The universal quantum invariant and colored ideal triangulations

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Topological invariants in low dimensional topology @ Shimane University
Introduction

Drinfeld double and Heisenberg double

Universal quantum invariant and its reconstruction

Extension

3-dim. descriptions
Introduction

- Background
- Ideas for reconstruction of quantum invariants
- State sum invariant with weights in a non-commutative ring
Background

1984 Jones polynomial

“Quantum invariants”

- Colored Jones polynomial
- Reshetkhin–Turaev invariant
- Universal quantum invariant
- Kontsevich integral
Background

KEY POINT FOR CONSTRUCTIONS

RIII move $\rightarrow$ “$R$-matrix”

$\rightarrow$ “hexagon identity”
Background

- Reshetkhin-Turaev invariant
  \[ R \in \text{End}(V \otimes V), \; V: \text{fin.dim. linear sp.} \]
  \[
  (1 \otimes R)(R \otimes 1)(1 \otimes R) = (R \otimes 1)(1 \otimes R)(R \otimes 1)
  \]

- Universal quantum invariant
  \[ R \in \mathcal{H} \otimes^2, \; \mathcal{H}: \text{ribbon Hopf algebra} \]
  \[
  R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}
  \]
Background

Definitions are combinatorial and diagrammatic

⇒ It is not easy to see topological properties of links from quantum invariants.
Background

What are “topological properties” of links?

- Properties defined using simple operations or surfaces. e.g. invertible, achiral, Brunnian, ribbon, boundary, etc.
- Properties defined by classical invariants. e.g. genus, homology, fundamental group, bridge number, Milnor invariants, etc.
Background

**TASK**

Find relationships between quantum invariants and topological properties of links!
Background

**METHODS**

Link (3-dim. obj.)
(w/ topological properties)

↓

Link diagram (2-dim. obj.) \(\rightsquigarrow\) Quantum invariants
(w/ planer properties)
Background

METHODS

Link (3-dim. obj.) $\rightarrow$ triangulation (3-dim. obj.)
(w/ topological properties)

$\downarrow$

Link diagram (2-dim. obj.) $\rightsquigarrow$ Quantum invariants
(w/ planer properties)
Ideas for reconstruction of quantum invariants

$A$: a fin-dim Hopf algebra/$k$

1. Drinfeld double $D(A) \sim_k A^* \otimes A$

   $\Rightarrow R \in D(A)^{\otimes 2}$ s.t.
   $$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in D(A)^{\otimes 3}.$$

2. Heisenberg double $H(A) \sim_k A^* \otimes A$

   $\Rightarrow S \in H(A)^{\otimes 2}$ s.t.
   $$S_{12}S_{13}S_{23} = S_{23}S_{12} \in H(A)^{\otimes 3}.$$
Ideas for reconstruction of quantum invariants

**Theorem (Kashaev ’97)**

There is an algebra embedding

\[ \phi: D(A) \to H(A) \otimes H(A)^{\text{op}}, \]

s.t.

\[ \phi \otimes^2 (R) = S''_{14} S_{13} \tilde{S}_{24} S'_{23}. \]

\( S', S'', \tilde{S} \): modifications of \( S \) satisfying pentagon relations
Ideas for reconstruction of quantum invariants

Octahedral triangulations of link complements
Ideas for reconstruction of quantum invariants

Octahedral triangulations of link complements

\[ \phi \otimes^2 (R) = S''_{14} S_{13} \tilde{S}_{24} S'_{23} \]
Ideas for reconstruction of quantum invariants

Pachner \((2, 3)\) move
Ideas for reconstruction of quantum invariants

Pachner \((2, 3)\) move

\[ S_{23}S_{12} = S_{12}S_{13}S_{23} \]
Ideas for reconstruction of quantum invariants

TO SUM UP...

Idea for the reconstruction

\[
\begin{array}{c}
\text{Pachner (2, 3) move} \\
\mapsto \\
\text{pentagon identity}
\end{array}
\]

\[
\begin{array}{c}
\text{→} \\
S\text{-tensor}
\end{array}
\]
Ideas for reconstruction of quantum invariants

TO SUM UP...

