On the universal sl_2 invariant of ribbon bottom tangles

Sakie Suzuki*

1 Introduction.

For each ribbon Hopf algebra H, Reshetikhin and Turaev [14] defined invariants of framed links colored by finite dimensional representations. A *universal invariant* [8, 9, 13] associated to H is an invariant of framed tangles and links, without using representations. The universal invariant has a universality property such that the colored link invariants constructed by Reshetikhin and Turaev are obtained from the universal invariants by taking trace in the representations attached to the components of links.

A quantized enveloping algebra $U_h(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ is a complete ribbon Hopf $\mathbb{Q}[[h]]$-algebra. By the *universal sl_2 invariant*, we mean the universal invariant associated to the quantized enveloping algebra $U_h := U_h(sl_2)$ of the Lie algebra sl_2. In [3], Habiro studied the universal invariant of *bottom tangles* (see Section 2, Figure 1) associated to an arbitrary ribbon Hopf algebra, and in [4], he studied the universal sl_2 invariant of bottom tangles. The universal sl_2 invariant of an n-component bottom tangle takes values in the n-fold completed tensor power $U_h^{\hat{\otimes} n}$ of U_h. For every oriented, ordered, framed link L, there is a bottom tangle whose *closure* is L (see Figure 2). The universal invariant of bottom tangles has a universality property such that the colored link invariants of a link L is obtained from the universal invariant of a bottom tangle T whose closure is L, by taking the quantum trace in the representations attached to the components of links. In particular, one can obtain the colored Jones polynomials of links from the universal sl_2 invariants of bottom tangles.

An n-component link L is called a *ribbon link* if it bounds a system of n ribbon disks in S^3 (see Section 2, Figure 3). A *ribbon bottom tangle* is defined as a bottom tangle whose closure is a ribbon link. In this paper, we study the universal sl_2 invariant of ribbon bottom tangles.

An n-component link $L = L_1 \cup \cdots \cup L_n$ is called a *boundary link* if it bounds a disjoint union of n orientable surfaces $F = F_1 \cup \cdots \cup F_n$ in S^3 so that L_i bounds F_i for $i = 1, \ldots, n$. There is certain similarities between ribbon links and boundary links. Eisermann [2] proved that the Jones polynomial $V(L) \in \mathbb{Z}[v, v^{-1}]$ of an n-component ribbon link L is divisible by the Jones polynomial $V(O^n) = (v + v^{-1})^n$ of the n-component unlink O^n, and Habiro [5] proved a similar result for boundary links. It is well known that all the Milnor $\bar{\mu}$-invariants [11, 12] vanish both for ribbon links and for boundary links. Also, both ribbon links and boundary links are link homotopic to unlinks, and, moreover, self

*sakie@kurims.kyoto-u.ac.jp

delta-equivalent to unlinks [15, 16].

1.1 Main result.

Set $v = \exp\frac{h}{2}$, $q = v^2$. We have $\mathbb{Z}[q,q^{-1}] \subset \mathbb{Z}[v,v^{-1}] \subset \mathbb{Q}[[h]]$. Let J_T denote the universal sl_2 invariant of a bottom tangle T.

In [4, Section 4], Habiro proved that, the universal sl_2 invariant J_T of an n-component, algebraically-split, 0-framed bottom tangle T is contained in a certain $\mathbb{Z}[q,q^{-1}]$-subalgebra $(\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}$ of $U_h^{\hat{\otimes} n}$ (see Subsection 7.3). In [4, Section 8], he defined another $\mathbb{Z}[q,q^{-1}]$-subalgebra $(\bar{U}_q^{ev})^{\sim\tilde{\otimes} n} \subset (\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}$ of $U_h^{\hat{\otimes} n}$ for $n \geq 0$, and stated the following.

Conjecture 1.1 (Habiro [4])**.** *Let $T \in BT_n$ be an n-component boundary bottom tangle with 0 framing. Then we have $J_T \in (\bar{U}_q^{ev})^{\sim\tilde{\otimes} n}$.*

Here, a *boundary bottom tangle* is a bottom tangle (cf. [3]) whose closure is a boundary link. We shall define a $\mathbb{Z}[q,q^{-1}]$-subalgebra $\hat{Q}_0^{(n)} \subset (\bar{U}_q^{ev})^{\sim\tilde{\otimes} n}$ of $U_h^{\hat{\otimes} n}$ for $n \geq 0$. The main result of the present paper is the following.

Theorem 1.2. *Let $T \in BT_n$ be an n-component ribbon bottom tangle with 0 framing. Then we have $J_T \in \hat{Q}_0^{(n)}$.*

1.2 Applications.

Here, we give several applications of Theorem 1.2 to the colored Jones polynomials. For finite dimensional representations $W_1, \ldots, W_n$ of U_h, we denote by $J_{L;W_1,\ldots,W_n}$ the colored Jones polynomial of an n-component, oriented, ordered, framed link L.

For $l \geq 1$, let V_l denote the l-dimensional irreducible representation of U_h. For a commutative ring $\mathcal{A}$, let $\mathcal{R}_{\mathcal{A}}$ denote the representation ring of U_h over $\mathcal{A}$, i.e., $\mathcal{R}_{\mathcal{A}}$ is the $\mathcal{A}$-algebra

$$\mathcal{R}_{\mathcal{A}} = \mathrm{Span}_{\mathcal{A}}\{V_l \mid l \geq 1\}$$

with the multiplication induced by the tensor product. Habiro [4] studied the following polynomials in V_2

$$P_l = \prod_{i=0}^{l-1}(V_2 - v^{2i+1} - v^{-2i-1}) \in \mathcal{R}_{\mathbb{Z}[v,v^{-1}]},$$
$$\tilde{P}_l' = v^l \frac{P_l}{\{l\}_q!} \in \mathcal{R}_{\mathbb{Q}(v)},$$

for $l \geq 0$. (See Subsection 3.1 for $\{l\}_q!$ and similar notations appearing below.) For $y_1, \ldots, y_n \in \mathcal{R}_{\mathbb{Q}(v)}$, we also consider the colored Jones polynomial $J_{L;y_1,\ldots,y_n}$ of an n-component link L (see Subsection 8.1). Habiro [4] proved the following theorem.

Theorem 1.3 (Habiro [4])**.** *Let L be an n-component, algebraically-split, 0-framed link. We have*

$$J_{L;\tilde{P}_{l_1}',\ldots,\tilde{P}_{l_n}'} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q}\mathbb{Z}[q,q^{-1}],$$

for $l_1, \ldots, l_n \geq 0$, where j is a number such that $l_j = \max\{l_i\}_{1\leq i\leq n}$.

Assuming Conjecture 1.1, Habiro [4] also proved the following theorem with a ribbon link replaced by a boundary link. Thus, this theorem follows from Theorem 1.2, the inclusion $\hat{Q}_0^{(n)} \subset (\bar{U}_q^{ev})^{\tilde{\otimes} n}$, and Habiro's argument in [4].

Theorem 1.4. *Let L be an n-component ribbon link with 0 framing. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q} I_{l_1} \cdots \hat{I}_{l_j} \cdots I_{l_n},$$

for $l_1,\dots,l_n \geq 0$, where j is a number such that $l_j = \max\{l_i\}_{1\leq i\leq n}$. Here, for $l \geq 0$, I_l is the ideal in $\mathbb{Z}[q,q^{-1}]$ generated by the elements $\{l-k\}_q!\{k\}_q!$ for $k=0,\dots,l$, and $\hat{I}_{l_j}$ denotes omission of I_{l_j}.

For every n-component link L, let $V(L) = J_{L;V_2,\dots,V_2} \in \mathbb{Z}[v,v^{-1}]$ be the Jones polynomial of L normalized so that $V(\emptyset) = 1$. Eisermann [2] proved the following theorem by using the Kauffman bracket. (In [2], a different normalization $V(\text{unknot}) = 1$ is used.)

Theorem 1.5 (Eisermann [2])**.** *For every n-component ribbon link L, we have*

$$V(L) \in (v+v^{-1})^n \mathbb{Z}[v,v^{-1}].$$

Note that $(v+v^{-1})^n$ is the Jones polynomial of the n-component unlink. We prove the following generalization of Theorem 1.5, using the universal sl_2 invariant.

Theorem 1.6. *For every n-component ribbon link L with 0 framings, we have*

$$J_{L;V_{2l_1},\dots,V_{2l_n}} \in (v+v^{-1})^n \mathbb{Z}[q,q^{-1}],$$

for $l_1,\dots,l_n \geq 1$.

We also prove the following result. A similar result for boundary links is stated in[4].

Theorem 1.7. *For every n-component ribbon link L with 0 framings, we have*

$$J_{L;P_{l_1},\cdots,P_{l_n}} \in (v+v^{-1})^{l_1+\cdots+l_n} \mathbb{Z}[q,q^{-1}],$$

for $l_1,\dots,l_n \geq 0$.

