of information updates (service completions)

$D_n,\ n \in \mathbb{Z}$: Delay of the packet whose service is completed at U_n

Example 1: Single Pushout Server (2/6: Stationarity)

Input Marked PP: $\Phi_S = \sum_{n\in\mathbb{Z}} \delta_{(T_n, S_n)}$

Output Marked PP: $\Psi_D = \sum_{n\in\mathbb{Z}} \delta_{(U_n, D_n)}$

$$\Psi_D(B \times E) = \sum_{n\in\mathbb{Z}} \mathbf{1}_B(T_n + S_n)\, \mathbf{1}_{E\cap[0,\tau_n]}(S_n)$$

Service is NOT interrupted

Lemma [Jiang & M '22]

Input marked PP Φ_S is simple & stationary with $\begin{cases} \lambda_\Phi \in (0,\infty) \\ \mathbb{P}^0_\Phi(S_0 < \infty) = 1 \\ \mathbb{P}^0_\Phi(S_0 \leq \tau_0) > 0 \end{cases}$

➜ Output marked PP Ψ_D is simple & stationary with $\begin{cases} \lambda_\Psi = \lambda_\Phi\, \mathbb{P}^0_\Phi(S_0 \leq \tau_0) \\ \mathbb{P}^0_\Psi(D_0 < \infty) = 1 \end{cases}$

Example 1: Single Pushout Server (3/6)

Lemma

1. $\lambda_\Psi \mathcal{L}_D(s) = \lambda_\Phi \mathbb{E}^0_\Phi\left[e^{-sS_0}\,\mathbf{1}_{\{S_0\le\tau_0\}}\right]$
2. $\lambda_\Psi \mathcal{L}_P(s) = \lambda_\Phi \mathbb{E}^0_\Phi\left[e^{-sP_1}\,\mathbf{1}_{\{S_0\le\tau_0\}}\right]$

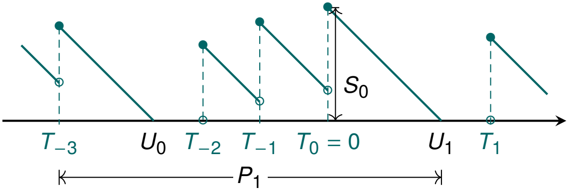
$\because$) 1

Exchange formula

$$\lambda_\Psi \mathbb{E}^0_\Psi[e^{-sD_0}] = \lambda_\Phi \mathbb{E}^0_\Phi\left[\sum_{n\in\mathbb{Z}} e^{-sD_n}\,\mathbf{1}_{[0,T_1)}(U_n)\right]$$

$1 \Leftrightarrow$ The packet arriving at $T_0 = 0$ is NOT interrupted

In this case, $D_1 = U_1 = S_0$



2 Similar

Example 1: Single Pushout Server (4/6: *GI*/*GI* Inputs)

$\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$: Stationary renewal process ($\tau_n = T_{n+1} - T_n$, $n \in \mathbb{Z}$: i.i.d.)

S_n, $n \in \mathbb{Z}$: Nonnegative & i.i.d. random variables

Note: $\begin{cases} (\tau_n, S_n),\ n \in \mathbb{Z}, \text{ are i.i.d.} \\ \tau_n \ \& \ S_n \text{ are possibly dependent} \end{cases}$

Proposition [Inoue et al. '19]

Laplace transform of stationary AoI for *GI*/*GI* pushout server:

$$\mathcal{L}_A(s) = \frac{\lambda_\Phi \, \mathbb{E}^0_\Phi\big[e^{-sS_0}\,\mathbf{1}_{\{S_0\le\tau_0\}}\big]\big(1-\mathcal{L}_\tau(s)\big)}{s\,\big(1-\mathbb{E}^0_\Phi\big[e^{-s\tau_0}\,\mathbf{1}_{\{S_0>\tau_0\}}\big]\big)}, \quad s>0$$

where $\mathcal{L}_\tau(s) := \mathbb{E}^0_\Phi[e^{-s\tau_0}]$

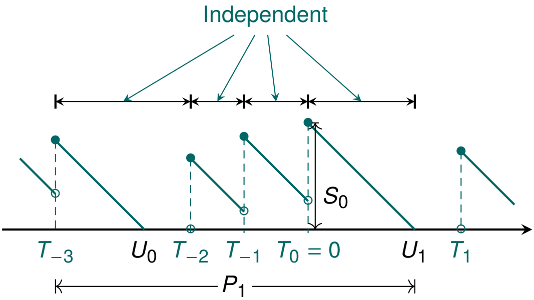
Expressed in terms of the joint distribution of τ_0 & S_0

