The universal quantum invariant and colored ideal triangulations

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Introduction

Drinfeld double and Heisenberg double

Universal quantum invariant and its reconstruction

Extension

3-dim. descriptions
Introduction

- Background
- Ideas for reconstruction of quantum invariants
- Remarks & Related topics
Background

1984 Jones polynomial

"Quantum invariants"

- Colored Jones polynomial
- Reshetkhin–Turaev invariant
- Universal quantum invariant
- Kontsevich integral
Background

KEY POINT FOR CONSTRUCTIONS

RIII move \quad \rightarrow \quad "R\text{-matrix"

\quad \rightarrow \quad "hexagon identity"
Background

- Reshetkhn-Turaev invariant
  \( R \in \text{End}(V \otimes V), \ V: \text{fin.dim. linear sp.} \)

  \[(1 \otimes R)(R \otimes 1)(1 \otimes R) = (R \otimes 1)(1 \otimes R)(R \otimes 1)\]

- Universal quantum invariant
  \( R \in \mathcal{R} \otimes^2, \ \mathcal{R}: \text{ribbon Hopf algebra} \)

  \[R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\]
Background

Definitions are **combinatorial** and **diagrammatic**

\[ \text{ \includegraphics{diagram}} \]

\[ \Rightarrow \text{ It is not easy to see **topological properties** of links from quantum invariants.} \]
Background

What are “topological properties” of links?

- Properties defined using **simple operations or surfaces**. e.g. invertible, achiral, Brunnian, ribbon, boundary, etc.
- Properties defined by **classical invariants**. e.g. genus, homology, fundamental group, bridge number, Milnor invariants, etc.
Background

**TASK**

Find relationships between *quantum invariants* and *topological properties* of links!
Background

METHODS

\[ \text{Link (3-dim. obj.)} \]
\[ \downarrow \]
\[ \text{Link diagram (2-dim. obj.)} \rightsquigarrow \text{Quantum invariants} \]
\[ \text{(w/ planer properties)} \]
Background

METHODS

Link (3-dim. obj.) → triangulation (3-dim. obj.)
(w/ topological properties)

Link diagram (2-dim. obj.) ⇝ Quantum invariants
(w/ planer properties)
Ideas for reconstruction of quantum invariants

$A$: a fin-dim Hopf algebra/$k$

1. Drinfeld double $D(A) \sim_k A^* \otimes A$

   $\Rightarrow R \in D(A)^{\otimes 2}$ s.t.
   
   $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in D(A)^{\otimes 3}$.

2. Heisenberg double $H(A) \sim_k A^* \otimes A$

   $\Rightarrow S \in H(A)^{\otimes 2}$ s.t.
   
   $S_{12}S_{13}S_{23} = S_{23}S_{12} \in H(A)^{\otimes 3}$. 
Ideas for reconstruction of quantum invariants

**Theorem (Kashaev ’97)**

There is an algebra embedding

\[ \phi: D(A) \to H(A) \otimes H(A)^{\text{op}}, \]

s.t.

\[ \phi \otimes^2 (R) = S''_{14} S_{13} \tilde{S}_{24} S'_{23}. \]

\( S', S'', \tilde{S} \): modifications of \( S \) satisfying pentagon relations
Ideas for reconstruction of quantum invariants

Octahedral triangulations of link complements
Ideas for reconstruction of quantum invariants

Octahedral triangulations of link complements

\[ \phi \otimes^2 (R) = S''_{14} S'_{13} \tilde{S}_{24} S''_{23} \]
Ideas for reconstruction of quantum invariants

Pachner $(2, 3)$ move
Ideas for reconstruction of quantum invariants

Pachner \((2, 3)\) move

\[ S_{23}S_{12} = S_{12}S_{13}S_{23} \]
Ideas for reconstruction of quantum invariants

TO SUM UP...

