Brief discussion of classical field theory:

What is it? to a manifold $M$ we associate a "space of fields" $F_M$.

- $F_M$ is usually the space of sections of some sheaf on $M$, which we would like to be Fréchet.

- Next ingredient is the action functional $S_M : F_M \to \mathbb{R}$, which is usually of the form

$$S_M[\gamma] = \int_M \left( L (x, \gamma(x), \dot{\gamma}(x), \ldots, \gamma^{(n)}(x)) \right) \, \text{d}t$$

(taken $M$ compact with some orientation) $L$ is the Lagrangian

Examples:

1. Classical mechanics: $M = [0,1]$, $F_M = \text{functions } M \to \text{some Riemannian mfld}$

$$S_M[\gamma] = \int_M \left( \frac{1}{2} \left( \frac{\text{d} \gamma(t)}{\text{d}t} \right)^2 - V(\gamma(t)) \right) \, \text{d}t, \quad \forall \in \mathbb{C}^\infty(N)$$

2. Thron-Simons Theory: $M = \text{cpt 3 mfld}$

$$F_M = \text{G-connections on the trivial bundle } M \times G \to M$$

$$S_M[A] = \int_M \text{tr} \left( \text{Ad}A + \frac{1}{2} A \wedge A \wedge A \right)$$

3. BF-Theory: $M = \text{cpt m-dim mfld equipped with G-bundle } P \to M$

$$F_M = \{ \text{connections on } P \} \oplus \Omega^m(P, \text{ad}^* P)$$

$$S_M[A, B] = \int_M \langle B, F_A \rangle$$

$F_A = \text{d}A + \frac{1}{2} [A, A]$ the curvature

$\langle \cdot, \cdot \rangle$ the dual pairing

The Variational Principle:

"Principle of Least Action" $\Rightarrow$ equations of motion are given by stationary points of the action.

We want to solve $\delta S = 0$, w/ $\delta$ the variational derivative

(if $F_M$ is Fréchet, this can be viewed as the Fréchet derivative)

When $S = \int_M L$ w/ $\delta M = \delta \phi$, $\delta S = 0 \iff \frac{\partial L}{\partial \dot{\phi}}$ Euler-Lagrange equations are satisfied.
Examples:  

Classical mechanics: 

\[
\mathcal{S}_{(1)}^{\text{cl}}[q] = \int_0^1 \left( \frac{1}{2} \dot{q}^2 - V(q) \right) \, dt
\]

\[\mathcal{S}_g = 0 \iff \lim_{\tau \to 0} \left[ \mathcal{S}_g + \tau \delta q \right] - \mathcal{S}[q] / \tau = 0\]

\[\int_0^1 \frac{d}{\tau} \left[ \frac{1}{2} \left( 2 \tau \delta q + \tau \ddot{q} \delta q^2 \right) + \tau \delta V_q \delta q + O(\tau^2) \right] = 0\]

\[\int_0^1 m \ddot{q} - \delta V_q \delta q = 0 \iff \int_0^1 \left( m \ddot{q} - \delta V_q \delta q \right) \, dt = 0 \quad \text{(subject to boundary conditions)}\]

\(\Rightarrow m \ddot{q} + \delta V_q = 0. \quad \text{Newton's 2nd Law}\)

\(\text{2. Chern-Simons Theory:} \quad \mathcal{S}_g = 0 \iff F_A = dA + A \wedge A = 0 \quad \text{(connection is flat)}\)

\(\text{3. BF Theory:} \quad \mathcal{S}_g^{\text{BF}}[A,B] = 0 \iff F_A = 0\)

\[d_A B = dB + [A,B] = 0.\]

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Quantum Field Theory:

I will discuss topological quantum field theory.

Nice axioms: Atiyah-Segal

A TQFT in dimension d is an assignment:

- closed oriented d-manifold \(\Sigma \mapsto \text{f.d. C-v.s.} \mathcal{Z}(\Sigma)\)
- oriented \((d+1)\)-manifold \(M \) w/ \( M = \Sigma \mapsto \mathcal{Z}(M) \in \mathcal{Z}(\Sigma)\)

Subject to the axioms:

1) \(\mathcal{Z}\) is functorial w.r.t. orientation-preserving diffeomorphisms of \(\Sigma, M\)

2) \(\mathcal{Z}\) is involutive: \(\mathcal{Z}(\Sigma^*) = \mathcal{Z}^*(\Sigma)\)

3) \(\mathcal{Z}\) is multiplicative; viz \(\mathcal{Z}(\Sigma_1 \cup \Sigma_2) = \mathcal{Z}(\Sigma_1) \otimes \mathcal{Z}(\Sigma_2)\)

moreover, \(\emptyset M_1 = \Sigma_1 \cup \Sigma_2, \emptyset M_2 = \Sigma_2 \cup \Sigma_3 \quad \text{then}\)