Example 1: Single Pushout Server (5/6: GI/GI Input)

(∵) Consider $\mathbb{E}^0_\Phi\big[e^{-sP_1}\,\mathbf{1}_{\{S_0\le\tau_0\}}\big]$

$$\mathbb{E}^0_\Phi\big[e^{-sP_1}\,\mathbf{1}_{\{S_0\le\tau_0\}}\big]$$

$$= \mathbb{E}^0_\Phi\big[e^{-s\tau_0}\,\mathbf{1}_{\{S_0\le\tau_0\}}\big]\underbrace{\sum_{n=0}^{\infty}\Big(\mathbb{E}^0_\Phi\big[e^{-s\tau_0}\,\mathbf{1}_{\{S_0>\tau_0\}}\big]\Big)^n}_{=\left(1-\mathbb{E}^0_\Phi[e^{-s\tau_0}\mathbf{1}_{\{S_0>\tau_0\}}]\right)^{-1}}\mathbb{E}^0_\Phi\big[e^{-sS_0}\,\mathbf{1}_{\{S_0\le\tau_0\}}\big]$$



Example 1: Single Pushout Server (6/6: M/GI Inputs)

$\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$: Homogeneous Poisson point process

S_n, $n \in \mathbb{Z}$: Nonnegative & i.i.d. random variables, independent of Φ

Proposition [Inoue et al. '19, Kesidis et al. '20]

Laplace transform of stationary AoI for M/GI pushout server:

$$\mathcal{L}_A(s) = \frac{\lambda_\Phi \, \mathcal{L}_S(s+\lambda_\Phi)}{s + \lambda_\Phi \, \mathcal{L}_S(s+\lambda_\Phi)}, \quad s > 0$$

where $\mathcal{L}_S(s) := \mathbb{E}_\Phi^0[e^{-sS_0}]$

($\because$) Substitute

- $\mathcal{L}_\tau(s) = \dfrac{\lambda_\Phi}{s+\lambda_\Phi}$
- $\mathbb{E}_\Phi^0[e^{-sS_0} \, \mathbf{1}_{\{S_0 \le \tau_0\}}] = \lambda_\Phi \, \mathcal{L}_S(s+\lambda_\Phi)$
- $\mathbb{E}_\Phi^0[e^{-s\tau_0} \, \mathbf{1}_{\{S_0 > \tau_0\}}] = \dfrac{\lambda_\Phi(1-\mathcal{L}_S(s+\lambda_\Phi))}{s+\lambda_\Phi}$

1 Stationary Point Processes & Palm Calculus
Stationary Point Processes
Introduction to Palm Calculus

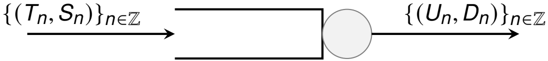
2 Application to AoI Analysis of Single-Source Systems
General Identities
Examples
Example 1: Single Pushout Server
Example 2: FCFS Single Server Queue

3 Application to AoI Analysis of Multi-Source Systems
General Identities
An Example
Example 3: Multi-source Single Pushout Server

4 Summary

Example 2: FCFS Single Server Queue (1/4: Model)

All generated packets are served (informative) in FCFS manner



Input Marked PP: $\Phi_S = \sum_{n\in\mathbb{Z}} \delta_{(T_n,S_n)}$

Output Marked PP: $\Psi_D = \sum_{n\in\mathbb{Z}} \delta_{(U_n,D_n)}$

Proposition (cf. [Loynes '62])