Idea for the reconstruction

\[
\text{Pachner (2, 3) move} \quad \mapsto \quad S\text{-tensor} \quad \mapsto \quad \text{pentagon identity}
\]

In this talk: w/ universal quantum invariant
State sum invariant with weights in a non-commutative ring

Turaev-Viro’s state sum invariant for \((M, T)\):

\[
Z(M) = w^{-\#\text{vertces}} \sum_{\lambda} w_{\lambda} \prod_{T} W(T; \lambda)
\]

- \(T\): a triangulation of \(M\)
- \(\lambda\): a color (giving an integer on each edge)
- \(T\): a tetrahedron in \(T\)
- \(W(T; \lambda) \in \mathbb{C}\): the weight on \(T\)

\[
\begin{array}{c}
 i \\
 l \\
 k \\
 m \\
 j \\
 n \\
\end{array}
\]
State sum invariant with weights in a non-commutative ring

Turaev-Viro’s state sum invariant for \((M, \mathcal{T})\):

\[
Z(M) = w^{-\#\text{vertices}} \sum_{\lambda} w_{\lambda} \prod_{T} W(T; \lambda)
\]

- \(\mathcal{T}\): a triangulation of \(M\)
- \(\lambda\): a color (giving an integer on each edge)
- \(T\): a tetrahedron in \(\mathcal{T}\)
- \(W(T; \lambda) \in \mathbb{C}\): the weight on \(T\) satisfying a pentagon identity.

\[\begin{array}{cccc}
  i & & k & \\
  & k & & j \\
  l & & m & \\
  & m & & n \\
\end{array} \quad \rightarrow \quad W(T; \lambda)\]
State sum invariant with weights in a non-commutative ring

1. [Turaev-Viro]
   (triangulation, quantum $6j$-symbol)

\[
\begin{vmatrix}
  j_1 & j_2 & j_3 \\
  i_1 & i_2 & i_3 \\
  k_1 & k_2 & k_3
\end{vmatrix} = \sum_n [n]_q \begin{vmatrix}
  i_1 & i_2 & j_3 \\
  k_2 & k_1 & n \\
  k_3 & k_2 & n
\end{vmatrix}
\begin{vmatrix}
  i_3 & i_1 & j_2 \\
  k_1 & k_3 & n
\end{vmatrix}
\]

2. [Baseilhac-Benedetti] QHI
   (ideal triangulation, quantum dilogarithm)

\[
\Psi(V)\Psi(U) = \Psi(U)\Psi(-UV)\Psi(V)
\]
State sum invariant with weights in a non-commutative ring

3. The universal quantum invariant
   (link diagram, the universal $R$-matrix)

$$R = \sum_{i\geq 0} \alpha_i \otimes \beta_i \in D(A)^{\otimes 2}$$

$$J: \begin{array}{c}
i \\ \circ \\
\circ \\ j
\end{array} \mapsto \sum_{i,j\geq 0} \beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u$$
State sum invariant with weights in a non-commutative ring

3. The universal quantum invariant
   (link diagram, the universal $R$-matrix )

\[ R = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in D(A)^{\otimes 2} \]

\[ J: \quad \sum_{i,j \geq 0} \beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u \]

state sum

products of weights
State sum invariant with weights in a non-commutative ring

3. The universal quantum invariant
   (link diagram, the universal $R$-matrix)

\[ R = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in D(A) \otimes^2 \]

\[ J: \begin{array}{c}
   \begin{tikzpicture}
   \filldraw[red] (0,0) circle (2pt);
   \filldraw[red] (1,0) circle (2pt);
   \draw[thick, -latex] (0,0) -- (1,0);
   \draw[thick, -latex] (1,0) -- (0,0);
   \end{tikzpicture}
\end{array} \rightarrow \begin{array}{c}
   \sum_{i,j \geq 0} \beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u
\end{array} \]

state sum \hspace{1cm} \text{products of weights}

The orientation of the link \Rightarrow The order of products of weights.
State sum invariant with weights in a non-commutative ring