Remark 1.8. *Theorem 1.7 does not follow from Theorem 1.4. For example, for a 3-component ribbon link L with 0-framing, Theorem 1.4 implies*

$$J_{L;P_1,P_1,P_1} \in (v-v^{-1})^8 (v+v^{-1})(v^2+1+v^{-2}) \mathbb{Z}[q,q^{-1}],$$

and Theorem 1.7 implies

$$J_{L;P_1,P_1,P_1} \in (v+v^{-1})^3 \mathbb{Z}[q,q^{-1}].$$

Consequently, we have

$$J_{L;P_1,P_1,P_1} \in (v-v^{-1})^8 (v+v^{-1})^3 (v^2+1+v^{-2}) \mathbb{Z}[q,q^{-1}].$$

Remark 1.9. *One can obtain Theorems 1.6 and 1.7 from Theorem 1.5, without using the universal sl_2 invariant. See Remark 8.7 for details.*

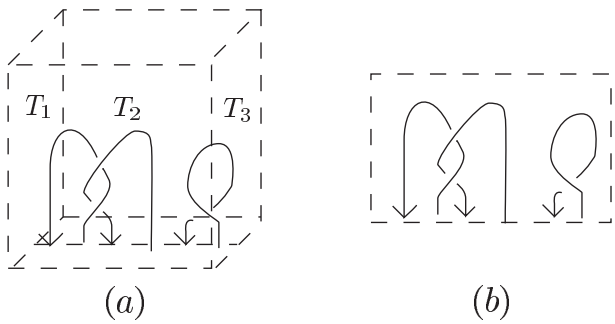

Figure 1: (a) A 3-component bottom tangle $T = T_1 \cup T_2 \cup T_3$. (b) A diagram of T in rectangle.

1.3 Organization of the paper.

The rest of the paper is organized as follows. In Section 2, we define bottom tangles and ribbon bottom tangles. In Section 3, we define the quantum enveloping algebra U_h, which is an h-adically complete ribbon Hopf $\mathbb{Q}[[h]]$-algebra. Then we define $\mathbb{Z}[q, q^{-1}]$-subalgebras $U_{\mathbb{Z},q}$, $U^{ev}_{\mathbb{Z},q}$, $\bar{U}_q$, and $\bar{U}^{ev}_q$ of U_h. In Section 4, we define the universal sl_2 invariant of bottom tangles. Then we discuss the universal sl_2 invariants of ribbon bottom tangles. In Section 5, we define $\mathbb{Z}[q, q^{-1}]$-submodules $Q^{(k)}_b$ $(k \geq 0, b \in \mathbb{Z})$ of $U_h^{\hat{\otimes} k}$. In fact, $Q^{(k)}_0$ $(k \geq 0)$ are $\mathbb{Z}[q, q^{-1}]$-subalgebras of $U_h^{\hat{\otimes} k}$. In Sections 6 and 7, we prove Theorem 1.2. Then we prove the inclusion $\hat{Q}^{(k)}_0 \subset (\bar{U}^{ev}_q)^{\tilde{\otimes} k}$ for $k \geq 0$. In Section 8, we prove Theorems 1.6 and 1.7. In Section 9, we consider the cases of the Borromean tangle and the Borromean rings.

2 Bottom tangles.

In this section, we recall from [3] the notions of *bottom tangles* and *ribbon bottom tangles*.

2.1 Bottom tangles.

An n-component *bottom tangle* $T = T_1 \cup \cdots \cup T_n$ is an oriented, ordered, framed tangle in a cube $[0, 1]^3$ consisting of n arcs $T_1, \ldots, T_n$, whose boundaries are in bottom $[0, 1] \times \{\frac{1}{2}\} \times \{0\}$, such that for each $i = 1, \ldots, n$, the component T_i runs from the $2i$th boundary points to the $(2i - 1)$th boundary points, where the boundary points are counted by the first coordinate. The $2i$th (resp. $(2i - 1)$th) boundary point is called the *start point* (resp. *end point*) of T_i. For simplicity, we draw a bottom tangle as a diagram in a rectangle as usual, see Figure 1. For each $n \geq 0$, let BT_n denote the set of the isotopy classes of n-component bottom tangles, and set $BT = \bigcup_{n \geq 0} BT_n$.

The *closure* of T is the link obtained from T by pasting a "$\cup$-shaped tangle" to each component of T, as depicted in Figure 2. There is a natural closure function $\mathrm{cl}\colon BT \to \mathcal{L}$, where we denote by $\mathcal{L}$ the set of the isotopy classes of oriented, ordered,

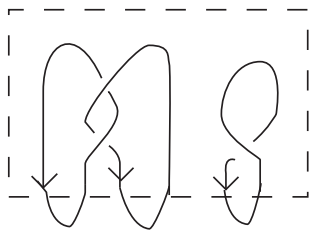

Figure 2: The closure of T.

framed links in S^3. The function $\mathrm{cl}\colon\ BT \to \mathcal{L}$ is surjective, i.e., for any link $L \in \mathcal{L}$, there is a bottom tangle $T \in BT$ such that $\mathrm{cl}(T) = L$.

The linking matrix $M(T)$ of a bottom tangle $T = T_1 \cup \cdots \cup T_n$ is defined as that of the closure of T. Thus, for $1 \leq i \neq j \leq n$, the linking number of T_i and T_j is defined as the linking number of the corresponding components in the closure of T, and, for $1 \leq i \leq n$, the framing of T_i is defined as the framing of the closure of T_i.

2.2 Ribbon bottom tangles.

Let us recall the definition of ribbon links. A system of n *ribbon disks* in S^3 is the image of a ribbon immersion

$$f\colon\ D_1 \cup \cdots \cup D_n \to S^3$$

of the disjoint union of n disks $D_1, \ldots, D_n$ into S^3. A *ribbon immersion* is an immersion whose singularities are ribbon singularities. A *ribbon singularity* A is a singularity whose preimage consists of two arcs, one of which is interior, see Figure 3 (a). An n-component link in S^3 is called a *ribbon link* if and only if it bounds a system of n ribbon disks in S^3. For example, see Figure 3 (b).

Definition 2.1. *A bottom tangle $T \in BT$ is called a ribbon bottom tangle if and only if the closure of T is a ribbon link.*

3 The quantized enveloping algebra U_h and its subalgebras.

We mostly follow the notations in [4].

3.1 q-integers.

Let h be an indeterminate, and let $\mathbb{Q}[[h]]$ denote the formal power series ring. Set

$$v = \exp\frac{h}{2} \in \mathbb{Q}[[h]], \quad q = v^2 = \exp h \in \mathbb{Q}[[h]].$$

We have $\mathbb{Z}[q, q^{-1}] \subset \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}[[h]]$.

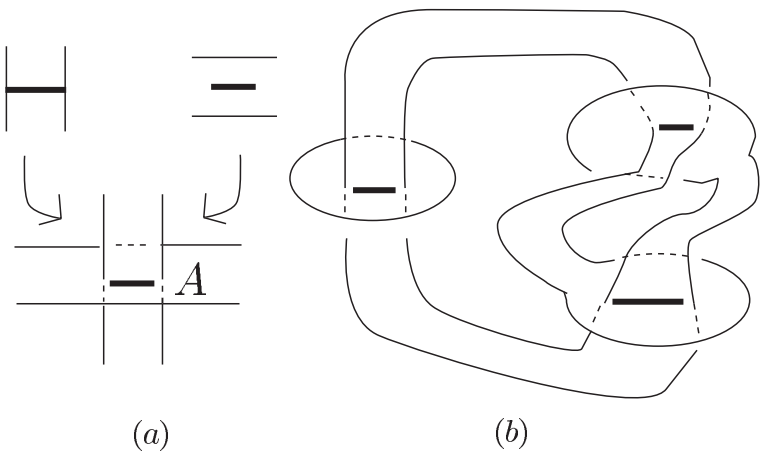

Figure 3: (a) A ribbon singularity. (b) A 2-component ribbon link.

We use the following q-integer notations.

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q\{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$
$$[i]_q = \{i\}_q/\{1\}_q, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n}/\{n\}_q!,$$

for $i \in \mathbb{Z}, n \geq 0$. These are elements in $\mathbb{Z}[q, q^{-1}]$. Sometimes we also use the following notations.

$$\{i\} = v^i - v^{-i}, \quad \{i\}_n = \{i\}\{i-1\} \cdots \{i-n+1\}, \quad \{n\}! = \{n\}_n,$$
$$[i] = \{i\}/\{1\}, \quad [n]! = [n][n-1] \cdots [1], \quad \begin{bmatrix} i \\ n \end{bmatrix} = \{i\}_n/\{n\}!,$$

for $i \in \mathbb{Z}, n \geq 0$. These are elements in $\mathbb{Z}[q, q^{-1}] \sqcup v\mathbb{Z}[q, q^{-1}] \subset \mathbb{Z}[v, v^{-1}]$. These two families of notations are the same up to multiplication by powers of v.