Input marked PP Φ_S is simple, stationary & ergodic with $\rho := \lambda_\Phi \mathbb{E}^0_\Phi[S_0] < 1$

➜ Output marked PP Ψ_D is simple, stationary & ergodic with
$$\begin{cases} \lambda_\Psi = \lambda_\Phi \\ \mathbb{P}^0_\Psi(D_0 < \infty) = 1 \end{cases}$$

R. M. Loynes. The stability of queues with non-independent inter-arrival and service times. *Proc. Cambridge Philos. Soc.*, **58**, 497–520, 1962

Example 2: FCFS Single Server Queue (2/4)

W_n, $n \in \mathbb{Z}$: Sojourn time of the packet arriving at T_n

Lemma

1. $\mathcal{L}_D(s) = \mathcal{L}_W(s) := \mathbb{E}^0_\Phi[e^{-sW_0}]$
2. $\mathcal{L}_P(s) = \mathbb{E}^0_\Phi[e^{-s(W_0 \vee \tau_0 + S_1)}]$ $\quad (a \vee b = \max\{a, b\})$

$(\because)$ 1

$\Xi = \sum_{n\in\mathbb{Z}} \delta_{R_n}$: PP of arrival times of packets finding the queue empty w/ intensity $\lambda_\Xi = \lambda_\Phi \mathbb{P}^0_\Phi(L(0-) = 0)$

$L(0-)$ = # of customers

Exchange Formula

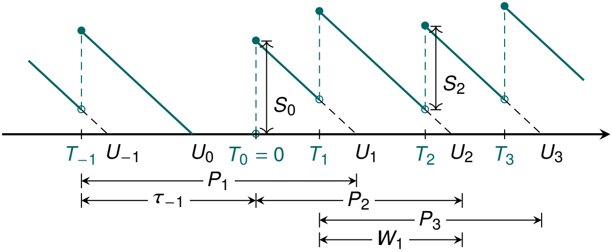
$$\lambda_\Psi \mathbb{E}^0_\Psi[e^{-sD_0}] = \lambda_\Xi \mathbb{E}^0_\Xi\left[\sum_{n\in\mathbb{Z}} e^{-sD_n} \mathbf{1}_{[0,R_1)}(U_n)\right] = \lambda_\Phi \mathbb{E}^0_\Phi[e^{-sW_0}]$$

$$\sum_{n\in\mathbb{Z}} e^{-sD_n} \mathbf{1}_{[0,R_1)}(U_n) = \sum_{n\in\mathbb{Z}} e^{-sW_n} \mathbf{1}_{[0,R_1)}(T_n)$$

$\lambda_\Psi = \lambda_\Phi$

Example 2: FCFS Single Server Queue (3/4)

❷ $P_1 =_d \max\{W_0, \tau_0\} + S_1$ in $\mathbb{P}^0_\Phi$



In GI/GI-input case

$\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$: Stationary renewal process (τ_n, $n \in \mathbb{Z}$: i.i.d.)

S_n, $n \in \mathbb{Z}$: Nonnegative & i.i.d. random variables

➜ $\mathcal{L}_P(s) = \mathcal{L}_S(s) \left(\mathbb{E}^0_\Phi[e^{-sW_0}\,\mathbf{1}_{\{W_0>\tau_0\}}] + \mathbb{E}^0_\Phi[e^{-s\tau_0}\,\mathbf{1}_{\{W_0\le\tau_0\}}]\right)$

$\mathcal{L}_S(s) = \mathbb{E}^0_\Phi[e^{-sS_0}]$

Example 2: FCFS Single Server Queue (4/4: M/GI Input)

$\Phi = \sum_{n\in\mathbb{Z}} \delta_{T_n}$: Homogeneous Poisson point process

S_n, $n \in \mathbb{Z}$: Nonnegative & i.i.d. random variables, independent of Φ

Proposition [Inoue et al. '19]