4. **Reconstruction** of the universal quantum invariant (colored ideal triangulation, the $S$-tensor)

\[ J' : \longrightarrow S = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in H(A)^{\otimes 2} \]
State sum invariant with weights in a non-commutative ring

4. **Reconstruction** of the universal quantum invariant (colored ideal triangulation, the $S$-tensor)

\[ J' : \rightarrow S = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in H(A) \otimes^2 \]

- invariant for “colored” 3-mfds
  (∃ a canonical choice of the color for a link ⇒ link inv.)
- invariant for closed 3-mfds if $A$ is involutory
Research topics in front of us

w/ Reconstruction:

- v.s. topological properties of links
- v.s. Volume conjecture
- v.s. Phys?
- “Quantum group theory” for Heisenberg double
Research topics in front of us

w/ Reconstruction:

► v.s. topological properties of links
► v.s. Volume conjecture
► v.s. Phys?
► “Quantum group theory” for Heisenberg double

w/ $J'$ for closed 3-mfds:

► v.s. WRT invariant
► v.s. Turaev-Viro invariant, QHI, and Kuperberg invariant
Drinfeld double and Heisenberg double
Quasi-triangular Hopf algebra

Quasi-triangular Hopf algebra \((\mathcal{H}, \eta, m, \varepsilon, \Delta, \gamma, R)\): Hopf algebra with the universal \(R\)-matrix \(R \in \mathcal{H} \otimes \mathcal{H}\) such that

\[
\Delta^\text{op}(x) = R \Delta(x) R^{-1} \quad \text{for} \ x \in \mathcal{H},
\]

\[
(\Delta \otimes 1)(R) = R_{13} R_{23}, \quad (1 \otimes \Delta)(R) = R_{13} R_{12}.
\]
Quasi-triangular Hopf algebra \((\mathcal{H}, \eta, m, \varepsilon, \Delta, \gamma, R)\): Hopf algebra with the universal \(R\)-matrix \(R \in \mathcal{H} \otimes^2\) such that

\[
\Delta^\text{op}(x) = R\Delta(x)R^{-1} \quad \text{for } x \in \mathcal{H},
\]

\[
(\Delta \otimes 1)(R) = R_{13}R_{23}, \quad (1 \otimes \Delta)(R) = R_{13}R_{12}.
\]

\(\Rightarrow\) invariant for braids.

\[
\begin{array}{c}
\leftrightarrow \\
\downarrow \quad \rightarrow \\
\end{array}
\quad \Rightarrow \quad R
Ribbons Hopf algebra

Ribbons Hopf algebra \((\mathcal{R}, \eta, m, \varepsilon, \Delta, \gamma, R, \theta)\): quasi-triangular Hopf algebra with the ribbon element \(\theta \in \mathcal{R}\) such that

\[
\theta^2 = u \gamma(u), \quad \gamma(\theta) = \theta, \quad \varepsilon(\theta) = 1, \quad \Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta),
\]

where \(u = \sum \gamma(\beta)\alpha\) with \(R = \sum \alpha \otimes \beta\).
Ribbon Hopf algebra

**Ribbon Hopf algebra** \((\mathcal{R}, \eta, m, \varepsilon, \Delta, \gamma, R, \theta)\): quasi-triangular Hopf algebra with the ribbon element \(\theta \in \mathcal{R}\) such that

\[
\theta^2 = u\gamma(u), \quad \gamma(\theta) = \theta, \quad \varepsilon(\theta) = 1, \quad \Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta),
\]

where \(u = \sum \gamma(\beta)\alpha\) with \(R = \sum \alpha \otimes \beta\).