\(\mathcal{Z}(M) = \langle \mathcal{Z}(M_1), \mathcal{Z}(M_2) \rangle\)

alt., if \(\emptyset M = \Sigma_1 \cup \Sigma_2^* \mapsto \mathcal{Z}(M) = \mathcal{Z}(\Sigma_1) \otimes \mathcal{Z}^*(\Sigma_2) = \text{Hom} (\mathcal{Z}(\Sigma_2), \mathcal{Z}(\Sigma_1))\)

4) \(\mathcal{Z}(\phi_d) = C, \quad \mathcal{Z}(\phi_{d+1}) = 1, \quad \mathcal{Z}(\Sigma \times [0,1]) = \text{id} \mathcal{Z}(\Sigma)\)

\[\text{5) } \mathcal{Z}(M^*) = \overline{\mathcal{Z}(M)}\]

why "topological" — there is no extra data (e.g. in physics you often must pick metrics)
A nice consequence: if \( M \sim (s^d) \) is a manifold with \( \partial M = \emptyset \) then its invariants of closed \((s^d)\)-manifolds are invariant of compact \((s^d)\)-manifolds.

If we cut \( M \) into \( M_1, M_2 \) along \( \Sigma \), then:

\[
Z(M_1) \in Z(\Sigma), \quad Z(M_2) \in Z(\Sigma^*)
\]

implies:

\[
Z(M) = \langle Z(M_1), Z(M_2) \rangle
\]

Also:

\[
Z(\Sigma \times S^1) = \text{dim} \ Z(\Sigma), \quad \text{for example, glue with \( id_{\Sigma} \)}
\]

1D TQFT - vector spaces

2D TQFT - Frobenius algebras

Idea: take a classical field theory and obtain a QFT (or TQFT)

How?

- geometric quantization ("canonical quantization")

- attempts to mimic quantum mechanics for symplectic manifolds

- Feynman path integrals

- algebraic prescription, fraught with difficulties from a mathematically
  perspective (but still somehow works?)

- deformation quantization

What are these things? Briefly...

Geometric quantization: motivating example: QM

On \( \mathbb{R}^d \), particle is described by position \( q \) and momentum \( p \).

Classically, evolution is given by

\[
\frac{df}{dt} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial \tau} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial \tau}
\]

(with \( H \) the "Hamiltonian", encoding the energy of the system)

\[
\{ \cdot, \cdot \}
\]

is the Poisson bracket

\[
\begin{align*}
\frac{\partial f}{\partial q} & = -p, & \frac{\partial f}{\partial p} & = q \\
\end{align*}
\]

* there is a general procedure Lagrangian \( \rightarrow \) Hamiltonian but this loses
  information generally ("Legendre transform")

An idea: EL equations are 2nd order

in Hamiltonian mechanics, equations are 1st order

\[
\begin{align*}
\frac{\partial L}{\partial \dot{\theta}} & = -p, & \frac{\partial L}{\partial \theta} & = q
\end{align*}
\]

(more equations, reduced complexity)

Update: Hamiltonian mechanics has a nice description in terms of symplectic geometry
In QM we have "pure states" \( \psi \in L^2(\mathbb{R}) \) with evolution governed by Schrödinger's equation:

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad \hat{H} = \text{quantum Hamiltonian, some operator on } L^2(\mathbb{R}), \text{ self-adjoint}
\]

They \( |\psi(t)|^2 \) = probability density.

The Schrödinger regime we go classical \( \rightarrow \) quantum

\[
p, q \rightarrow \hat{p} := -i\hbar \frac{\partial}{\partial q} \quad \hat{\hat{p}} := \hat{q} \quad [\hat{q}, \hat{p}] = i\hbar
\]

M \rightarrow \hat{H} (\hat{\hat{p}})

Geometric quantization attempts to mimic this.

Action + Lagrangian \( \rightarrow \) Hamiltonian \( \rightarrow \) "Quantum Hilbert Space".

Ingredient is a symplectic manifold \((M, \omega)\)

Feynman integration: gives a way of computing "expectation values" of "observables"

- black box: we don't have the actual quantum Hilbert space (mostly)
- for a theory with action \( S \) and observable \( \mathcal{O} \)

\[
\langle \mathcal{O} \rangle = \int e^{i\frac{\alpha}{\hbar} S[\phi]} \mathcal{O}[\phi] \, D\phi
\]

integral over space of all fields \( \phi \)

There is a general algebraic method to "approximating" this "integral".

Idea: expand around critical points of \( S \)

(i.e. "biggest" quantum contributions come from those that are "close" to the classical solutions)
Thorn-Simons a la Witten: is a TQFT

\[ S_M[A] = \int_M \frac{1}{4} (F^2 + \frac{2}{3} A \wedge F \wedge F) \]

Feynman integral: \[ \mathcal{Z} = \int \exp \left( \frac{i}{\hbar} S_M[A] \right) \mathcal{D}A \]

Invariant of closed 3-manifolds.

To actually compute: need to regularize (but that's another story)

How do we get knot invariants?