Laplace transform of stationary AoI for FCFS $M/GI/1$ queue:

$$\mathcal{L}_A(s) = \lambda(1-\rho)\mathcal{L}_S(s)\left\{\frac{1-\mathcal{L}_S(s)}{s-\lambda_\Phi(1-\mathcal{L}_S(s))} + \frac{\mathcal{L}_S(s+\lambda_\Phi)}{s+\lambda_\Phi\mathcal{L}_S(s+\lambda_\Phi)}\right\}, s > 0$$

($\because$) Substitute

- $\mathcal{L}_W(s) = \dfrac{(1-\rho)s\,\mathcal{L}_S(s)}{s-\lambda_\Phi(1-\mathcal{L}_S(s))}$ (Pollaczek-Khinchine Formula)
- $\mathbb{E}^0_\Phi[e^{-sW_0}\,\mathbf{1}_{\{W_0>\tau_0\}}] = \mathcal{L}_W(s) - \mathcal{L}_W(s+\lambda_\Phi)$
- $\mathbb{E}^0_\Phi[e^{-s\tau_0}\,\mathbf{1}_{\{W_0\le\tau_0\}}] = \dfrac{\lambda_\Phi\,\mathcal{L}_W(s+\lambda_\Phi)}{s+\lambda_\Phi}$

1 Stationary Point Processes & Palm Calculus
Stationary Point Processes
Introduction to Palm Calculus

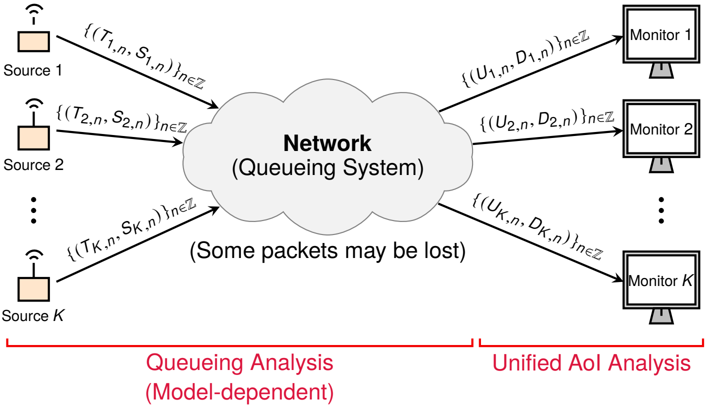
2 Application to AoI Analysis of Single-Source Systems
General Identities
Examples
Example 1: Single Pushout Server
Example 2: FCFS Single Server Queue

3 Application to AoI Analysis of Multi-Source Systems
General Identities
An Example
Example 3: Multi-source Single Pushout Server

4 Summary

General Multi-source Network Model

- K sources share a network facility, $\mathcal{K} = \{1, 2, \ldots, K\}$
- Each source has its dedicated monitor



Input & Output Processes

Input to Network:

$\{T_n\}_{n\in\mathbb{Z}}$: Times at which packets are generated & timestamped

$c_n, n \in \mathbb{Z}$: Source of the packet generated at T_n

$S_n, n \in \mathbb{Z}$: Required service time of the packet generated at T_n

$$\Phi := \sum_{n\in\mathbb{Z}} \delta_{T_n}, \qquad \Phi_{c,S} := \sum_{n\in\mathbb{Z}} \delta_{(T_n,c_n,S_n)}$$

Output from Network:

$\{U_n\}_{n\in\mathbb{Z}}$: Service completion times of informative packets (= Times of information updates); $\cdots < U_{-1} < U_0 \le 0 < U_1 < \cdots$

$C_n, n \in \mathbb{Z}$: Source of the packet whose service is completed at U_n

$D_n, n \in \mathbb{Z}$: Delay of the packet whose service is completed at U_n

$$\Psi := \sum_{n\in\mathbb{Z}} \delta_{U_n}, \qquad \Psi_{C,D} := \sum_{n\in\mathbb{Z}} \delta_{(U_n,C_n,D_n)}$$

Status information updated at time U_n is generated & timestamped at $U_n - D_n$ by source C_n