3.2 The quantized enveloping algebra U_h.

We denote by U_h the h-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by the elements H, E, and F, satisfying the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$

where we set

$$K = v^H = \exp \frac{hH}{2}.$$

For $p \in \mathbb{Z}$, let $\Gamma_p(U_h)$ denote the complete $\mathbb{Q}[[h]]$-submodule of U_h topologically spanned by the elements $F^iH^jE^k$ with $i,j,k \geq 0, k-i=p$. This gives a topological $\mathbb{Z}$-graded algebra structure for U_h

$$U_h = \big(\bigoplus_{p\in\mathbb{Z}} \Gamma_p(U_h)\big)\hat{}\,, \tag{1}$$

where $()\hat{}$ denotes h-adic completion. The elements of $\Gamma_p(U_h)$ are said to be homogeneous of degree p. For a homogeneous element x of U_h, the degree of x is denoted by $|x|$.

The algebra U_h has a complete ribbon Hopf algebra structure with the comultiplication $\Delta\colon\ U_h \to U_h\hat{\otimes}U_h$, the counit $\varepsilon\colon\ U_h \to \mathbb{Q}[[h]]$ and the antipode $S\colon\ U_h \to U_h$ defined by

$$\begin{aligned}
&\Delta(H) = H\otimes 1 + 1\otimes H, \quad \varepsilon(H)=0, \quad S(H) = -H,\\
&\Delta(E) = E\otimes 1 + K\otimes E, \quad \varepsilon(E)=0, \quad S(E) = -K^{-1}E,\\
&\Delta(F) = F\otimes K^{-1} + 1\otimes F, \quad \varepsilon(F)=0, \quad S(F) = -FK,
\end{aligned}$$

where $\hat{\otimes}$ denotes the h-adically completed tensor product. The universal R-matrix and its inverse are given by

$$R = D\Big(\sum_{n\geq 0} v^{\frac{1}{2}n(n-1)}\frac{(v-v^{-1})^n}{[n]!}F^n\otimes E^n\Big), \tag{2}$$

$$R^{-1} = \Big(\sum_{n\geq 0}(-1)^n v^{-\frac{1}{2}n(n-1)}\frac{(v-v^{-1})^n}{[n]!}F^n\otimes E^n\Big)D^{-1}, \tag{3}$$

where $D = v^{\frac{1}{2}H\otimes H} = \exp\big(\frac{1}{4}H\otimes H\big) \in U_h^{\hat{\otimes}2}$. The ribbon element and its inverse are given by

$$r = \sum \bar{\alpha}K^{-1}\bar{\beta} = \sum \bar{\beta}K\bar{\alpha}, \quad r^{-1} = \sum \alpha K\beta = \sum \beta K^{-1}\alpha,$$

where $R = \sum \alpha\otimes\beta$, and $R^{-1} = (S\otimes 1)R = \sum \bar{\alpha}\otimes\bar{\beta}$.

We use notations $D = \sum D^+_{[1]}\otimes D^+_{[2]}$, and $D^{-1} = \sum D^-_{[1]}\otimes D^-_{[2]}$. We shall use the following formulas associated to D.

$$\sum D^+_{[2]}\otimes D^+_{[1]} = D, \quad (\Delta\otimes 1)D = D_{13}D_{23}, \tag{4}$$

$$(\varepsilon\otimes 1)(D) = 1, \quad (1\otimes S)D = (S\otimes 1)D = D^{-1}, \tag{5}$$

$$D(1\otimes x) = (K^{|x|}\otimes x)D, \tag{6}$$

where $D_{13} = \sum D^+_{[1]}\otimes 1\otimes D^+_{[2]}$, $D_{23} = 1\otimes D$, and x is a homogeneous element of U_h.

3.3 Subalgebras $U_{\mathbb{Z},q}$ and $U^{ev}_{\mathbb{Z},q}$ of U_h.

Let $U_{\mathbb{Z}}$ denote the Lusztig's integral form of U_h (cf. [10]), which is defined to be the $\mathbb{Z}[v,v^{-1}]$-subalgebra of U_h generated by K, K^{-1}, $E^{(n)} = E^n/[n]!$, and $F^{(n)} = F^n/[n]!$ for $n \geq 1$.

Set

$$\tilde{E}^{(n)} = (v^{-1}E)^n/[n]_q! = v^{-\frac{1}{2}n(n+1)}E^{(n)},$$
$$\tilde{F}^{(n)} = F^n K^n/[n]_q! = v^{-\frac{1}{2}n(n-1)}F^{(n)}K^n,$$

for $n \geq 0$. Let $U_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_h generated by $K, K^{-1}, \tilde{E}^{(n)}$, and $\tilde{F}^{(n)}$ for $n \geq 1$. Note that

$$U_{\mathbb{Z}} = U_{\mathbb{Z},q} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[v,v^{-1}].$$

Let $U^{ev}_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by $K^2, K^{-2}, \tilde{E}^{(n)}$, and $\tilde{F}^{(n)}$ for $n \geq 1$. $U_{\mathbb{Z},q}$ is equipped with a $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{Z}[q,q^{-1}]$-algebra structure

$$U_{\mathbb{Z},q} = U^{ev}_{\mathbb{Z},q} \oplus KU^{ev}_{\mathbb{Z},q}.$$

$U_{\mathbb{Z},q}$ inherits from U_h a Hopf $\mathbb{Z}[q,q^{-1}]$-algebra structure such that

$$\Delta(K^i) = K^i \otimes K^i, \quad S^{\pm 1}(K^i) = K^{-i}, \tag{7}$$
$$\Delta(\tilde{E}^{(n)}) = \sum_{j=0}^{n} \tilde{E}^{(n-j)}K^j \otimes \tilde{E}^{(j)}, \quad \Delta(\tilde{F}^{(n)}) = \sum_{j=0}^{n} \tilde{F}^{(n-j)}K^j \otimes \tilde{F}^{(j)}, \tag{8}$$
$$S^{\pm 1}(\tilde{E}^{(n)}) = (-1)^n q^{\frac{1}{2}n(n\mp 1)}K^{-n}\tilde{E}^{(n)}, \quad S^{\pm 1}(\tilde{F}^{(n)}) = (-1)^n q^{-\frac{1}{2}n(n\mp 1)}K^{-n}\tilde{F}^{(n)}, \tag{9}$$
$$\varepsilon(K^i) = 1, \quad \varepsilon(\tilde{E}^{(n)}) = \varepsilon(\tilde{F}^{(n)}) = \delta_{n,0}, \tag{10}$$

for $n \geq 0$.

We have

$$K\tilde{E}^{(m)} = q^m \tilde{E}^{(m)}K, \quad K\tilde{F}^{(m)} = q^{-m}\tilde{F}^{(m)}K, \tag{11}$$
$$\tilde{E}^{(m)}\tilde{F}^{(n)} = \sum_{p=0}^{\min(m,n)} q^{-n(m-p)}\tilde{F}^{(n-p)} \begin{bmatrix} H-m-n+2p \\ p \end{bmatrix}_q \tilde{E}^{(m-p)}, \tag{12}$$

for $m, n \geq 0$. Here, for $i \in \mathbb{Z}$ and $p \geq 0$, we set

$$\begin{bmatrix} H+i \\ p \end{bmatrix}_q = \{H+i\}_{q,p}/\{p\}_q!,$$

where

$$\{H+i\}_{q,p} = \{H+i\}_q\{H+i-1\}_q \cdots \{H+i-p+1\}_q,$$

and

$$\{H+j\}_q = q^{H+j} - 1 = q^j K^2 - 1,$$

for $j \in \mathbb{Z}$.

Let $U^0_{\mathbb{Z},q}$ denote the $\mathbb{Z}[v,v^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by K, K^{-1}, and $\begin{bmatrix} H+i \\ p \end{bmatrix}_q$ for $i \in \mathbb{Z}$, $p \geq 0$. Let $U^{ev0}_{\mathbb{Z},q}$ denote the $\mathbb{Z}[v,v^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by K^2, K^{-2}, and $\begin{bmatrix} H+i \\ p \end{bmatrix}_q$ for $i \in \mathbb{Z}$, $p \geq 0$. The following lemma, which is a $\mathbb{Z}[q,q^{-1}]$-version of a well known result for $U_{\mathbb{Z}}$ (cf. [10]), can be proved by using the formulas (11) and (12).

Lemma 3.1. *$U_{\mathbb{Z},q}$ (resp. $U^{ev}_{\mathbb{Z},q}$) is $\mathbb{Z}[q,q^{-1}]$-spanned by the elements $\tilde{F}^{(i)} g \tilde{E}^{(j)}$ with $i,j \geq 0$ and $g \in U^0_{\mathbb{Z},q}$ (resp. $g \in U^{ev0}_{\mathbb{Z},q}$).*

3.4 Subalgebras $\bar{U}_q$ and $\bar{U}^{ev}_q$ of U_h.

Let $\bar{U}$ denote the $\mathbb{Z}[v,v^{-1}]$-subalgebra of U_h generated by the elements K, K^{-1}, $(v - v^{-1})E$, and $(v - v^{-1})F$.