Idea for the reconstruction

Pachner $(2, 3)$ move $\mapsto$ pentagon identity

$\mapsto S$-tensor
Ideas for reconstruction of quantum invariants

TO SUM UP...

Idea for the reconstruction

\[
\begin{align*}
\text{Pachner (2, 3) move} & \quad \rightarrow \quad S\text{-tensor} \\
\end{align*}
\]

In this talk: w/ universal quantum invariant
Remarks

Turaev-Viro’s state sum invariant for \((M, \mathcal{T})\):

\[ Z(M) = w^{-\#\{\text{vertices}\}} \sum_{\lambda} w_\lambda \prod_{T} W(T; \lambda) \]

- \(\mathcal{T}\): a triangulation of \(M\)
- \(\lambda\): a color (giving an integer on each edge)
- \(T\): a tetrahedron in \(\mathcal{T}\)
- \(W(T; \lambda) \in \mathbb{C}\): the weight on \(T\)
Remarks

Turaev-Viro’s state sum invariant for $(M, \mathcal{T})$:  

$$Z(M) = w^{-\#\text{\{vertices\}}} \sum_{\lambda} w_{\lambda} \prod_{T} W(T; \lambda)$$

- $\mathcal{T}$: a triangulation of $M$
- $\lambda$: a color (giving an integer on each edge)
- $T$: a tetrahedron in $\mathcal{T}$
- $W(T; \lambda) \in \mathbb{C}$: the weight on $T$ satisfying a pentagon identity.
Remarks

1. [Turaev-Viro]  
   (triangulation, quantum $6j$-symbol)

$$|j_1 \ j_2 \ j_3|_{i_1 \ i_2 \ i_3 \ k_1 \ k_2 \ k_3} = \sum_n [n]_q |i_1 \ i_2 \ i_3|_{k_2 \ k_1 \ n} |i_2 \ i_3 \ j_1|_{k_3 \ k_2 \ n} |i_3 \ i_1 \ j_2|_{k_1 \ k_3 \ n}$$

2. [Baseilhac-Benedetti] QHI  
   (ideal triangulation, quantum dilogarithm)

$$\Psi(V)\Psi(U) = \Psi(U)\Psi(-UV)\Psi(V)$$
Remarks

3. The universal quantum invariant
   (link diagram, the universal $R$-matrix )

\[
R = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in D(A)^{\otimes 2}
\]

\[
J: \quad i \quad \mapsto \quad \sum_{i, j \geq 0} \beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u
\]
Remarks

3. The universal quantum invariant (link diagram, the universal $R$-matrix)

\[ R = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in D(A)^{\otimes 2} \]

\[ J: \quad \text{state sum} \quad \sum_{i,j \geq 0} \beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u \quad \text{products of weights} \]
Remarks

3. The universal quantum invariant
(link diagram, the universal $R$-matrix )

$$R = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in D(A) \otimes^2$$

$$J: \quad \sum_{i,j \geq 0} \beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u$$

The orientation of the link $\Rightarrow$ The order of products of weights.
Remarks

4. Reconstruction of the universal quantum invariant (colored ideal triangulation, the $S$-tensor)

$$J' : \quad \mapsto \quad S' = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in H(A)^{\otimes 2}$$
Remarks

4. **Reconstruction** of the universal quantum invariant
   (colored ideal triangulation, the $S$-tensor)

\[ J' : \quad \mapsto \quad S = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in H(A) \otimes^2 \]

- invariant for “colored” 3-mfds
  (\( \exists \) a canonical choice of the color for a link \( \Rightarrow \) link inv.)
- invariant for closed 3-mfds if \( A \) is involutory
Related topics; w/ the reconstruction $J'$

1. To study topological properties of links using $J'$.
Related topics; w/ the reconstruction $J'$

1. To study **topological properties** of links using $J'$.

   e.g. Obtain the **volume** of the link complement from $J'$!
   (=Restate the volume conjecture using $J'$)
Related topics; w/ the reconstruction $J'$