Output Sub-processes

Sub-processes: For $k \in \mathcal{K} = \{1, 2, \ldots, K\}$

$\{U_{k,n}\}_{n\in\mathbb{Z}}$: Times of information updates (service completions) of source k; $\cdots < U_{k,0} \le 0 < U_{k,1} < \cdots$ (Renumbered)

$D_{k,n}$, $n \in \mathbb{Z}$: Delay of source k packet whose service is completed at $U_{k,n}$

$$\Psi_k := \sum_{n\in\mathbb{Z}} \delta_{U_{k,n}} = \sum_{n\in\mathbb{Z}} \delta_{(U_n,C_n)}(\cdot \times \{k\})$$

$$\Psi_{k,D} := \sum_{n\in\mathbb{Z}} \delta_{(U_{k,n},D_{k,n})} = \sum_{n\in\mathbb{Z}} \delta_{(U_n,C_n,D_n)}(\cdot \times \{k\} \times \cdot)$$

Note:

- Ψ_k, $k \in \mathcal{K}$, have no common points & $\Psi = \sum_{k\in\mathcal{K}} \Psi_k$
- Marked PP $\Psi_C = \sum_{n\in\mathbb{Z}} \delta_{(U_n,C_n)}$ is simple & stationary with

$$\begin{cases} \lambda_\Psi \in (0,\infty) \\ \mathbb{P}^0_\Psi(C_0 = k) > 0 \end{cases} \rightarrow \begin{cases} \lambda_{\Psi_k} = \lambda_\Psi\, \mathbb{P}^0_\Psi(C_0 = k) \in (0,\infty) \\ \text{Palm probab. } \mathbb{P}^0_{\Psi_k} \text{ is well-defined} \end{cases}$$

AoI Processes of Respective Sources

$\{A_k(t)\}_{t\in\mathbb{R}}, k \in \mathcal{K}$: AoI process for source $k \in \mathcal{K}$

$$A_k(t) = D_{k,n} + t - U_{k,n}, \quad t \in [U_{k,n}, U_{k,n+1}),\ n \in \mathbb{Z}$$

$\{P_{k,n}\}_{n\in\mathbb{Z}}, k \in \mathcal{K}$: Sequence of peak AoI for source $k \in \mathcal{K}$

$$P_{k,n+1} = D_{k,n} + (U_{k,n+1} - U_{k,n}), \quad n \in \mathbb{Z}$$

Lemma [Jiang & M '22]

Marked PP $\Psi_{C,D}$ is simple & stationary with $\begin{cases} \lambda_\Psi \in (0,\infty) \\ \mathbb{P}^0_\Psi(C_0 = k) > 0 \\ \mathbb{P}^0_{\Psi_k}(D_0 < \infty) = 1 \end{cases}$

$^\forall k \in \mathcal{K}$ ➜

- $\left.\begin{array}{l}\text{Sub-marked PP } \Psi_{k,D}, k \in \mathcal{K} \\ \text{AoI processes } \{A_k(t)\}_{t\in\mathbb{R}}, k \in \mathcal{K}\end{array}\right\}$ are jointly stationary with $\Psi_{C,D}$
- $\mathbb{P}\big(A_k(0) < \infty\big) = 1,\ k \in \mathcal{K}$

Marginal AoI Distributions

Corollary

Marked PP $\Psi_{C,D}$ is simple & stationary with $\begin{cases} \lambda_\Psi \in (0, \infty) \\ \mathbb{P}^0_\Psi(C_0 = k) > 0 \\ \mathbb{P}^0_{\Psi_k}(D_0 < \infty) = 1 \end{cases}$

➜

Marginal AoI Distribution:

$$\mathbb{P}\big(A_k(0) \leq x\big) = \lambda_{\Psi_k} \int_0^x \big(\mathbb{P}^0_{\Psi_k}(P_{k,0} > y) - \mathbb{P}^0_{\Psi_k}(D_{k,0} > y)\big)\, \mathrm{d}y, \quad x \geq 0$$