Set

$$e = v^{-1}(q-1)E, \quad f = (q-1)FK.$$

Let $\bar{U}_q$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by the elements K, K^{-1}, e and f. Note that

$$\bar{U} = \bar{U}_q \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[v,v^{-1}].$$

Let $\bar{U}^{ev}_q$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U^{ev}_{\mathbb{Z},q}$ generated by the elements K^2, K^{-2}, e and f. $\bar{U}^{ev}_q$ is equipped with a $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{Z}[q,q^{-1}]$-algebra structure

$$\bar{U}_q = \bar{U}^{ev}_q \oplus K\bar{U}^{ev}_q.$$

Let $\bar{U}^0_q$ (resp. $\bar{U}^{ev0}_q$) denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_q generated by K, K^{-1} (resp. K^2, K^{-2}).

$\bar{U}_q$ inherits from U_h a Hopf $\mathbb{Z}[q,q^{-1}]$-algebra structure such that

$$\Delta(e^n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q e^{n-j}K^j \otimes e^j, \quad \Delta(f^n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{-j(n-j)} f^{n-j}K^j \otimes f^j, \tag{13}$$

$$S^{\pm 1}(e^n) = (-1)^n q^{\frac{1}{2}n(n\mp 1)} K^{-n} e^n, \quad S^{\pm 1}(f^n) = (-1)^n q^{-\frac{1}{2}n(n\mp 1)} K^{-n} f^n, \tag{14}$$

$$\varepsilon(e^n) = \varepsilon(f^n) = \delta_{n,0}, \tag{15}$$

for $n \geq 0$.

We have

$$Ke = qeK, \quad Kf = q^{-1}fK, \tag{16}$$

$$e^m f^n = \sum_{p=0}^{\min(m,n)} q^{\frac{1}{2}p(p+1)-nm} \{p\}_q! \begin{bmatrix} m \\ p \end{bmatrix}_q \begin{bmatrix} n \\ p \end{bmatrix}_q f^{n-p} \{H - m - n + 2p\}_{q,p} e^{m-p}, \tag{17}$$

for $m, n \geq 0$.

The following lemma, which is a $\mathbb{Z}[q,q^{-1}]$-version of a well known result for $\bar{U}$ by De Concini and Procesi [1], can be proved by using the formula (16) and (17).

Lemma 3.2. *$\bar{U}_q$ (resp. $\bar{U}_q^{ev}$) is freely $\mathbb{Z}[q,q^{-1}]$-spanned by the elements $f^iK^je^k$ (resp. $f^iK^{2j}e^k$) with $i,k\geq 0$ and $j\in\mathbb{Z}$.*

We can rewrite the R-matrix (2) and its inverse (3) as

$$R^{\pm 1}=D^{\pm 1}\sum_{n\geq 0}R_n^{\pm}, \tag{18}$$

where $R_n^{\pm}\in(U_{\mathbb{Z},q}\otimes\bar{U}_q)\cap(\bar{U}_q\otimes U_{\mathbb{Z},q})$ are defined by

$$\begin{aligned} R_n^+ &= q^{\frac{1}{2}n(n-1)}\tilde{F}^{(n)}K^{-n}\otimes e^n \\ &= q^{n(n-1)}f^nK^{-n}\otimes\tilde{E}^{(n)}, \end{aligned} \tag{19}$$

$$\begin{aligned} R_n^- &= (-1)^n\tilde{F}^{(n)}\otimes K^{-n}e^n \\ &= (-1)^nq^{\frac{1}{2}n(n-1)}f^{(n)}\otimes K^{-n}\tilde{E}^{(n)}. \end{aligned} \tag{20}$$

We use the notations $R_n=\sum R_{n[1]}^+\otimes R_{n[2]}^+$ and $R_n^-=\sum R_{n[1]}^-\otimes R_{n[2]}^-$.

3.5 Adjoint action.

We regard U_h as a left U_h-module via the left adjoint action defined by

$$\mathrm{ad}(a\otimes b):=\sum a'bS(a''),$$

where $\Delta(a)=\sum a'\otimes a''$. We also use the notation $a\triangleright b:=\mathrm{ad}(a\otimes b)$.

For each homogeneous element $x\in U_h$, we have

$$K^l\triangleright x=q^{l|x|}x, \tag{21}$$

$$\tilde{E}^{(n)}\triangleright x=\sum_{j=0}^{n}(-1)^jq^{\frac{1}{2}j(j-1)+j|x|}\tilde{E}^{(n-j)}x\tilde{E}^{(j)}, \tag{22}$$

$$\tilde{F}^{(n)}\triangleright x=\sum_{j=0}^{n}(-1)^jq^{-\frac{1}{2}j(j-1)+j|x|}\tilde{F}^{(n-j)}x\tilde{F}^{(j)}, \tag{23}$$

for $l\in\mathbb{Z},n\geq 0$. We shall use the following lemma.

Lemma 3.3. *For $a,b\in U_h$, we have*

$$\begin{aligned} U_{\mathbb{Z},q}\triangleright ab &\subset (U_{\mathbb{Z},q}\triangleright a)(U_{\mathbb{Z},q}\triangleright b), \\ \bar{U}_q\triangleright ab &\subset (\bar{U}_q\triangleright a)(\bar{U}_q\triangleright b). \end{aligned}$$

Proof. The assertion follows from the Hopf algebra structures of $U_{\mathbb{Z},q}$ and $\bar{U}_q$ and the formula

$$x\triangleright ab=\sum(x'\triangleright a)(x''\triangleright b),$$

where $\Delta(x)=\sum x'\otimes x''$. □

The following proposition is suggested by Habiro. In fact, Habiro and Le [6] prove a generalization of a $\mathbb{Z}[v, v^{-1}]$-version of (i) of the following proposition to quantized enveloping algebras for all simple Lie algebras.

Proposition 3.4. *We have*

$$(i)\ \ U_{\mathbb{Z},q} \triangleright \bar{U}_q^{ev} \subset \bar{U}_q^{ev}, \quad \textit{and} \quad (ii)\ \ U_{\mathbb{Z},q} \triangleright K\bar{U}_q^{ev} \subset K\bar{U}_q^{ev}.$$

Proof. We prove (i), then (ii) is similar. In view of Lemma 3.2, it is enough to prove that $x \triangleright f^{i_1}K^{2i_2}e^{i_3} \in \bar{U}_q^{ev}$ for every $x \in \{K, K^{-1}, \tilde{E}^{(n)}, \tilde{F}^{(n)} \mid n \geq 0\}$ and $i_1, i_3 \geq 0$, $i_2 \in \mathbb{Z}$. Using the formulas (21)–(23), we have

$$K^{\pm 1} \triangleright f^{i_1}K^{2i_2}e^{i_3} = q^{\pm(i_3-i_1)}f^{i_1}K^{2i_2}e^{i_3}, \tag{24}$$

$$\begin{aligned}
&\tilde{E}^{(n)} \triangleright f^{i_1}K^{2i_2}e^{i_3} \\
&\quad = \sum_{p=0}^{\min(i_1,n)} (-1)^n q^{\frac{1}{2}p(p+1)-n(i_1+2i_2)+2i_2p} \begin{bmatrix} i_1 \\ p \end{bmatrix}_q f^{i_1-p}K^{2i_2}g(i_1,i_2,i_3,n,p)e^{i_3+n-p},
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\tilde{F}^{(n)} \triangleright f^{i_1}K^{2i_2}e^{i_3} \\
&\quad = \sum_{p=0}^{\min(i_3,n)} q^{\frac{1}{2}p(p+1)-n(i_1+2i_2)+2i_2p} \begin{bmatrix} i_3 \\ p \end{bmatrix}_q f^{n+i_1-p}K^{2i_2}g(i_3,i_2,i_1,n,p)e^{i_3-p},
\end{aligned} \tag{26}$$

where

$$g(i_1,i_2,i_3,n,p) = \sum_{s=0}^{p}(-1)^s q^{\frac{1}{2}s(s+1)-s(n-p+i_1)} \begin{bmatrix} p \\ s \end{bmatrix}_q \begin{bmatrix} n-p+2i_2+i_3+s-1 \\ n-p \end{bmatrix}_q K^{2s}. \tag{27}$$

The right hand sides of (24)–(27) are all contained in $\bar{U}_q^{ev}$, hence we have the assertion. □

4 The universal sl_2 invariant of bottom tangles.

In this section, we define the universal sl_2 invariant of bottom tangles by using a notion of *decorated diagrams*. Then we discuss the universal sl_2 invariant of ribbon bottom tangles.

4.1 Decorated diagrams.

A *generic tangle diagram* is a diagram of tangle obtained from copies of the fundamental tangles, see Figure 4, by pasting horizontally and vertically. In what follows, we consider generic tangle diagrams only, hence we call them diagrams. A *decorated diagram* of a bottom tangle $T \in BT$ is a diagram P_T of T together with finitely many dots on strands, each labeled by an element of U_h. We also allow pairs of dots, each connected by an oriented dashed line, which is labeled by an element of $U_h^{\hat{\otimes}2}$ so that the first

Figure 4: Fundamental tangles. The orientations of the strands are arbitrary.

