Brief discussion of classical field theory:

What is it? to a manifold M we associate a "space of fields" F_M

- F_M is usually the space of sections of some sheaf on M, which we would like to be Fréchet.
- next ingredient is the *action functional* $S_M : F_M \to \mathbb{R}$, which is usually of the form
$$S_M[\varphi] = \int_M \mathcal{L}(x, \varphi(x), \dot{\varphi}(x), \dots, \varphi^{(n)}(x))$$

(taken M cpct w/ some orientation) $\mathcal{L}$ is the Lagrangian

examples: ① Classical mechanics: $M = [0,1]$, F_M = functions $M \to$ some Riemannian mfld
$$S_M[q] = \int_M \left(\frac{m}{2} g(\dot{q}(t), \dot{q}(t)) - V(q(t))\right) dt, \qquad V \in \mathcal{C}^\infty(N)$$

② Chern-Simons theory: M = cpct 3 mfld

F_M = G-connections on the trivial bundle $M \times G \to M$
$$S_M[A] = \int_M \mathrm{tr}\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right)$$

③ BF-theory: $M \sim$ cpct m-dim mfld equipped w/ G-bundle $P \to M$
$$F_M = \underbrace{\{\text{connections on } P\}}_{A} \oplus \underbrace{\Omega^{m-2}(M, \mathrm{ad}^* P)}_{B}$$
$$S_M[A,B] = \int_M \langle B, F_A \rangle$$
w/ $F_A = dA + \frac{1}{2}[A,A]$ the curvature

$\langle \, , \rangle$ the dual pairing.

The Variational Principle:

"Principle of Least Action" $\rightsquigarrow$ equations of motion are given by stationary points of the action.

We want to solve $\delta S = 0$, w/ δ the variational derivative

(if F_M is Fréchet, this can be viewed as the Fréchet derivative)

When $S = \int_M \mathcal{L}$ w/ $\partial M = \emptyset$ $\quad \delta S = 0 \iff \frac{\partial \mathcal{L}}{\partial \varphi}$ Euler-Lagrange equations are satisfied.

examples: ① Classical mechanics: $S_{[0,1]}[q] = \int_0^1 \left(\frac{m}{2}\dot{q}^2 - V(q)\right)dt$

$$\delta S = 0 \hookrightarrow \lim_{\tau\to 0} S[q+\tau\delta q] - S[q] / \tau = 0$$

$$\hookrightarrow \lim_{\tau\to 0} \int_0^1 \frac{1}{\tau}\left[\frac{m}{2}\left(2\tau\dot{q}\,\delta\dot{q} + \tau^2(\delta\dot{q})^2\right) - \tau\, dV/dq\,\delta q + \mathcal{O}(\tau^2)\right] = 0$$

$$\hookrightarrow \int_0^1 m\dot{q}\,\delta\dot{q} - dV/dq\,\delta q = 0 \iff \int_0^1 (m\ddot{q}\,\delta q - dV/dq\,\delta q)\,dt = 0$$ (subject to boundary conditions)

$$\hookrightarrow m\ddot{q} + dV/dq = 0.$$ Newton's 2^{nd} Law

② Chern-Simons Theory: $\delta S_M[A] = 0 \hookrightarrow F_A = dA + A\wedge A = 0$ (connection is flat)

③ BF Theory: $\delta S_M[A,B] = 0 \hookrightarrow F_A = 0$

$$d_A B = dB + [A,B] = 0.$$

Quantum Field Theory:

I will discuss topological quantum field theory.

Nice axioms: Atiyah - Segal

a TQFT in dimension d is an assignment:
- closed oriented d-mfld $\Sigma \mapsto$ f.d. $\mathbb{C}$-v.s. $Z(\Sigma)$
- oriented $(d+1)$-mfld M w/ $\partial M = \Sigma \mapsto Z(M) \in Z(\Sigma)$

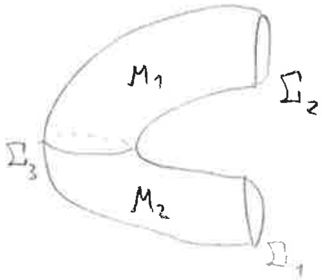
subject to the axioms:

1) Z is functorial w.r.t. orientation-preserving diffeomorphisms of Σ, M
2) Z is involutory; (involutive) $Z(\Sigma^*) = Z(\Sigma)^*$
3) Z is multiplicative; viz $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$

moreover, if $\partial M_1 = \Sigma_1 \cup \Sigma_3$, $\partial M_2 = \Sigma_2 \cup \Sigma_3^*$

and $M = M_1 \cup_{\Sigma_3} M_2$ then

$$Z(M) = \langle Z(M_1), Z(M_2)\rangle$$



alt. if $\partial M = \Sigma_1 \cup \Sigma_0^*$

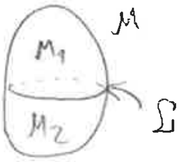
$$\Rightarrow Z(M) \in Z(\Sigma_0)^* \otimes Z(\Sigma_1) = \mathrm{Hom}(Z(\Sigma_0), Z(\Sigma_1)).$$