2. v.s. Phys?
Related topics; w/ the reconstruction $J'$

2. v.s. Phys?

\[ \text{TQFT} \xrightarrow{J'} \]

\[ H(A)^{\otimes n} \]

\[ H(A)^{\otimes m} \]

e.g. Give a TQFT construction of $J'$!
Related topics; w/ the reconstruction $J'$

3. v.s. Kontsevich invariant

$Z(D) \leq A(0) = \text{Span}_e \langle 0, \Theta, \Theta, \ldots \rangle$

$R = \exp \frac{i}{2}, \quad \exists W_q \text{ s.t. } J_q(k) = W_q \circ Z(k)$
Related topics; w/ the reconstruction $J'$

3. v.s. Kontsevich invariant

\[ Z\left( \sum_{\mathfrak{C}} \right) \leq A(0) = \text{Span}_\mathbb{C} \langle 0, \Omega, \Omega, \ldots \rangle \]

\[ R = \begin{cases} \exp \frac{1}{2} \end{cases} \quad , \quad \exists W_q \text{ s.t. } J_q(k) = W_q \circ Z(k) \]

Extend the reconstruction of the universal invariant to that of Kontsevich invariant!
Related topics; w/ $J'$ for colored 3-mfds

1. v.s. WRT invariant (with $\text{Un}(g)$)

$$M = S^3_L, \quad T_3(M) = TR_3(J_L)$$

survey presentation
Related topics; w/ $J'$ for colored 3-mfds

1. v.s. WRT invariant (with $\mathcal{U}_h(g)$)

$$M = S^3_L, \quad T_3(M) = TR_3(J_L)$$

Survey presentation

Find direct relationship between $J'$ and WRT invariant!
Related topics; w/ $J'$ for colored 3-mfds

2. v.s. Turaev-Viro invariant, QHI, and Kuperberg invariant

by triangulations.

\[ \leftrightarrow \]

6-$j$ symbol, (T-V) quantum dilog, (QHI)

"Hopf diagram" (Kuperberg)
Related topics; w/ $J'$ for colored 3-mfds

2. v.s. Turaev-Viro invariant, QHI, and Kuperberg invariant

by triangulations.

$\leftrightarrow$ 6-j symbol, (T-V) quantum dilog, (OHI)

"Hopf diagram" (Kuperberg)

Find direct relationship between $J'$ and these invariants!
2. Invariant using an associator  
(working project with Anderson Vera)

\[ \Phi \in A(3\downarrow 3) = \text{Span}_{\mathbb{C}} \langle H_{11}, H_{12}, \ldots \rangle / I_{\text{H}_{\text{S}_{\text{T_{U}}_{\text{AS}}}}}, \]

satisfies a “pentagon” relation.

[Bar-Natan, Dancso, 12] \( Z(\begin{array}{ccc} x_1 & x_2 \\ x_2 & x_3 & x_1 \end{array}) = \Phi \)
Related topics; w/ $J'$ for colored 3-mfds

2. Invariant using an associator
(working project with Anderson Vera)

$$\Phi \in A(\mbbox{3}) = \text{Span}_a \langle \mbbox{3}, \mbbox{3}, \ldots \rangle / \text{AS}$$

satisfies a "pentagon" relation.

$$[[\text{Bar-Natan}, \text{Dancso}], \mbbox{12}] \ Z(\mbbox{3}) = \Phi$$

Construct an invariant using an associator! then find direct relationship with LMO invariant!
Related topics; algebraic aspect

“Quantum group theory” for Heisenberg double

\[ \text{Un(q)} \begin{cases} \cdot \text{R-matrix} & \text{knot theory} \\ \cdot \text{crystal basis} & \text{motivated} \end{cases} \]
Related topics; algebraic aspect

“Quantum group theory” for Heisenberg double

\[ \text{Un} \{ \begin{array}{c} \text{R-matrix} \\ \text{crystal basis} \end{array} \} \]

\[ \text{D}(B(\text{Un})) \]