\(\Rightarrow\) invariant for tangles.
Notation

\[ A = (A, \eta, m, \varepsilon, \Delta, \gamma) \]: a fin-dim Hopf algebra over a field \( k \), with basis \( \{e_\alpha\}_\alpha \).

\[ A^{\text{op}} = (A, \eta, m^{\text{op}}, \varepsilon, \Delta, \gamma^{-1}) \]: the opposite Hopf algebra of \( A \),

\[ (A^{\text{op}})^* = (A^*, \varepsilon^*, \Delta^*, \eta^*, (m^{\text{op}})^*, (\gamma^{-1})^*) \]: the dual of \( A^{\text{op}} \).
Drinfeld double and Heisenberg double

The Drinfeld double (quasi-triangular Hopf algebra):

\[ D(A) = ((A^{\text{op}})^* \otimes A, \eta_{D(A)}, m_{D(A)}, \varepsilon_{D(A)}, \Delta_{D(A)}, \gamma_{D(A)}, R) \]

The universal \( R \)-matrix \( R = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in D(A)^{\otimes 2} \) satisfies

\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in D(A)^{\otimes 3}. \]

The Heisenberg double (algebra with the \( S \)-tensor):

\[ H(A) = (A^* \otimes A, \eta_{H(A)}, m_{H(A)}) \]

The \( S \)-tensor \( S = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in H(A)^{\otimes 2} \) satisfies

\[ S_{12}S_{13}S_{23} = S_{23}S_{12} \in H(A)^{\otimes 3}. \]
Drinfeld double and Heisenberg double

Set

\[ S' = \sum (1 \otimes \tilde{e}_a) \otimes (e^a \otimes 1) \in H(A)^{\text{op}} \otimes H(A), \]

\[ S'' = \sum (1 \otimes e_a) \otimes (\tilde{e}^a \otimes 1) \in H(A) \otimes H(A)^{\text{op}}, \]

\[ \tilde{S} = \sum (1 \otimes \tilde{e}_a) \otimes (\tilde{e}^a \otimes 1) \in H(A)^{\text{op}} \otimes H(A)^{\text{op}}, \]

where \( \tilde{e}_a = \gamma(e_a) \) and \( \tilde{e}^b = (\gamma^*)^{-1}(e^b) \).
Drinfeld double and Heisenberg double

**Theorem (Kashaev ’97)**

We have $\phi: D(A) \rightarrow H(A) \otimes H(A)^{\text{op}}$ such that

$$\phi^{\otimes 2}(R) = S''_{14} S'_{13} \tilde{S}_{24} S'_{23}.$$
Drinfeld double and Heisenberg double

\( D(A) \): Drinfeld double of \( A \).
We have a ribbon Hopf algebra

\[
\mathcal{R} = D(A)[\theta]/(\theta^2 - u\gamma(u)),
\]

where \( u = \sum \gamma^*(e^a) \otimes e_a \).
We also consider the algebra

\[
\mathcal{H} = (H(A) \otimes H(A)^{\text{op}})[\bar{\theta}]/(\bar{\theta}^2 - \phi(u\gamma(u))),
\]

and extend the embedding \( \phi: D(A) \rightarrow H(A) \otimes H(A)^{\text{op}} \) to the map \( \bar{\phi}: \mathcal{R} \rightarrow \mathcal{H} \) by \( \bar{\phi}(\theta) = \bar{\theta} \).
Universal quantum invariant and its reconstruction
Universal quantum invariant for tangles in a cube

(1) Choose a diagram

(2) Put labels

(3) Read labels

\[ J(C) = \sum \gamma(\alpha)\gamma(\beta')u\theta^{-1} \otimes \alpha'\beta \in \bar{\mathcal{R}} \otimes \mathcal{R}. \]

\[ (R = \sum \alpha \otimes \beta = \sum \alpha' \otimes \beta') \]
Reconstruction of the universal quantum invariant

(1) Modify diagram

- Exchange $\wedge$ and $\div$ with $\ast$ and $\circ$, resp.
- Duplicate stracds
- Thicken the left strands
Reconstruction of the universal quantum invariant

(2) Put labels

\[
S \quad S^{-1} \quad S' \quad (S')^2 \quad \tilde{S} \quad \tilde{S}^{-1} \quad S'' \quad (S'')^2
\]