4) $Z(\phi_d) = \mathbb{C}$, $Z(\phi_{d+1}) = 1$, $Z(\Sigma \times [0,1]) = id_{Z(\Sigma)}$

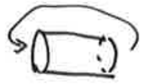
[5) $Z(M^*) = \overline{Z(M)}$]

why "topological" ~ there is no extra data (e.g. in physics you often must pick metrics)

a nice consequence: - if M ~ $(d+1)$ mfld w/ $\partial M = \emptyset$ => $Z(M) \in \mathbb{C}$
↳ invariants of closed $(d+1)$-mflds



- if we cut M into M_1, M_2 along Σ then
$Z(M_1) \in Z(\Sigma)$, $Z(M_2) \in Z(\Sigma^*)$
$\rightsquigarrow$ $Z(M) = \langle Z(M_1), Z(M_2) \rangle$

- also: $Z(\Sigma \times S^1) = \dim Z(\Sigma)$, for example  glue with id_Σ

1D TQFT ~ vector spaces 2D TQFT ~ frobenius algebras

Idea: take a classical field theory and obtain a QFT (or TQFT)

how?
- geometric quantization ("canonical quantization")
  ↳ attempts to mimic quantum mechanics for symplectic manifolds
- Feynman path integrals
  ↳ algebraic prescription, fraught with difficulties from a mathematical perspective (but still somehow works?)
- (Deformation quantization)

What are these things? briefly...

Geometric quantization: motivating example: QM
on $\mathbb{R}$, particle is described by position q and momentum p.
classically, evolution is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$$

(with H the "Hamiltonian", encoding the energy of the system)

$\{\ ,\ \}$ is the Poisson bracket $\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$

* there is a general procedure Lagrangian → Hamiltonian but this loses information generally ("Legendre transform")

* idea: EL equations are 2^{nd} order
in Hamiltonian mechanics, equations are 1^{st} order $\frac{\partial H}{\partial q} = -\dot{p}$, $\frac{\partial H}{\partial p} = \dot{q}$
(more equations, reduced complexity)

Upshot: Hamiltonian mechanics has a nice description in terms of symplectic geometry physical system is described by variables p, q and we have $\{q, p\} = 1$

In 1D QM we have "pure states" $\psi \in \mathcal{L}^2(\mathbb{R})$ w/ evolution governed by

Schrödinger's equation: $\underline{\underline{i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi}}$ $\hat{H}$ = quantum Hamiltonian, some operator on $\mathcal{L}^2(\mathbb{R})$, self adjoint

"quantum" ~ consider $\|\psi\|_{L^2} = 1$
then $|\psi(x)|^2$ = probability density.

in the Schrödinger regime we go classical $\to$ quantum

$p, q \longmapsto \hat{p} := -i\hbar\, \partial/\partial q$, $\hat{q} := q$ $\qquad [\hat{q}, \hat{p}] = i\hbar$

$H \longmapsto \hat{H}$ (!)

note: like a 0-d ~~T~~QFT

$\psi(t) = \exp(-\frac{i}{\hbar}\hat{H}t)\psi(0)$

$\mathcal{H}$ —— $\mathcal{H}$

to $[0,t] \sim \exp(-{}^{i}/_{\hbar}\hat{H}t)$

Geometric quantization attempts to mimic this.

Action + Lagrangian $\xrightarrow{\text{Legendre}}$ Hamiltonian $\xrightarrow[\text{geometric quantization}]{(!)}$ "Quantum Hilbert Space".

ingredient is a symplectic manifold (M, ω)

pick a "prequantization" and "polarization"...

· Feynman integration: gives a way of computing "expectation values" of "observables"

· black box: we don't have the actual quantum Hilbert space (mostly)

· for a theory with action S, observable O

$$\langle O \rangle = \int e^{i/\hbar\, S[\varphi]}\, O[\varphi]\, D\varphi$$

integral over space of all fields φ

there is a general algebraic method to "approximating" this "integral".

idea: expand around critical points of S

(i.e: "biggest" quantum contributions come from those that are "close" to the classical solutions)

Chern-Simons à la Witten: ... is a TQFT

$M\times G = P \circlearrowleft G$

$\downarrow$

M

$$S_M[A] = \int_M \mathrm{tr}\left(A\wedge dA + \tfrac{2}{3} A\wedge A\wedge A\right)$$

Feynman integral: $$Z = \int \exp\left(\frac{ik}{4\pi} S_M[A]\right) \mathcal{D}A$$ invariant of closed 3-mflds.