Laplace Transform of Marginal AoI:

$$\mathcal{L}_{A_k}(s) := \mathbb{E}[e^{-sA_k(0)}] = \frac{\lambda_{\Psi_k}}{s} \big(\mathcal{L}_{D_k}(s) - \mathcal{L}_{P_k}(s)\big), \quad s > 0$$

where $\mathcal{L}_{D_k}(s) := \mathbb{E}^0_{\Psi_k}[e^{-sD_{k,0}}]$, $\mathcal{L}_{P_k}(s) := \mathbb{E}^0_{\Psi_k}[e^{-sP_{k,0}}]$

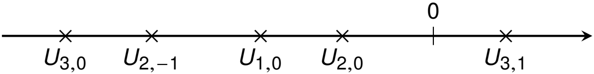
The results from single-source systems can be applied to the marginal distribution of multi-source systems

Joint Performance (Preliminary: Last Update up to 0)

$U_{k,0}$, $k \in \mathcal{K}$: Time at which the status information of source k was last updated up to time 0

$(\eta_1, \ldots, \eta_K)$: Random permutation of $(1, \ldots, K)$ s.t.

$$U_0 = U_{\eta_1,0} > U_{\eta_2,0} > \cdots > U_{\eta_K,0}$$



$(\eta_1, \eta_2, \eta_3) = (2, 1, 3)$ in this sample

At time 0, monitor η_j displays the jth newest information
(Monitor η_1 [resp. η_K] displays the newest [rest. oldest] information)

$$U_{\eta_k,0} = U_{\eta_1,0} - \sum_{j=1}^{k-1} \left(U_{\eta_j,0} - U_{\eta_{j+1},0}\right), \quad k = 1, 2, \ldots, K$$

Joint Laplace Transform

Theorem [Jiang & M '22]

Marked PP $\Psi_{C,D}$ is simple & stationary with $\begin{cases} \lambda_\Psi \in (0,\infty) \\ \mathbb{P}^0_\Psi(C_0 = k) > 0 \\ \mathbb{P}^0_{\Psi_k}(D_0 < \infty) = 1 \end{cases}$

${}^\forall k \in \mathcal{K}$

➜

$$\mathcal{L}_{\boldsymbol{A}}(\boldsymbol{s}) := \mathbb{E}\left[\exp\left(-\sum_{k=1}^{K} s_k\, A_k(0)\right)\right]$$

$$= \frac{\lambda_\Psi}{\overline{s}}\, \mathbb{E}^0_\Psi\left[(1 - e^{-\overline{s}U_1}) \prod_{k=1}^{K-1} \exp\left\{-s_{\eta_k} D_{\eta_k,0} - \overline{s}_{\eta[k+1]}(U_{\eta_k,0} - U_{\eta_{k+1},0})\right\}\right.$$

$$\left. \times \exp(-s_{\eta_K} D_{\eta_K,0})\right], \quad {}^\forall \boldsymbol{s} = (s_1, \ldots, s_K) \in [0,\infty)^K \text{ s.t. } \overline{s} > 0$$

where $\overline{s} = \sum_{k=1}^{K} s_k,\ \overline{s}_{\eta[k]} = \sum_{j=k}^{K} s_{\eta_j}$

Outline of Proof

Inversion Formula

$$\mathcal{L}_{\boldsymbol{A}}(\boldsymbol{s}) = \lambda_\Psi \mathbb{E}^0_\Psi\left[\int_{U_0=0}^{U_1} \exp\left(-\sum_{k=1}^{K} s_k \underbrace{A_k(t)}_{=\,D_{k,0}+t-U_{k,0}}\right) \mathrm{d}t\right]$$

$$= \lambda_\Psi \mathbb{E}^0_\Psi\left[\prod_{k=1}^{K} e^{-s_k(D_{k,0}-U_{k,0})} \underbrace{\int_0^{U_1} e^{-\bar{s}t}\,\mathrm{d}t}_{=\,(1-e^{-\bar{s}U_1})/\bar{s}}\right]$$

$$\prod_{k=1}^{K} e^{-s_k(D_{k,0}-U_{k,0})} = \prod_{k=1}^{K} \exp\bigl(-s_{\eta_k}(D_{\eta_k,0} - U_{\eta_k,0})\bigr)$$