Construct “quantum group theory” for Heisenberg double (with $S$-tensor and crystal basis?)
Drinfeld double and Heisenberg double

- Quasi-triangular Hopf algebra and Ribbon Hopf algebra
- Drinfeld double and Heisenberg double
Quasi-triangular Hopf algebra

**Quasi-triangular Hopf algebra** \((\mathcal{H}, \eta, m, \varepsilon, \Delta, \gamma, R)\): Hopf algebra with the universal \(R\)-matrix \(R \in \mathcal{H} \otimes 2\) such that

\[
\Delta^{\text{op}}(x) = R \Delta(x) R^{-1}
\quad \text{for} \; x \in \mathcal{H},
\]

\[
(\Delta \otimes 1)(R) = R_{13} R_{23}, \quad (1 \otimes \Delta)(R) = R_{13} R_{12}.
\]
Quasi-triangular Hopf algebra

Quasi-triangular Hopf algebra \((\mathcal{H}, \eta, m, \varepsilon, \Delta, \gamma, R)\): Hopf algebra with the universal \(R\)-matrix \(R \in \mathcal{H} \otimes 2\) such that

\[
\Delta^{\text{op}}(x) = R \Delta(x) R^{-1} \quad \text{for } x \in \mathcal{H},
\]

\[
(\Delta \otimes 1)(R) = R_{13} R_{23}, \quad (1 \otimes \Delta)(R) = R_{13} R_{12}.
\]

⇒ invariant for braids.
Ribbon Hopf algebra

Ribbon Hopf algebra \((\mathcal{R}, \eta, m, \varepsilon, \Delta, \gamma, R, \theta)\): quasi-triangular Hopf algebra with the ribbon element \(\theta \in \mathcal{R}\) such that

\[
\theta^2 = u\gamma(u), \quad \gamma(\theta) = \theta, \quad \varepsilon(\theta) = 1, \quad \Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta),
\]

where \(u = \sum \gamma(\beta)\alpha\) with \(R = \sum \alpha \otimes \beta\).
Ribbon Hopf algebra

Ribbon Hopf algebra $\left(\mathcal{R}, \eta, m, \varepsilon, \Delta, \gamma, R, \theta\right)$: quasi-triangular Hopf algebra with the ribbon element $\theta \in \mathcal{R}$ such that

$$\theta^2 = u\gamma(u), \quad \gamma(\theta) = \theta, \quad \varepsilon(\theta) = 1, \quad \Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta),$$

where $u = \sum \gamma(\beta)\alpha$ with $R = \sum \alpha \otimes \beta$.

$\Rightarrow$ invariant for tangles.

$\xrightarrow{\theta}$
**Notation**

\[ A = (A, \eta, m, \varepsilon, \Delta, \gamma): \text{ a fin-dim Hopf algebra over a field } k, \]

with basis \( \{e_\alpha\}_\alpha \).

\[ A^{\text{op}} = (A, \eta, m^{\text{op}}, \varepsilon, \Delta, \gamma^{-1}): \text{ the opposite Hopf algebra of } A, \]

\[ (A^{\text{op}})^* = (A^*, \varepsilon^*, \Delta^*, \eta^*, (m^{\text{op}})^*, (\gamma^{-1})^*): \text{ the dual of } A^{\text{op}}. \]
Drinfeld double and Heisenberg double

The Drinfeld double $D(A)$ is a quasi-triangular Hopf algebra with the universal $R$-matrix

$$R = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in D(A)^{\otimes 2}$$

satisfying

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \in D(A)^{\otimes 3}. $$

The Heisenberg double $H(A)$ is an algebra with the $S$-tensor

$$S = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in H(A)^{\otimes 2}$$

satisfying

$$S_{12} S_{13} S_{23} = S_{23} S_{12} \in H(A)^{\otimes 3}. $$
Drinfeld double and Heisenberg double