integral over space of connections

to actually compute: need to <u>regularize</u> (but that's another story)

how do we get <u>knot invariants?</u>

look at observables: Wilson lines

pick an irred. rep R of G, circle C in M

$$W_R(C) = \mathrm{tr}_R(\mathrm{hol}(A,C))$$

with a link L (components ℓ_i, $i=1,\dots,r$) labelled by irred. reps

$$Z(M; C_i, R_i) = \int \exp(iS[A]) \prod_{i=1}^{r} W_{R_i}(\ell_i)$$

When $M = S^3$ $\rightsquigarrow$ invariants from Jones theory and its generalizations.

example: $G = U(1)$ over S^3

pick circles C_a labelled by numbers n_a $\leftrightarrow$ irred. reps of $U(1)$

$$\rightsquigarrow \langle W\rangle = Z(S^3; C_a, n_a) = \exp\left(\frac{i}{2k}\sum_{a,b} n_a n_b \int_{C_a} dx^i \int_{C_b} dy^j\, \varepsilon_{ijk}\cdot \frac{(x-y)^k}{|x-y|^3}\right)$$ (Polyakov)

Gauss linking for $a\neq b$

$a=b$? this gives self-linking

we need to pick a framing of C_a, displace along this framing to get C_a' and compute this linking number

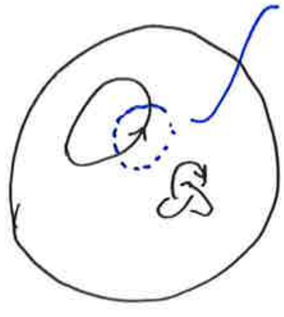
working on S^3: choose framing s.t. Gauss linking $\# = 0$

(makes self-linking contribution $=0$ in $U(1)$ case but not in general)

General idea:

2-mfld Σ $\rightsquigarrow$ geometric quantization gives a Hilbert space $\mathcal{H}_\Sigma$

in general:



want to cut out a 2-mfld w/ some marked pts representing Wilson lines ~ points P_i carrying "charges" (reps) R_i

gives $\mathcal{H}_{\Sigma; R_i, P_i}$ (incoming $= R$, outgoing $= R^*$)

For S^2:

(i) no marked points : $\dim \mathcal{H}_{S^2} = 1$

(ii) 1 marked pt carrying rep R : $\dim \mathcal{H}_{S^2; R, P} = \begin{cases} 1 & \text{if } R \text{ trivial} \\ 0 & \text{otherwise} \end{cases}$

(iii) 2 pts w/ reps R_i, R_j

$$\dim \mathcal{H}_{S^2; R_i, P_i} = \begin{cases} 1 & \text{if } R_i = R_j^* \\ 0 & \text{otherwise} \end{cases}$$

$M = 3$-mfld

<u>important case</u>: 4 pts w/ reps R, R, R^*, R^*

($G = SU(N)$, $R =$ defining rep.)

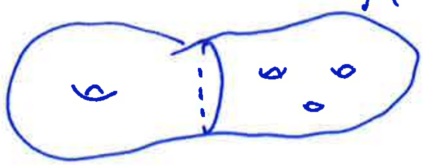
then: $\dim \mathcal{H} = 2$.

So far: only part of the ingredients for 2D TQFT

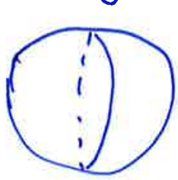
What do we get from 3-mflds? vectors in the associated Hilbert space (physically, via path integrals)

Example:

M



S^3

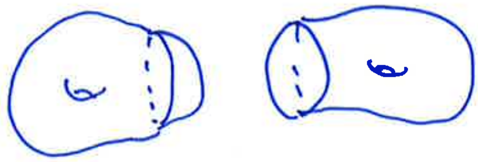


$\xrightarrow{Z}$ $Z(M) \cdot Z(S^3)$

$\|$

cut M into M_1, M_2 along S^2

$\rightsquigarrow$



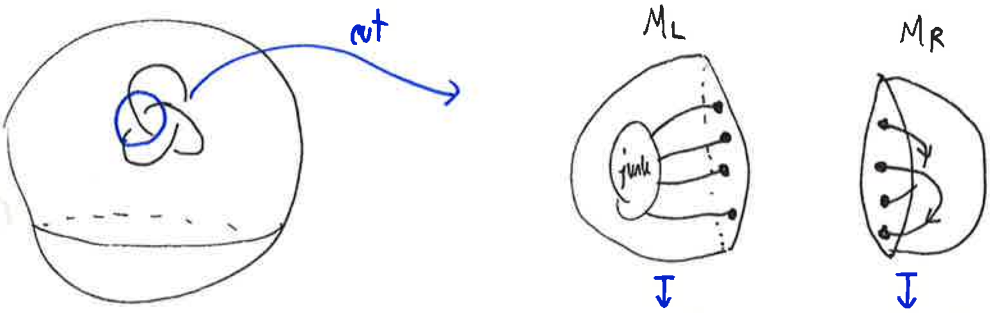
$\xrightarrow{Z}$ $Z(M_1) \cdot Z(M_2)$

in particular: $\dfrac{Z(M)}{Z(S^3)} = \dfrac{Z(M_1)}{Z(S^3)} \cdot \dfrac{Z(M_2)}{Z(S^3)}$

special case: pick reps R_i ~ ~~unknotted~~ unlinked knots c_i
excise the circles, then