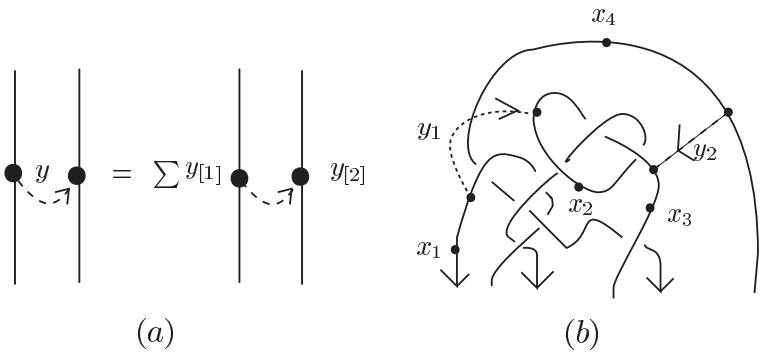

Figure 5: (a) How to label an element $y = \sum y_{[1]} \otimes y_{[2]}$ to the connected dots. (b) A decorated diagram P.

tensorand is attached to the start point of the line, see Figure 5(a). We can ignore the orientation of a dashed line if the element $y = \sum y_{[1]} \otimes y_{[2]} \in U_h^{\hat{\otimes} 2}$ on it is symmetric, i.e., if $\sum y_{[1]} \otimes y_{[2]} = \sum y_{[2]} \otimes y_{[1]}$. For every decorated diagram P_T for an n-component bottom tangle $T = T_1 \cup \cdots \cup T_n \in BT_n$, we define an element $J(P_T) \in U_h^{\hat{\otimes} n}$ as follows. The ith component of $J(P_T)$ is defined to be the product of the elements put on the component corresponding to T_i, where the elements are read off along each component reversing the orientation of P_T, and written from left to right. For example, for the decorated diagram P depicted in Figure 5 (b), we have

$$J(P) = \sum x_1 y_{1[1]} \otimes x_2 y_{1[2]} y_{2[2]} x_3 \otimes x_4 y_{2[1]},$$

where $x_i \in U_h$ and $y_j = \sum y_{j[1]} \otimes y_{j[2]} \in U_h^{\hat{\otimes} 2}$ for each i, j. In what follows, sometimes we identify the decorated diagram and its image of J. For example, the picture depicted in Figure 6 represents the formula (6).

4.2 The universal sl_2 invariant of bottom tangles.

For $T = T_1 \cup \cdots \cup T_n \in BT_n$, we define the universal sl_2 invariant $J_T \in U_h^{\hat{\otimes} n}$ of T as follows. We choose a diagram P_T of T. We denote by $C(P_T)$ the set of the crossings of P_T. We call a map

$$s\colon\ C(P_T) \ \rightarrow\ \{0, 1, 2, \ldots\}$$

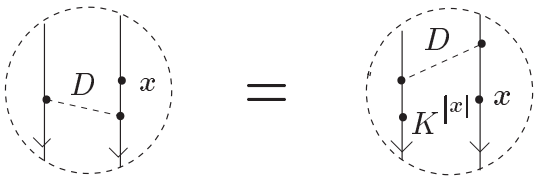

Figure 6: The formula (6): $D(1 \otimes x) = (K^{|x|} \otimes x)D$. Two pictures above imply two decorated diagrams of a bottom tangle which are identical outside the dotted circles.

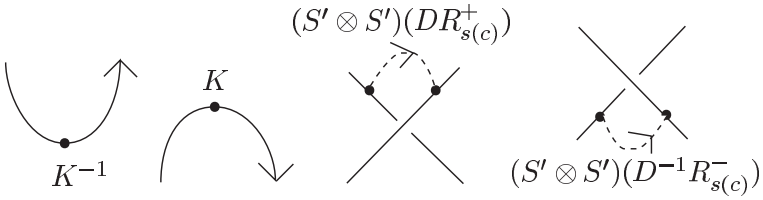

Figure 7: How to place elements in U_h on the fundamental tangles.

a *state*. We denote by $S(P_T)$ the set of states for P_T. For each state $s \in S(P_T)$, we define a decorated diagram (P_T, s) (by abusing the notation) as follows. Recall the notation (18) $R^{\pm 1} = D^{\pm 1} \sum_{l \geq 0} R^{\pm}_l$. For each fundamental tangle in P_T, we attach elements following the rule described in Figure 7, where "S'" should be replaced with id if the string is oriented downward, and by S otherwise, see Figure 8. Set

$$J_T = \sum_{s \in S(P_T)} J(P_T, s).$$

As is well known [13], J_T does not depend on the choice of the diagram P_T, and defines an isotopy invariant of bottom tangles.

For example, let us compute the universal sl_2 invariant J_C of a bottom tangle C as depicted in Figure 9 (a), where c_1 (resp. c_2) denotes the upper (resp. lower) crossing of P_C. The decorated diagram (P_C, s) for the state $s \in S(P)$ is depicted in Figure 9 (b), where we set $m = s(c_1), n = s(c_2)$. We have

$$\begin{aligned} J(P_C, s) &= \sum S(D^+_{[1]} R^+_{m[1]}) S(D'^+_{[2]} R^+_{n[2]}) \otimes D'^+_{[1]} R^+_{n[1]} D^+_{[2]} R^+_{m[2]} \\ &= \sum q^{\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)} S(\tilde{F}^{(m)} K^{-m}) D^-_{[1]} S(e^n) D'^-_{[2]} \otimes D'^-_{[1]} \tilde{F}^{(n)} K^{-n} D^-_{[2]} e^m \\ &= (-1)^{m+n} q^{-n-2mn} D^{-2} (K^{-2n} \tilde{F}^{(m)} e^n \otimes K^{-2m} \tilde{F}^{(n)} e^m), \end{aligned}$$

where $D^{\pm 1} = \sum D^{\pm}_{[1]} \otimes D^{\pm}_{[2]} = \sum D'^{\pm}_{[1]} \otimes D'^{\pm}_{[2]}$. Hence we have

$$\begin{aligned} J_C &= \sum_{s \in S(P_C)} J(P_C, s) \\ &= \sum_{m,n \geq 0} (-1)^{m+n} q^{-n-2mn} D^{-2} (K^{-2n} \tilde{F}^{(m)} e^n \otimes K^{-2m} \tilde{F}^{(n)} e^m). \end{aligned}$$

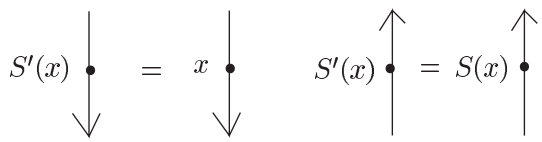

Figure 8: The definition of S'.

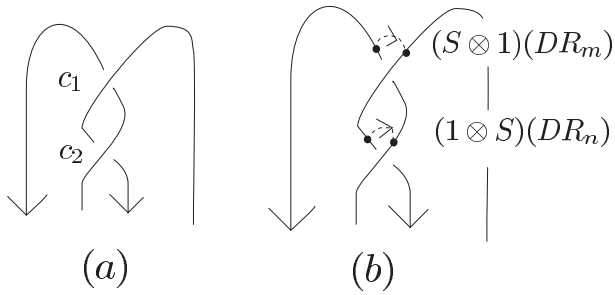

Figure 9: (a) A diagram P_C of $C \in BT_2$. (b) The decorated diagram (P_C, s).

4.3 The universal sl_2 invariant of ribbon bottom tangles.

In this subsection, we discuss the universal sl_2 invariant of ribbon bottom tangles. Habiro [4] studied the universal sl_2 invariant of 1-component ribbon bottom tangles. We generalize those to n-component ribbon bottom tangles for $n \geq 1$.

For $T \in BT_{i+j+2}$, $i, j \geq 0$, let $(\mathrm{ad}_b)_{i,j}(T) \in BT_{i+j+1}$ and $(\mu_b)_{(i,j)}(T) \in BT_{i+j+1}$ denote the bottom tangles as depicted in Figure 10. We use the following lemma.