Set

\[ S'' = \sum (1 \otimes \tilde{e}_a) \otimes (e^a \otimes 1) \in H(A)^{op} \otimes H(A), \]

\[ S''' = \sum (1 \otimes e_a) \otimes (\tilde{e}^a \otimes 1) \in H(A) \otimes H(A)^{op}, \]

\[ \tilde{S} = \sum (1 \otimes \tilde{e}_a) \otimes (\tilde{e}^a \otimes 1) \in H(A)^{op} \otimes H(A)^{op}, \]

where \( \tilde{e}_a = \gamma(e_a) \) and \( \tilde{e}^b = (\gamma^*)^{-1}(e^b) \).
Drinfeld double and Heisenberg double

**Theorem (Kashaev ’97)**

We have \( \phi: D(A) \rightarrow H(A) \otimes H(A)^{\text{op}} \) such that

\[
\phi \otimes^2 (R) = S''_{14} S_{13} \tilde{S}_{24} S'_{23}.
\]
Drinfeld double and Heisenberg double

$D(A)$: Drinfeld double of $A$.
We have a ribbon Hopf algebra

$$\mathcal{R} = D(A)[\theta]/(\theta^2 - u\gamma(u)),$$

where $u = \sum \gamma^*(e^a) \otimes e_a$.

We also consider the algebra

$$\mathcal{H} = (H(A) \otimes H(A)^{\text{op}})[\bar{\theta}]/(\bar{\theta}^2 - \phi(u\gamma(u))),$$

and extend the embedding $\phi: D(A) \to H(A) \otimes H(A)^{\text{op}}$ to the map $\bar{\phi}: \mathcal{R} \to \mathcal{H}$ by $\bar{\phi}(\theta) = \bar{\theta}$. 
Universal quantum invariant and its reconstruction
Universal quantum invariant for tangles in a cube

(1) Choose a diagram

(2) Put labels

(3) Read labels

\[
J(C) = \sum \gamma(\alpha) \gamma(\beta') u \theta^{-1} \otimes \alpha' \beta \in \bar{R} \otimes R.
\]

\[
( R = \sum \alpha \otimes \beta = \sum \alpha' \otimes \beta' )
\]
Reconstruction of the universal quantum invariant

(1) Modify diagram

- Exchange \( \leftrightarrow \) and \( \leftrightarrow \) with \( \circ \) and \( \circ \), resp.
- Duplicate strands
- Thicken the left strands
Reconstruction of the universal quantum invariant

(2) Put labels

\[
\begin{array}{cccc}
S & S^{-1} & S' & (S')^3 \\
\tilde{S} & \tilde{S}^{-1} & S'' & (S'')^3
\end{array}
\]

(3) Read the labels

\[
J'(C') = (\bar{\theta} \otimes 1)\phi^2(J(C')) \in \tilde{\mathcal{H}} \otimes \mathcal{H}.
\]
Sketch of proof

\[ J : \begin{array}{c}
1 \\
\swarrow \downarrow \searrow \\
2
\end{array} \rightarrow R \]

\[ J' : \begin{array}{c}
1 & 2 & 3 & 4 \\
\swarrow \downarrow \downarrow \swarrow \searrow \\
\end{array} \rightarrow S''_{14}S_{13}\tilde{S}_{24}S'_{23} \]
Sketch of proof

\[ J : \quad \begin{array}{c}
    1 \quad 2 \\
   \downarrow \quad \downarrow \\
\end{array} \quad \rightarrow \quad R \\

\[ J' : \quad \begin{array}{c}
    1 \quad 2 \quad 3 \quad 4 \\
   \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\end{array} \quad \rightarrow \quad S''_{14}S_{13}\tilde{S}_{24}S'_{23} = \phi \otimes^2 (R) \]
Extension of the universal quantum invariant

- Colored diagrams
- Colored moves
- Invariance of the universal quantum invariant
Colored diagrams

