Bing doubling and the colored Jones polynomial

Sakie Suzuki

RIMS

2012.12.24

結び目の数学 V @ Waseda University
Introduction

Colored Jones polynomial

Unified WRT invariant

Results
Introduction
Quantum invariants for links

\[ L = L_1 \cup \cdots \cup L_n: \text{framed link} \]

- **Kontsevich inv.**
  \[ Z_L \in \hat{A}(\bigcup_n S^1) \]

- **Universal \( sl_2 \) inv.**
  \[ J_L \in U_h(sl_2)^{\otimes n}/I \]

- **Colored Jones poly.**
  \[ J_{L;V_1,\ldots,V_n} \in \mathbb{Z}[q^{1/4}, q^{-1/4}] \]
Quantum invariants for 3-mfds

$M$: Integral homology sphere (≡ IHS)

LMO inv. \[ Z_M \in \hat{\mathcal{A}}(\emptyset) \]

Unified WRT inv. \[ J_M \in \mathbb{Z}[q] \]

WRT inv. \[ \tau_M^\zeta \in \mathbb{Z}[\zeta] \]
Quantum invariants

Links
- Kontsevich inv.
- Universal $sl_2$ inv.
- Colored Jones poly.

IHS
- LMO inv.
- Unified WRT inv.
- WRT inv.
Quantum invariants

Links
- Kontsevich inv.
- Universal $sl_2$ inv.
- Colored Jones poly.

IHS
- LMO inv.
- Unified WRT inv.
- WRT inv.
What is “Topological” ? ~ Classical

- Fundamental groups
  (Alexander poly., Milnor $\mu$ inv., …)
- Coverings
  (Alexander poly., …)
- Seifert surfaces
  (Alexander poly., boundary links, …)
- Cobordisms
  (slice, ribbon, …)
- Local moves
  (crossing change, mutant, satellite, …)
What is “Topological” ? ~ Classical

- **Fundamental groups**
  (Alexander poly., Milnor $\mu$ inv., …)

- **Coverings**
  (Alexander poly., …)

- **Seifert surfaces**
  (Alexander poly., boundary links, …)

- **Cobordisms**
  (slice, ribbon, …)

- **Local moves**
  (crossing change, satellite, mutation, …)
Bing doubling

\[ K = \begin{array}{c}
\end{array} \quad B(K) = \begin{array}{c}
\end{array} \]
Bing doubling and Link concordance

Fact: $K$ is slice $\Rightarrow B(K)$ is slice.
Bing doubling and Link concordance

Fact: $K$ is slice $\implies B(K)$ is slice.

Q1: Does the converse hold?

(Harvey, Teichner, ...)

L: a link obtained from Borromean rings by a sequence of Bing doublings.

Q2: Are Whitehead doubles of $L$ slice?

(Freedman, Lin), "Surgery conjecture" in 4-dim. topology.
Bing doubling and Link concordance

Fact: $K$ is slice $\Rightarrow B(K)$ is slice.

Q1: Does the converse hold?

(Harvey, Teichner, ...)

L: a link obtained from Borromean rings by a sequence of Bing doublings.
Bing doubling and Link concordance

Fact: $K$ is slice $\Rightarrow B(K)$ is slice.

Q1: Does the converse hold?
   (Harvey, Teichner, ...)

L: a link obtained from Borromean rings by a sequence of Bing doublings.

Q2: Are Whitehead doubles of $L$ slice?
   (Freedman, Lin)
Bing doubling and Link concordance

Fact: $K$ is slice $\Rightarrow B(K)$ is slice.

Q1: Does the converse hold?

(Harvey, Teichner, ...)

L: a link obtained from Borromean rings by a sequence of Bing doublings.

Q2: Are Whitehead doubles of $L$ slice?

(Freedman, Lin)

“Surgery conjecture” in 4-dim. topology.
Bing doubling and Milnor $\bar{\mu}$ invariant

Roughly

Milnor $\bar{\mu}$ invariants of length $l \geq 2$

$\parallel$

“linking numbers of degree $l$”

linking number of degree $2 = \text{number of } \text{disjoint components}
Bing doubling and Milnor $\overline{\mu}$ invariant
Bing doubling and Milnor $\bar{\mu}$ invariant
Bing doubling and Milnor $\bar{\mu}$ invariant
Bing doubling and Milnor $\bar{\mu}$ invariant

Milnor $\bar{\mu}$ invariants count the following parts:

- Length 2
- Length 3
- Length 4
Bing doubling and Finite type invariants

The set of $A$-finite type invariant $\bigcup_{n \geq 0} V_n$ with the filtration

$$V_0 = V_1 \subset V_2 \subset \cdots$$

induces the filtration

$$\mathcal{K} = \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$$

where $\mathcal{K} := \text{Span}_A \{ \text{isotopy classes of knots} \}$. 
Bing doubling and Finite type invariants

Theorem (Habiro)

\[ K_1, K_2: \text{knots} \]

\[ K_1 - K_2 \in \mathcal{K}_i \iff K_1 \sim_{C_i} K_2. \]

Here \( \sim_{C_i} \) is the equivalent relation generated by
Colored Jones polynomial
Colored Jones polynomial

\[ W_1, \ldots, W_5 \in \text{Mod}_f(U_q(sl_2)). \]