$$= \prod_{k=1}^{K} \exp(-s_{\eta_k} D_{\eta_k,0}) \underbrace{\prod_{k=1}^{K}\prod_{j=1}^{k-1} \exp\bigl(-s_{\eta_k}(U_{\eta_j,0} - U_{\eta_{j+1},0})\bigr)}_{=\,\prod_{j=1}^{K-1}\exp\left(-\sum_{k=j+1}^{K} s_{\eta_k}(U_{\eta_j,0}-U_{\eta_{j+1},0})\right)}$$

$$\begin{cases} U_{\eta_k,0} = -\sum_{j=1}^{k-1}(U_{\eta_j,0} - U_{\eta_{j+1},0}) \\ (\because)\ \ U_{\eta_1,0} = U_0 = 0 \text{ in } \mathbb{P}^0_\Psi \end{cases}$$

$$\prod_{j=1}^{K-1} \exp\left(-\sum_{k=j+1}^{K} s_{\eta_k}(U_{\eta_j,0} - U_{\eta_{j+1},0})\right), \qquad \sum_{k=j+1}^{K} s_{\eta_k} = \bar{s}_{\eta[j+1]}$$

1 Stationary Point Processes & Palm Calculus
Stationary Point Processes
Introduction to Palm Calculus

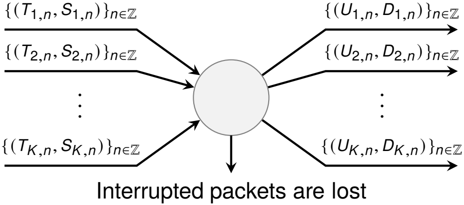
2 Application to AoI Analysis of Single-Source Systems
General Identities
Examples
Example 1: Single Pushout Server
Example 2: FCFS Single Server Queue

3 Application to AoI Analysis of Multi-Source Systems
General Identities
An Example
Example 3: Multi-source Single Pushout Server

4 Summary

Example 3: Single Pushout Server (1/4: Model)

- A generated packet is immediately served without waiting
- At the generation time of a packet, another packet in service (if any) is interrupted & lost



Input to Server:

$\{T_n\}_{n\in\mathbb{Z}}$: Times at which packets are generated & timestamped
$\tau_n = T_{n+1} - T_n,\ n \in \mathbb{Z}$ (interarrival times)

$c_n,\ n \in \mathbb{Z}$: Source of the packet generated at T_n

$S_n,\ n \in \mathbb{Z}$: Required service time of the packet generated at T_n

$$\Phi := \sum_{n\in\mathbb{Z}} \delta_{T_n}, \qquad \Phi_{c,S} := \sum_{n\in\mathbb{Z}} \delta_{(T_n, c_n, S_n)}$$

Example 3: Single Pushout Server (2/4: Sub-processes)

Sub-processes: For $k \in \mathcal{K} = \{1, 2, \dots, K\}$

$\{T_{k,n}\}_{n\in\mathbb{Z}}$: Times at which source k packets are generated & timestamped; $\cdots < T_{k,0} \le 0 < T_{k,1} < \cdots$ (Renumbered)

$S_{k,n}$, $n \in \mathbb{Z}$: Required service time of the source k packet generated at $T_{k,n}$

$$\Phi_k := \sum_{n\in\mathbb{Z}} \delta_{T_{k,n}} = \sum_{n\in\mathbb{Z}} \delta_{(T_n,c_n)}(\cdot \times \{k\})$$

$$\Phi_{k,S} := \sum_{n\in\mathbb{Z}} \delta_{(T_{k,n},S_{k,n})} = \sum_{n\in\mathbb{Z}} \delta_{(T_n,c_n,S_n)}(\cdot \times \{k\} \times \cdot)$$

Note:

- Φ_k, $k \in \mathcal{K}$, have no common points & $\Phi = \sum_{k\in\mathcal{K}} \Phi_k$
- Marked PP $\Phi_c = \sum_{n\in\mathbb{Z}} \delta_{(T_n,c_n)}$ is simple & stationary with

$$\begin{cases} \lambda_\Phi \in (0,\infty) \\ \mathbb{P}^0_\Phi(c_0 = k) > 0 \end{cases} \rightarrow \begin{cases} \lambda_{\Phi_k} = \lambda_\Phi\, \mathbb{P}^0_\Phi(c_0 = k) \in (0,\infty) \\ \text{Palm probab. } \mathbb{P}^0_{\Phi_k} \text{ is well-defined} \end{cases}$$

Example 3: Single Pushout Server (3/4: Stationarity)

Input Marked PP: $\Phi_{c,S} = \sum_{n \in \mathbb{Z}} \delta_{(T_n, c_n, S_n)}$

Output Marked PP: $\Psi_{C,D} = \sum_{n \in \mathbb{Z}} \delta_{(U_n, C_n, D_n)}$

$$\Psi_{C,D}(B \times \{k\} \times E) = \sum_{n \in \mathbb{Z}} \mathbf{1}_B(T_n + S_n)\, \mathbf{1}_{\{k\}}(c_n)\, \mathbf{1}_{E \cap [0, \tau_n]}(S_n)$$

Service is NOT interrupted

Lemma [Jiang & M '22]

Marked PP $\Phi_{c,S}$ is simple & stationary with $\begin{cases} \lambda_\Phi \in (0, \infty) \\ \mathbb{P}^0_\Phi(c_0 = k) > 0 \\ \mathbb{P}^0_{\Phi_k}(S_0 < \infty) = 1 \\ \mathbb{P}^0_{\Phi_k}(S_0 \le \tau_0) > 0 \end{cases}$

$^\forall k \in \mathcal{K}$

➜ Marked PP $\Psi_{C,D}$ is simple & stationary with $\begin{cases} \lambda_\Psi \in (0, \infty) \\ \mathbb{P}^0_\Psi(C_0 = k) > 0 \\ \mathbb{P}^0_{\Psi_k}(D_0 < \infty) = 1 \end{cases}$

Example 3: Single Pushout Server (4/4: M/GI-Input)

For simplicity, $\lambda = \lambda_\Phi, \quad \lambda_k = \lambda_{\Phi_k}, \ k \in \mathcal{K}$

Theorem [Jiang & M '22]

Joint Laplace transform of stationary AoIs for multi-source M/GI pushput server:

$$\mathcal{L}_{\boldsymbol{A}}(\boldsymbol{s}) = \lambda_1 \cdots \lambda_K \sum_{(j_1,\ldots,j_K)\in\sigma[\mathcal{K}]} \prod_{k=1}^{K} \frac{\mathcal{L}_{S,j_k}(\overline{s}_{j[k]} + \lambda)}{\overline{s}_{j[k]} + \overline{\lambda}_{j[k]} \mathcal{L}_{S,j[k]}(\overline{s}_{j[k]} + \lambda)}$$

where

$\mathcal{L}_{S,k}(s) := \mathbb{E}^0_{\Phi_k}[e^{-sS_0}]$

$\sigma[\mathcal{K}]$: Set of all permutations of $\mathcal{K} = \{1, 2, \ldots, K\}$

$\overline{s}_{j[k]} := \sum_{m=k}^{K} s_{j_m}$ for $(j_1, \ldots, j_K) \in \sigma[\mathcal{K}]$

$\overline{\lambda}_{j[k]} := \sum_{m=k}^{K} \lambda_{j_m}$ for $(j_1, \ldots, j_K) \in \sigma[\mathcal{K}]$

Summary

Palm Calculus as a Unified Tool: Provides a robust, model-agnostic framework for AoI analysis by simply assuming stationarity

Bridging the Two Averages: In particular, the Inversion Formula rigorously connects time-average performance (user's view) with event-driven characteristics (system's view)

General Structural Identities: The stationary distribution of AoI is fundamentally characterized by the distributions of Delays (D_n) and Peak AoIs (P_n)

From Theory to Practice: Combining this framework with specific assumptions (e.g., M/GI-input, etc.) leads to closed/computable-form results for diverse systems

Change your measure to simplify your analysis