Lemma 4.1 (Habiro [3]). *For every bottom tangle $T \in BT_{i+j+2}$, $i, j \geq 0$, we have*

$$J_{(\mathrm{ad}_b)_{i,j}(T)} = \mathrm{ad}_{i,j}(J_T),$$
$$J_{(\mu_b)_{i,j}(T)} = \mu_{i,j}(J_T),$$

where we set

$$\mathrm{ad}_{i,j} = \mathrm{id}^{\otimes i} \otimes \mathrm{ad} \otimes \mathrm{id}^{\otimes j} \colon\ U_h^{\hat{\otimes} i+j+2} \to U_h^{\hat{\otimes} i+j+1},$$
$$\mu_{i,j} = \mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes j} \colon\ U_h^{\hat{\otimes} i+j+2} \to U_h^{\hat{\otimes} i+j+1}.$$

For a $2k$-component bottom tangle $W = W_1 \cup \cdots \cup W_{2k} \in BT_{2k}, k \geq 0$, set

$$W^{ev} = \bigcup_{i=1}^{k} W_{2i} \ \text{ and } \ W^{odd} = \bigcup_{i=1}^{k} W_{2i-1}.$$

For a diagram P_W of W, let P_W^{ev} (resp. P_W^{odd}) denote the part of the diagram P_W corresponding to W^{ev} (resp. W^{odd}). We say a bottom tangle $W \in BT_{2k}$ is *even-trivial* if W^{ev} is a trivial bottom tangle, see Figure 11. We also say a diagram P_W of W is

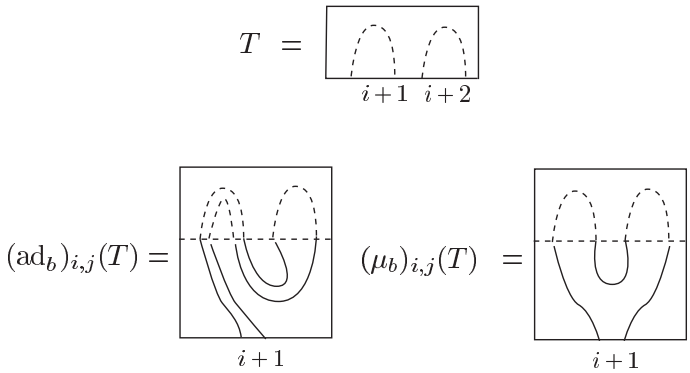

Figure 10: A bottom tangle $T \in BT_{i+j+2}$ and the bottom tangles $(\mathrm{ad}_b)_{(i,j)}(T)$, $(\mu_b)_{(i,j)}(T) \in BT_{i+j+1}$. We depict only the $(i+1), (i+2)$th components of T, and the $(i+1)$th components of $(\mathrm{ad}_b)_{(i,j)}(T)$,$(\mu_b)_{(i,j)}(T)$.

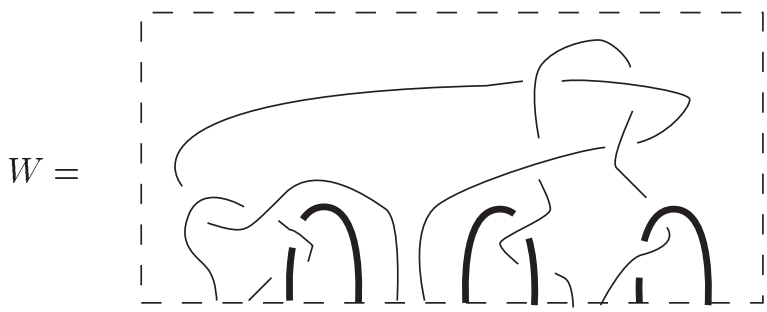

Figure 11: An even-trivial bottom tangle $W \in BT_6$. Here W^{ev} is depicted with thick lines.

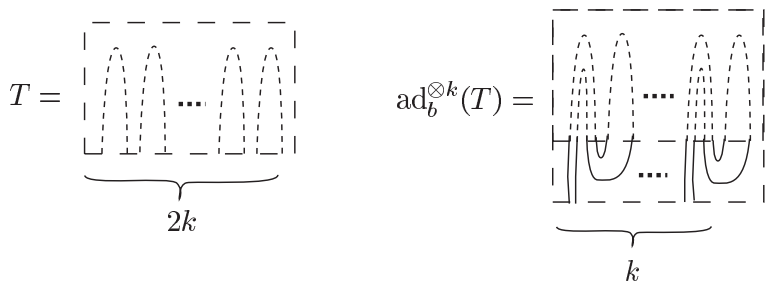

Figure 12: A bottom tangle $T \in BT_{2k}$ and the bottom tangle $\mathrm{ad}_b^{\otimes k}(T) \in BT_k$.

even-trivial if and only if P_W^{ev} has no self crossings. Note that a bottom tangle W has an even-trivial diagram if and only if W is even-trivial.

The following lemma is almost the same as [3, Theorem 11.5].

Lemma 4.2. *For any bottom tangle $T \in BT_n$, the following conditions are equivalent.*

(1) T is a ribbon bottom tangle.

(2) There is an even-trivial bottom tangle $W \in BT_{2k}, k \geq 0$, and there are integers $N_1, \dots, N_n \geq 0$ satisfying $N_1 + \cdots + N_n = k$, such that

$$T = \mu_b^{[N_1,\dots,N_n]} ad_b^{\otimes k}(W), \tag{28}$$

where

$$\mathrm{ad}_b^{\otimes k} \colon\ BT_{2k} \to BT_k$$

is as depicted in Figure 12, and

$$\mu_b^{[N_1,\dots,N_n]} \colon\ BT_{N_1+\cdots+N_n} \to BT_n$$

is as depicted in Figure 13.

If (28) holds, then we call $(W; N_1, \dots, N_n)$ a *ribbon data* for T. For example, for the even-trivial bottom tangle $W \in BT_3$ in Figure 11, the ribbon bottom tangle $\mu^{[1,2,0]}(ad_b)^{\otimes 3}(W) \in BT_3$ is as depicted in Figure 14.

For $n \geq 1$, let

$$\mu^{[n]} \colon\ U_h^{\hat{\otimes} n} \to U_h, \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 x_2 \cdots x_n$$

denote the n-input multiplication. For integers $N_1, \dots, N_n \geq 0$, $N_1 + \cdots + N_n = k$, set

$$\mu^{[N_1,\dots,N_n]} = \mu^{[N_1]} \otimes \cdots \otimes \mu^{[N_n]} \colon\ U_h^{\hat{\otimes} k} \to U_h^{\hat{\otimes} n}.$$

By Lemmas 4.1 and 4.2, we have the following proposition.

Proposition 4.3. *Let $T \in BT_n$ be a ribbon bottom tangle and $(W \in BT_{2k}; N_1, \dots, N_n)$ a ribbon data for T. Then we have*

$$J_T = \mu^{[N_1,\dots,N_n]}\, \mathrm{ad}^{\otimes k}(J_W).$$

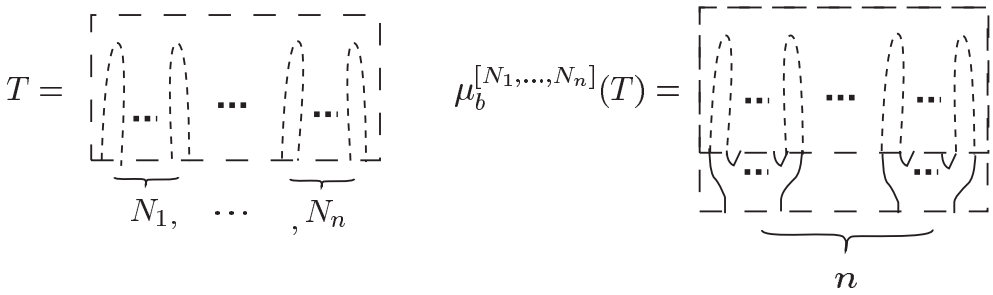

Figure 13: A bottom tangle $T \in BT_k$ and the bottom tangle $\mu_b^{[N_1,\ldots,N_n]}(T) \in BT_n$.

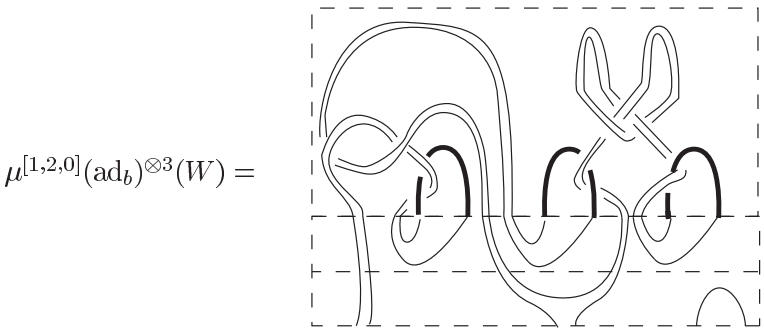

Figure 14: The ribbon bottom tangle $\mu^{[1,2,0]}(ad_b)^{\otimes 3}(W) \in BT_3$ for the even-trivial bottom tangle $W \in BT_3$ in Figure 11.

5 The $\mathbb{Z}[q,q^{-1}]$-subalgebra $Q_0^{(k)}$ of $(\bar{U}_q^{ev})^{\otimes k}$.

In this section we define a $\mathbb{Z}[q,q^{-1}]$-subalgebra $Q_0^{(k)}$ of $(\bar{U}_q^{ev})^{\otimes k}$ for $k \geq 0$. Proposition 5.2 is the main result in this section, which we use later in the proof of Theorem 1.2.

5.1 The $\mathbb{Z}[q,q^{-1}]$-submodule $Q_b^{(k)}$ of $(\bar{U}_q^{ev})^{\otimes k}$.

In this subsection, we define $\mathbb{Z}[q,q^{-1}]$-submodules $Q_b^{(k)}$ ($b \in \mathbb{Z}$) of $(\bar{U}_q^{ev})^{\otimes k}$ for $k \geq 0$. In fact, $Q_0^{(k)}$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebras of $(\bar{U}_q^{ev})^{\otimes k}$.