- As Operator invariant
- By Skein relation
- From Kontsevich invariant
- From Universal \( sl_2 \) invariant
Colored Jones polynomial (Operator invariant)

\[ V, W \in \text{Mod}_f(U_q(sl_2)) \]

\[ \mathbb{C}(q^{1/4}) \]

\[ \text{coev}^* \otimes \text{coev} \]

\[ V^* \otimes V \otimes W \otimes W^* \]

\[ 1 \otimes R_{V,W} \otimes 1 \]

\[ V^* \otimes W \otimes V \otimes W^* \]

\[ 1 \otimes R_{W,V} \otimes 1 \]

\[ V^* \otimes V \otimes W \otimes W^* \]

\[ \text{ev} \otimes \text{ev}^* \]

\[ \mathbb{C}(q^{1/4}) \]

\[ J_{L;V,W} \]
Generalized colored Jones polynomial

Set

\[ \mathcal{R} = \text{Span}_{Q(q^{1/2})} \{ V_m : m\text{-dim. irr. rep.} \mid m \geq 1 \}. \]

**Definition**

For a link \( L = L_1 \cup \cdots \cup L_n \) and

\[ X_i = \sum_{j_i} x_{j_i}^{(i)} V_{j_i} \in \mathcal{R}, \quad x^{(i)} \in Q(q^{1/2}), \]

set

\[ J_{L;X_1,\ldots,X_n} = \sum_{j_1,\ldots,j_n} x_{j_1}^{(1)} \cdots x_{j_n}^{(n)} J_{L;V_{j_1},\ldots,V_{j_n}}. \]
For $l \geq 0$, set

$$P^\prime_l = \frac{1}{\{l\}!} \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R},$$

$$\tilde{P}^\prime_l = q^{-\frac{1}{4} l(l-1)} P^\prime_l \in \mathcal{R},$$

$$\mathcal{P}_k = \text{Span}_{\mathbb{Z}[q,q^{-1}]} \{ \tilde{P}^\prime_l \mid l \geq k \},$$

$$\hat{\mathcal{P}} = \lim_{k \geq 0} \mathcal{P}_0 / \mathcal{P}_k,$$

$$\omega^{\pm 1} = \sum_{l=0}^{\infty} (\pm 1)^l q^{\pm \frac{1}{4} l(l+3)} P^\prime_l \in \hat{\mathcal{P}}.$$
Theorem (Habiro)

$L = L_1 \cup \cdots \cup L_n$: algebraically-split link

$$J_{L;\omega^\epsilon_1,\ldots,\omega^\epsilon_n} = \sum_{l_1,\ldots,l_n=0}^{\infty} \left( \prod_{i=1}^{n} \epsilon_i^l q^{\epsilon_i \frac{1}{4} l_i (l_i+3)} \right) J_{L;P'_1,\ldots,P'_n} \in \widehat{\mathbb{Z}}[q]$$

Here $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ and

$$\widehat{\mathbb{Z}}[q] = \lim_{n \geq 0} \mathbb{Z}[q] / (((1 - q)(1 - q^2) \cdots (1 - q^n)))$$
Unified WRT invariant
Unified WRT invariant

\[ L = L_1 \cup \cdots \cup L_n \] link with framings \( \epsilon_1, \ldots, \epsilon_n \in \{ \pm 1 \} \)

\[ M = S^3_L \] IHS obtained by surgery along \( L \) in \( S^3 \)

**Definition (Unified WRT invariant)**

Set

\[ J_M = J_{L^0, \omega^{\epsilon_1}, \ldots, \omega^{\epsilon_n}}, \in \widehat{\mathbb{Z}}[q]. \]

(\( L^0 \) : \( L \) with all framings 0.)
Results
Notation

We use the following $q$-integer notations:

\[
\{i\} = q^{\frac{i}{2}} - q^{-\frac{i}{2}}, \\
\{i\}_n = \{i\}\{i-1\}\cdots\{i-n+1\}, \\
\{n\}! = \{n\}_n, \\
\binom{i}{n} = \{i\}_n/\{n\}!,
\]

for $i \in \mathbb{Z}, n \geq 0.$
Results

Theorem (S)

Let $K$ be a knot with 0-framing. For $i, j \geq 0$, we have

$$J_{B(K);P'_i,P'_j} = \sum_{l \geq 0} a_{i,j}^{(l)} J_{K;P'_l},$$

where

$$a_{i,j}^{(l)} = \delta_{i,j} (-1)^i \frac{\{2i + 1\}!\{l\}!}{\{2l + 1\}!} \lambda_{l,i},$$

and

$$\lambda_{l,i} = \sum_{k=0}^{l} (-1)^k \binom{2l + 1}{k} \left[ \frac{2l + i - 2k + 1}{2i + 1} \right].$$
Example

\[ \Phi_m \in \mathbb{Z}[q]: m\text{-th cyclotomic polynomial} \]

\[ (\Phi_1 = q - 1, \quad \Phi_2 = q + 1, \quad \Phi_3 = q^2 + q + 1) \]

\[ J_{M_n; P'_1, \ldots, P'_1} = (-1)^n q^{-2n+4} \Phi_1^{n-2} \Phi_2^{n-2} \Phi_3 \Phi_4^{n-3} \]
Theorem (S)

Let $K$ be a knot with 0-framing and $M$ the integral homology sphere obtained by surgery along $B(K)$ with $\pm 1$ framing in $S^3$. We have

$$J_M - 1 \in \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_6 \mathbb{Z}[q].$$