: tangle diagrams obtained from the following parts

We can define the map $J'$ on colored diagrams in a similar way.
Colored moves

- Colored Pachner \((2, 3)\) moves

Here, the orientation of each strand is arbitrary, and the thickness of each strand with \(*\)-mark is arbitrary.
Colored moves

- Colored \((0, 2)\) moves

Here, the orientation and thickness of each strand are arbitrary.
Colored moves

- Colored symmetry moves

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\begin{tikzpicture}
\draw (0,0) -- (0,2);
\draw (0,1) -- (1,1);
\draw (1,0) -- (1,1);
\draw (1,2) -- (1,1);
\end{tikzpicture}}
\end{array}
\end{array} & = \\
\begin{array}{c}
\begin{array}{c}
\text{\begin{tikzpicture}
\draw (0,0) -- (0,2);
\draw (0,1) -- (1,1);
\draw (1,0) -- (1,1);
\draw (1,2) -- (1,1);
\end{tikzpicture}}
\end{array}
\end{array}
\end{align*}
\]

Here, the orientation and thickness of each strand are arbitrary.
Colored moves

- Planer isotopies

\[
\begin{align*}
\downarrow &= \quad = \quad , \\
\times &= \quad \times = \quad , \\
\times &= \quad \times = \quad
\end{align*}
\]

Here, the orientation and thickness of each strand are arbitrary.
Invariance of the universal quantum invariant

\( \mathcal{CD} \): the set of colored diagrams

\( \sim_c \): the equivalence relation on \( \mathcal{CD} \) generated by colored moves.

**Theorem (S)**

*If* \( \gamma^2 = 1 \), *then the map* \( J' \) *is an invariant under* \( \sim_c \).
Invariance of the universal quantum invariant

\[ \sim'_{c}: \text{the equivalence relation on } CD \text{ generated by colored moves except for} \]

\[ \begin{aligned}
\times^* &= \times^*, & \langle \times \rangle &= \langle \times \rangle, \\
\times^* &= \times^*, & \times &= \times.
\end{aligned} \]

**Theorem (S)**

*The map \( J' \) is an invariant under \( \sim'_{c} \).*
3-dimensional descriptions

- Colored singular triangulations
- Colored moves
- v.s. link complements
Colored tetrahedron

: a tetrahedron with an ordering $f_1, f_2, f_3, f_4$ of its faces

There are eight types of colored tetrahedra:
Colored singular triangulation $C(Z)$

Define $C(Z)$ for a colored diagram $Z$ as follows.

1. Place tetrahedra
Colored singular triangulation $C(Z)$

(2) Define star-vertices

![Diagram showing colored singular triangulation](image-url)
Colored singular triangulation $C(Z)$

(2) Attach the tetrahedra
Colored moves

colored Pachner (2,3) move

pentagon relation

\[ J' \]

Pachner (2,3) move
v.s. link complements in $S^3 \setminus \{\pm \infty\}$

The octahedral decomposition $\mathcal{O}(D)$:

1. Place an octahedron at each crossing
v.s. link complements in $S^3 \setminus \{\pm \infty\}$

(2) Attach the octahedra

the boundary of the octahedron
v.s. link complements in $S^3 \setminus \{\pm \infty\}$

**Theorem (S)**

*The octahedral triangulation $\mathcal{O}(D)$ admits a colored ideal triangulation $\mathcal{C}(Z(D))$.*

**Sketch of the proof**

\[ D \rightarrow Z(D) \]
Remarks

- $\gamma^2 = 1 \Rightarrow J'$ is an inv. of closed 3-mfd.

- ( Conj) $\gamma^2 \neq 1 \Rightarrow J'$ is an inv. of framed 3-mfd.

- The colored diagrams form a strict monoidal category and $J'$ is formulated as a functor.

- Hoping to get TQFT if we take $L^2(\mathbb{R})$ as a module of $H(B_q(sl_2))$, which may give $\text{Vol}(M) + i\text{CS}(M)$. 