Let Ψ denote the $\mathbb{Z}[q,q^{-1}]$-algebra generated by the elements $x^2, x^{-2}, y_+^{(1)}, y_+^{(2)}, y_-^{(1)}, y_-^{(2)}$, and satisfying the relations $x^2x^{-2} = x^{-2}x^2 = 1$. We equip Ψ with $\mathbb{Z}_{\geq 0}$-graded algebra structure so that $\deg(x^{\pm 2}) = \deg(y_\pm^{(2)}) = 0$ and $\deg(y_\pm^{(1)}) = 1$.

Let $^-\colon\ \Psi \to \bar{U}_q^{ev}$ be the algebra homomorphism defined by

$$\bar{x}^2 = K^2, \quad \bar{x}^{-2} = K^{-2}, \quad \bar{y}_+^{(i)} = e^i, \quad \bar{y}_-^{(i)} = f^i,$$

for $i = 1, 2$. For $X \in \Psi/\{0\}$, let $m(X) \in \{0, 1, 2, \ldots\}$ be the multiplicity of zero at $q = -1$, i.e., $m(X) = n$ if and only if $X \in (q+1)^n\Psi \backslash (q+1)^{n-1}\Psi$. We set $m(0) = \infty$.

For $b \in \mathbb{Z}$, let Q_b denote the $\mathbb{Z}[q,q^{-1}]$-submodule of $\bar{U}_q^{ev}$ spanned by

$$\{\bar{w} \in \bar{U}_q^{ev} \mid w \text{ is homogeneous in } \Psi, \deg(w) - 2m(w) \leq b\}.$$

We have

$$Q_a \cdot Q_b \subset Q_{a+b}, \tag{29}$$

for $a, b \in \mathbb{Z}$, and

$$Q_a \subset Q_b, \tag{30}$$

for $a, b \in \mathbb{Z}, a \leq b$.

We define $Q_b^{(0)} = \mathbb{Z}[q,q^{-1}]$ for $b \in \mathbb{Z}$. For $k \geq 1$ and $b \in \mathbb{Z}$, let $Q_b^{(k)}$ denote a $\mathbb{Z}[q,q^{-1}]$-submodule of $(\bar{U}_q^{ev})^{\otimes k}$ defined by

$$Q_b^{(k)} = \sum_{\substack{i_1,\ldots,i_k \in \mathbb{Z} \\ i_1+\cdots+i_k = b}} Q_{i_1} \otimes \cdots \otimes Q_{i_k}.$$

Note that $Q_0^{(k)}$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $(\bar{U}_q^{ev})^{\otimes k}$ for $k \geq 0$.

5.2 The filtration $\{Q^d\}_{d\geq 0}$ of $U_{\mathbb{Z},q}$ and the adjoint action on Q_b.

Let Q^1 denote the two-sided ideal in $U_{\mathbb{Z},q}$ generated by $\tilde{E}^{(2n-1)}$ and $\tilde{F}^{(2n-1)}$ for $n \geq 1$. $U_{\mathbb{Z},q}$ is equipped with a filtered algebra structure

$$U_{\mathbb{Z},q} = Q^0 \supset Q^1 \supset \cdots \supset Q^d \supset \cdots.$$

Let $\Delta^{[l]}\colon\ U_h \to U_h^{\hat{\otimes} l}$, $l \geq 0$, denote the l-output comultiplication defined by $\Delta^{[0]} = \varepsilon, \Delta^{[1]} = \Delta$, and

$$\Delta^{[l]} = (\Delta \otimes 1^{\otimes(l-2)}) \cdots (\Delta \otimes 1)\Delta$$

for $l \geq 2$.

Lemma 5.1. *For $l \geq 1, d \geq 0$, we have*

$$\Delta^{[l]}(Q^d) \subset \sum_{\substack{i_1,\dots,i_l \in \mathbb{Z} \\ i_1+\cdots+i_l=d}} Q^{i_1} \otimes \cdots \otimes Q^{i_l}.$$

Proof. By (8), we obtain

$$\Delta(Q^1) \subset Q^1 \otimes U_{\mathbb{Z},q} + U_{\mathbb{Z},q} \otimes Q^1.$$

The assertion follows this inclusion. □

Proposition 5.2. *For $b \in \mathbb{Z}$, $d \geq 0$, we have*

$$Q^d \triangleright Q_b \subset Q_{b-d}.$$

Proof. It is enough to prove that

$$w \triangleright \bar{z} \in Q_{b-d},$$

for $w \in Q^d$ with $d = 0, 1$, and for $\bar{z} = \bar{z}_1 \cdots \bar{z}_l \in Q_b$, $l \geq 0$, where $z_i \in \{x^{\pm 2}, y_\pm^{(1)}, y_\pm^{(2)}\}$ for $i = 1, \dots, l$. We have

$$w \triangleright z = \sum (w_{(1)} \triangleright z_1) \cdots (w_{(l)} \triangleright z_l),$$

if $\Delta^{[l]}(w) = \sum w_{(1)} \otimes \cdots \otimes w_{(l)}$. Thus, by Lemma 5.1 and (29), we have only to prove that

$$w \triangleright \bar{z}_i \in \begin{cases} Q_0 & \text{if } w \in \{K^{\pm 1}, \tilde{E}^{(2l)}, \tilde{F}^{(2l)} \mid l \geq 1\}, \\ Q_{-1} & \text{if } w \in \{\tilde{E}^{(2l-1)}, \tilde{F}^{(2l-1)} \mid l \geq 1\}, \end{cases}$$

for $z_i = x^2, x^{-2}, y_+^{(2)}, y_-^{(2)}$, and

$$w \triangleright \bar{z}_i \in \begin{cases} Q_1 & \text{if } w \in \{K^{\pm 1}, \tilde{E}^{(2l)}, \tilde{F}^{(2l)} \mid l \geq 1\}, \\ Q_0 & \text{if } w \in \{\tilde{E}^{(2l-1)}, \tilde{F}^{(2l-1)} \mid l \geq 1\}, \end{cases}$$

for $z_i = y_+^{(1)}, y_-^{(1)}$. Now the rest of the proof is straightforward. For example, if $w = \tilde{E}^{(2l-1)}$ and $z = K^{\pm 2} \in Q_0$, then we have

$$\tilde{E}^{(2l-1)} \triangleright K^{\pm 2} = -q^{\mp 2(2l-1)} \begin{bmatrix} 2l \pm 2 - 2 \\ 2l - 1 \end{bmatrix}_q K^{\pm 2} e^{2l-1} \in (q+1)Q_1 \subset Q_0.$$

If $w = \tilde{E}^{(2l-1)}$ and $z = e^2 \in Q_0$, then we have

$$\tilde{E}^{(2l-1)} \triangleright e^2 = -[2l]_q e^{2l+1} \in (q+1)Q_1 \subset Q_0.$$

If $w = \tilde{E}^{(2l-1)}$ and $z = e \in Q_1$, then we have

$$\tilde{E}^{(2l-1)} \triangleright e = -e^{2l} \in Q_0.$$

We can prove the other cases similarly. Hence we have the assertion. □

6 Proof of Theorem 1.2.

In this and the next sections, we prove Theorems 1.2. First of all, let us prepare some notions.

For $n \geq 1, 1 \leq i \leq n$, and for every element $X \in U_h$, we define $X_i \in U_h^{\hat{\otimes} n}$ by

$$X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1,$$

where X is at the ith position.

For $n \geq 2, 0 \leq i \neq j \leq n$, and for every element $Y = \sum y_1 \otimes y_2 \in U_h^{\hat{\otimes} 2}$, we define $Y_{ij} \in U_h^{\hat{\otimes} n}$ by

$$Y_{ij} = \sum 1 \otimes \cdots \otimes y_1 \otimes \cdots \otimes y_2 \otimes \cdots \otimes 1,$$

where y_1 is at the ith position and y_2 is at the jth position.

For $n \geq 1, 1 \leq i \leq n$ and for every element $Y = \sum y_1 \otimes y_2 \in U_h^{\hat{\otimes} 2}$, we define $Y_{ii} \in U_h^{\hat{\otimes} n}$ by

$$Y_{ii} = \sum 1 \otimes \cdots \otimes y_1 y_2 \otimes \cdots \otimes 1,$$

where $y_1 y_2$ is at the ith position.

For every symmetric integer matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ of size $n \geq 1$, we define two invertible elements $D^M \in U_h^{\hat{\otimes} n}, \tilde{D}^M \in U_h^{\hat{\otimes} n}$ by

$$D^M = \prod_{1 \leq i,j \leq n} D_{ij}^{m_{ij}} = \prod_{1 \leq i < j \leq n} D_{ij}^{2m_{ij}} \prod_{1 \leq i \leq n} (v^{H^2/2})_i^{m_{ii}},$$
$$\tilde{D}^M = D^M \prod_{1 \leq i \leq n} K_i^{m_{ii}} = \prod_{1 \leq i < j \leq n} D_{ij}^{2m_{ij}} \prod_{1 \leq i \leq n} (v^{H^2/2} K)_i^{m_{ii}}.$$

6.1 The outline of the proof of Theorem 1.2.

Let $T \in BT_n$, $n \geq 0$, be a ribbon bottom tangle, and $(W \in BT_{2k}; N_1, \ldots, N_n)$, $k \geq 0$, a ribbon data for T. Let $M(W)$ be the linking matrix of W. Let P_W be an even-trivial diagram of W, and choose a state $s \in S(P_W)$. We use this assumptions until the end of Section 7. The outline of the proof of Theorem 1.2 is as follows.