Look at observables: Wilson lines

Pick an irrep \( R \) of \( G \), circle \( C \) in \( M \)

\[ W_R (c) = tr_R (\text{hol}(A,c)) \]

With a link \( L \) (components \( e_i, i=1,\ldots,r \)) labelled by irreps \( \gamma \)

\[ \mathcal{Z} (M; c_i, R_i) = \int \exp (i S[A]) \prod_{i=1}^r W_{R_i} (e_i) \]

When \( M = S^3 \) no invariants from Jones theory and its generalizations.

Example: \( G = SL(2,\mathbb{C}) \) over \( S^3 \)

Pick circles \( \partial a \) labelled by numbers \( n_a \to \) irrep reps of \( SU(2) \)

\[ \langle W \rangle = \mathcal{Z} (S^3; C_a, n_a) = \exp \left( \frac{1}{2\hbar} \sum_{a,b} n_a n_b \int_{C_a} \int_{C_b} \epsilon_{ijk} \frac{(x-y)^k}{|x-y|^3} \right) \] (Polyakov)

\( a = b \): this gives self-linking

We need to pick a framing of \( C_a \), displace along this framing to get \( C'_a \) and compute this linking number

Working on \( S^3 \): choose framing s.t. Gauss linking \( \# = 0 \)

(makes self-linking contribution \( = 0 \) in \( SL(2,\mathbb{C}) \) case but not in general)
General idea:

$2$-mfd $\to$ geometric quantization gives a Hilbert space $H_\Sigma$

In general:

- Want to cut out a $2$-mfd w/ some marked pts representing Wilson lines $\sim$ points $P_i$: carrying "charges" (reps) $R_i$ gives $H_\Sigma; R_i; P_i$ (incoming $= R_i$; outgoing $= R^*)$

For $S^2$:

(i) no marked points: $\dim H_{S^2} = 1$

(ii) 1 marked pt carrying rep $R$: $\dim H_{S^2; R, P} = \begin{cases} 1 & \text{if } R \text{ trivial} \\ 0 & \text{otherwise} \end{cases}$

(iii) 2 pts w/ reps $R_i, R_j$

$$\dim H_{S^2; R_i, P_i} = \begin{cases} 1 & \text{if } R_i = R_j^* \\ 0 & \text{otherwise} \end{cases}$$

Important case: 4 pts w/ reps $R, R, R^*, R^*$

($G = SU(N)$, $R$ defining rep.)

Then: $\dim H = 2$.

So far: only part of the ingredients for $2D$ TQFT

What do we get from $3$-mfd's? vectors in the associated Hilbert space (physically, via path integrals)

Example:

Cut $M$ into $M_1, M_2$

along $S^2$

$Z(M) \cdot Z(S^3)$
In particular:
\[
\frac{Z(M)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)} \cdot \frac{Z(M_2)}{Z(S^3)}
\]

Special case: pick reps \( R_i \) unknotted unlinked disks \( C_i \)
excise the circles, then
\[
\frac{Z(S^3; C_i, R_i)}{Z(S^3)} = \prod_{k=1}^5 \frac{Z(S^3; C_i, R_k)}{Z(S^3)}
\]

Shin relation

\( B \sim \text{diff} \text{ of } M_R \text{ giving a "half twist" of the top marked pts} \)

\[ \dim H_R = 2 \Rightarrow \alpha \psi + \beta B \psi + \gamma B^2 \psi = 0 \]

\[ \Rightarrow (\ast) \quad \alpha \langle \chi, \psi \rangle + \beta \langle \chi, B \psi \rangle + \gamma \langle \chi, B^2 \psi \rangle = 0 \quad \text{(giving back to } M_L) \]

i.e.: \[ \alpha \chi + \beta \chi + \gamma \chi = 0 \]

\( B \) acts on a 2D v.s. so \[ B^2 - Tr(B) \cdot B + \det(B) \cdot I = 0 \]

\( \Rightarrow \quad \alpha = \det B \)
\[ \beta = - Tr(B) \]
\[ \gamma = 1 \]
if we are labelled by the $N$-dim rep of $SU(N)$ we can find the e-vals of $B$; after multiplying by some (relevant factor) can be written as:

\[
\begin{align*}
q^{\frac{N}{2}} & \quad + \quad \beta \quad + \quad (q^{\frac{N}{2}} - q^{-\frac{N}{2}}) \quad + \quad q^{-\frac{N}{2}} = 0 \\
\Rightarrow \quad (q + \beta) & \quad \tau (S^3; C) + \beta \tau (S^3; C^2) = 0 \\
\Rightarrow \quad \langle \text{unknot in } S^3 \rangle & \quad = \quad q^{\frac{N}{2}} - q^{-\frac{N}{2}} \\
& \quad \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{N}{2}} - q^{-\frac{N}{2}}} \\
\end{align*}
\]

so if $N=2$: $(SU(2))$

\[
\begin{align*}
\text{we get:} \quad \tau (S^2 - t^{-\frac{1}{2}}) & \quad \tau = t^{\frac{N}{2}} - \frac{1}{t} \\
(\text{unknot}) & \quad \langle \text{unknot} \rangle = 1
\end{align*}
\]

Willen: QFT and Jones Poly.

Ahiah: TQFT