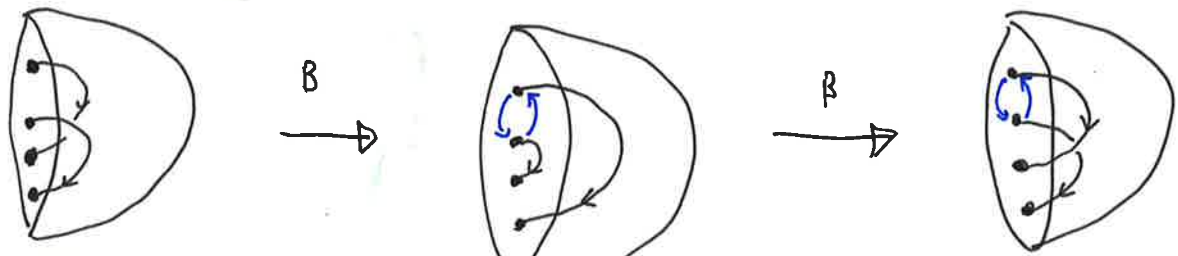
$$\frac{Z(S^3; C_i, R_i)}{Z(S^3)} = \prod_{k=1}^{s} \frac{Z(S^3; C_k, R_k)}{Z(S^3)}$$

Skein relation



$\mapsto \chi \in \mathcal{H}_L$ $\qquad$ $\mapsto \psi \in \mathcal{H}_R$

B ~ diffeo of M_R giving a "half twist" of the top marked pts

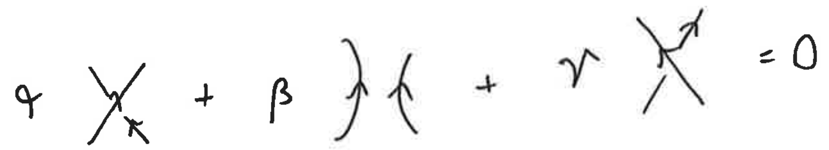


$\mapsto \psi$ $\qquad$ $\mapsto B\psi$ $\qquad$ $\mapsto B^2\psi$

$\dim \mathcal{H}_R = 2 \Rightarrow \quad \alpha\psi + \beta B\psi + \gamma B^2\psi = 0$

$\Rightarrow \quad (*) \quad \alpha\langle \chi, \psi\rangle + \beta\langle \chi, B\psi\rangle + \gamma\langle \chi, B^2\psi\rangle = 0$ (gluing back to M_L)

i.e.:

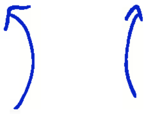


B acts on a 2D v.s. so $B^2 - \mathrm{Tr}(B)\cdot B + \det(B)\cdot I = 0$

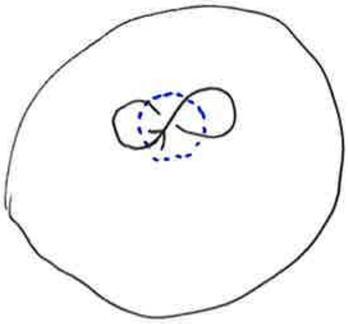
$\rightsquigarrow \quad \alpha = \det B$
$\beta = -\mathrm{tr}\, B$
$\gamma = 1$

if we are labelled by the N-dim rep of $SU(N)$ we can find the e-vals of B, after multiplying by some (irrelevant factor $(q^{?})$) by

can be written as:

$\underbrace{-q^{+N/2}}_{\alpha}$  $+ \underbrace{(q^{1/2} - q^{-1/2})}_{\beta}$  $+ \underbrace{q^{-N/2}}_{\gamma}$  $= 0$

and,



$$\Rightarrow \quad (\alpha + \gamma) Z(S^3; C) + \beta Z(S^3; C^2) = 0$$

$$\Rightarrow \quad \langle \text{unknot in } S^3 \rangle = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}$$

So if $N=2$: $(SU(2))$

we get: $(q \leftrightarrow t)$

$(t^{1/2} - t^{-1/2})$  $= t$  $- \frac{1}{t}$ 

and $\langle \text{unknot} \rangle = 1$

Jones Polynomial.

Witten: QFT and Jones Poly.

Atiyah: TQFT