(3) Read the labels

\[
J'(C') = (\tilde{\theta} \otimes 1)\phi^2(J(C')) \in \tilde{\mathcal{H}} \otimes \mathcal{H}.
\]
Sketch of proof

\[ J : \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \quad 2
\end{array}
\end{array}
\end{array} \quad \mapsto \quad R \]

\[ J' : \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3 \quad 4
\end{array}
\end{array}
\end{array} \quad \mapsto \quad S''_{14} S_{13} \tilde{S}_{24} S'_{23} \]
Sketch of proof

\[ J : \quad 1 \quad 2 \quad \mapsto \quad R \]

\[ J' : \quad 1 \quad 2 \quad 3 \quad 4 \quad \mapsto \quad S_{14}' S_{13} \tilde{S}_{24} S_{23}' = \phi \otimes^2 (R) \]
Extension of the universal quantum invariant

- Colored diagrams
- Colored moves
- Invariance of the universal quantum invariant
Colored diagrams

: tangle diagrams obtained from the following parts

\[ \underline{\text{new}} \]

We can define the map $J'$ on colored diagrams in a similar way.
Colored moves

- Colored Pachner \((2, 3)\) moves

\[\begin{align*}
\star & \quad \star & \quad \star & \quad \star \\
\star & \quad \star & \quad \star & \quad \star
\end{align*}\]

Here, the orientation of each strand is arbitrary, and the thickness of each strand with \(*\)-mark is arbitrary.
Colored moves

- Colored \((0,2)\) moves

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{colored_moves.png}}
\end{array}
\]

Here, the orientation and thickness of each strand are arbitrary.
Colored moves

▶ Colored symmetry moves

Here, the orientation and thickness of each strand are arbitrary.
Colored moves

- Planar isotopies

Here, the orientation and thickness of each strand are arbitrary.
Invariance of the universal quantum invariant

$\mathcal{CD}$: the set of colored diagrams

$\sim_c$: the equivalence relation on $\mathcal{CD}$ generated by colored moves.

**Theorem (S)**

*If $\gamma^2 = 1$, then the map $J'$ is an invariant under $\sim_c$.***
Invariance of the universal quantum invariant

$\sim'_c$: the equivalence relation on $CD$ generated by colored moves except for

\[
\begin{align*}
\star &\quad \sim'_c \quad \star \\
\X &\quad \sim'_c \quad \X \\
\X &\quad \sim'_c \quad \X
\end{align*}
\]

Theorem (S)

The map $J'$ is an invariant under $\sim'_c$. 
3-dimensional descriptions

- Colored singular triangulations
- Colored moves
- v.s. link complements
Colored tetrahedron

: a tetrahedron with an ordering $f_1, f_2, f_3, f_4$ of its faces

There are eight types of colored tetrahedra:
Colored singular triangulation $\mathcal{C}(Z)$

Define $\mathcal{C}(Z)$ for a colored diagram $Z$ as follows.

(1) Place tetrahedra
Colored singular triangulation $\mathcal{C}(\mathbb{Z})$

(2) Define star-vertices
Colored singular triangulation $\mathcal{C}(Z)$

(2) Attach the tetrahedra
Colored moves

colored Pachner (2,3) move

pentagon relation

\( J' \)
v.s. link complements in $S^3 \setminus \{ \pm \infty \}$

The octahedral decomposition $\mathcal{O}(D)$:

1. Place an octahedron at each crossing
v.s. link complements in $S^3 \setminus \{\pm \infty\}$

(2) Attach the octahedra

the boundary of the octahedron
v.s. link complements in $S^3 \setminus \{\pm \infty\}$

**Theorem (S)**

*The octahedral triangulation $O(D)$ admits a colored ideal triangulation $C(Z(D))$.*

**Sketch of the proof**
Remarks

- $\gamma^2 = 1 \Rightarrow J'$ is an inv. of closed 3-mfd.

- (Conj) $\gamma^2 \neq 1 \Rightarrow J'$ is an inv. of framed 3-mfd.

- The colored diagrams form a strict monoidal category and $J'$ is formulated as a functor.

- Hoping to get TQFT if we take $L^2(\mathbb{R})$ as a module of $H(B_q(sl_2))$, which may give $\text{Vol}(M) + i\text{CS}(M)$. 