First, we prove the following proposition.

Proposition 6.1. *We have*

$$J(P_W, s) \in \tilde{D}^{M(W)} (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}.$$

Then we consider the contribution of $\tilde{D}^{M(W)}$ to the adjoint action, and we prove there is an element $\tilde{J}(P_W, s) \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}$ such that

$$\mathrm{ad}^{\otimes k}(J(P_W, s)) = \mathrm{ad}^{\otimes k}(\tilde{J}(P_W, s)). \tag{31}$$

Using the proof of Proposition 6.1, we prove the following lemma.

Lemma 6.2. *There are elements $w_{2i-1} \in Q^{d_i}$ and $w_{2i} \in Q_{b_i}$ for $i = 1, \dots k$, such that*

$$\tilde{J}(P_W, s) = w_1 \otimes \cdots \otimes w_{2k}, \tag{32}$$

$$\sum_i^k (b_i - d_i) \leq 0. \tag{33}$$

By Proposition 5.2, we have

$$w_{2i-1} \triangleright w_{2i} \in Q_{b_i - d_i}.$$

Thus, by (31) and Lemma 6.2, we have

$$\mathrm{ad}^{\otimes k}(J(P_W, s)) = \mathrm{ad}^{\otimes k}(\tilde{J}(P_W, s)) \in Q^{(k)}_{\sum_i^k (b_i - d_i)} \subset Q_0^{(k)}. \tag{34}$$

In the next section, we define a completion $\hat{Q}_0^{(k)}$ of $Q_0^{(k)}$ and prove

$$\sum_{s \in S(P_W)} \mathrm{ad}^{\otimes k}(J(P_W, s)) \in \hat{Q}_0^{(k)}.$$

Since $\mu^{[N_1, \dots, N_n]} \hat{Q}_0^{(k)} \subset \hat{Q}_0^{(n)}$, we have Theorem 1.2, i.e.,

$$J_T = \sum_{s \in S(P_W)} \mu^{[N_1, \dots, N_n]} \mathrm{ad}^{\otimes k}(J(P_W, s)) \in \hat{Q}_0^{(n)}.$$

6.2 Proof of Proposition 6.1.

Let $\vec{\cap}$, $\vec{\cup}$ and $\cap$ denote the fundamental tangles defined by

$\vec{\cap} =$  , $\vec{\cup} =$  , $\cap =$  .

We use the following lemma.

Lemma 6.3. *For every diagram P_K of a bottom tangle $K \in BT_1$ with framing $r(K) \in \mathbb{Z}$, let $u(P_K) \in \mathbb{Z}_{\geq 0}$ be the total number of the copies of $\vec{\cap}$ and $\vec{\cup}$ which are contained in P_K. Then, the sum $u(P_K) + r(K)$ is even.*

Proof. Note that the parity of $u(P_K) + r(K)$ does not change by the Reidemeister moves RI, RII, RIII, and crossing changes as depicted in Figure 15. Since P_K is equal to the bottom tangle $\cap$ up to those moves, we have

$$u(P_K) + r(K) \equiv u(\cap) + r(\cap) = 0 \pmod 2.$$

This completes the proof. □

Before proving Proposition 6.1, we modify the dots of the decorated diagram (P_W, s). Then we define three decorated diagrams $(P_W, s)^\circ$, $(P_W, s)^\bullet$, and $(P_W, s)^\diamond$, which we use in the proof of Proposition 6.1.

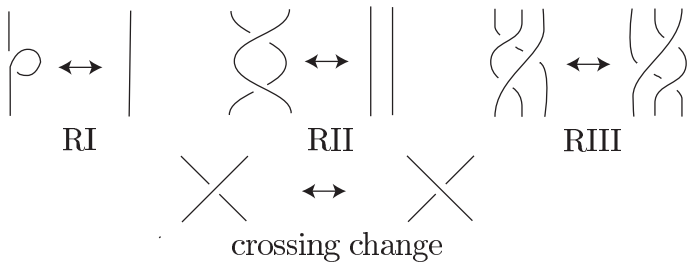

Figure 15: The Reidemeister moves RI, RII, RIII, and the crossing change.

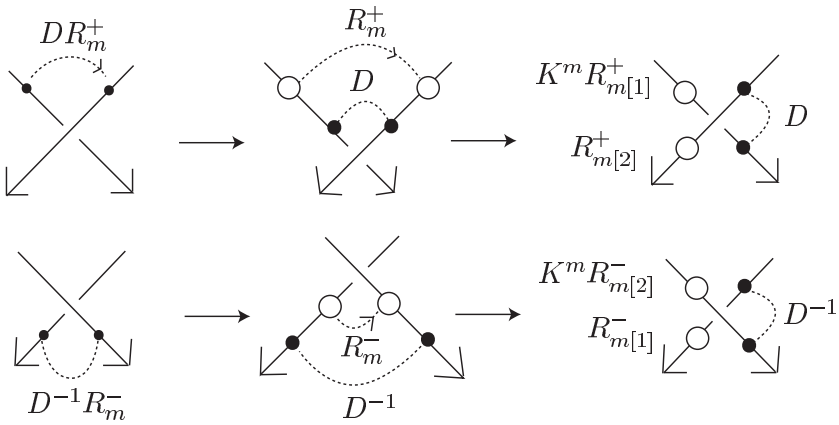

Figure 16: The modification process of (P_W, s) on positive and negative crossings.

In the modification process, for simplicity, we assume that all crossings of P_W are oriented downward. (After we attach elements to the dots on P_W, we work up to isotopy in $\mathbb{R}^2$.) In fact, we can work with an arbitrary even-trivial diagram. The modification process goes as follows. Let c be a crossing of (P_W, s) and set $m = s(c)$. As depicted in Figure 16, we separate the two dots labeled by $D^{\pm 1}R_m^{\pm}$ to two black dots and two white dots labeled by $D^{\pm 1}$ and $R_m^{\pm}$, respectively. Then we slide the black (resp. white) dots to the right hand side (resp. left hand side) of the crossings, and put the produced element K^m into the same dot of $R_m^{\pm}$. Here the transformation follows from the formulas

$$\begin{aligned} DR_m^+ &= \sum D_{[1]}R_{m[1]}^+ \otimes D_{[2]}R_{m[2]}^+ \\ &= \sum D_{[1]}K^m R_{m[1]}^+ \otimes R_{m[2]}^+ D_{[2]}. \end{aligned}$$

and

$$\begin{aligned} D^{-1}R_m^- &= \sum D_{[1]}^- R_{m[1]}^- \otimes D_{[2]}^- R_{m[2]}^- \\ &= \sum R_{m[1]}^- D_{[1]}^- \otimes D_{[2]}^- K^m R_{m[2]}^-. \end{aligned}$$

By (19) and (20), we have

$$\begin{aligned} K^m R_{m[1]}^+ \otimes R_{m[2]}^+ &\sim \tilde{F}^{(m)} \otimes e^m \\ &\sim f^m \otimes \tilde{E}^{(m)}, \\ R_{m[1]}^- \otimes K^m R_{m[2]}^- &\sim \tilde{F}^{(m)} \otimes e^m \\ &\sim f^{(m)} \otimes \tilde{E}^{(m)}, \end{aligned}$$

where $\sim$ means equality up to multiplication by $\pm q^j, K^{2j} (j \in \mathbb{Z})$ on any tensorands. In what follows, we can work up to multiplication by $\pm q^j, K^{2j} (j \in \mathbb{Z})$. Since P_W is even-trivial, the set $C(P_W)$ of the crossings of P_W is the disjoint union of two subsets

$$C^{eo} = \{ \text{ crossings of } P_W^{ev} \text{ with } P_W^{odd} \},$$

and

$$C^{oo} = \{ \text{ crossings of } P_W^{odd} \text{ with } P_W^{odd} \}.$$

By this observation and the fact

$$K^m R_{m[1]}^+ \otimes R_{m[2]}^+, R_{m[1]}^- \otimes K^m R_{m[2]}^- \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev}) \cap (\bar{U}_q^{ev} \otimes U_{\mathbb{Z},q}^{ev}),$$

we can assume that the element attached to the white dot on P_W^{ev} (resp. P_W^{odd}) is contained in $\bar{U}_q^{ev}$ (resp. $U_{\mathbb{Z},q}^{ev}$). For example, we attach elements to positive crossings as depicted in Figure 17. We have completed the modification. By abusing the notation, we denote by (P_W, s) the decorated diagram obtained from the modification.

We define the decorated diagrams $(P_W, s)^\circ$, $(P_W, s)^\bullet$, and $(P_W, s)^\diamond$ as follows.

(1) Let $(P_W, s)^\circ$ denote the diagram P_W together with the white dots on crossings of (P_W, s). Note that

$$J(P_W, s)^\circ \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}. \tag{35}$$

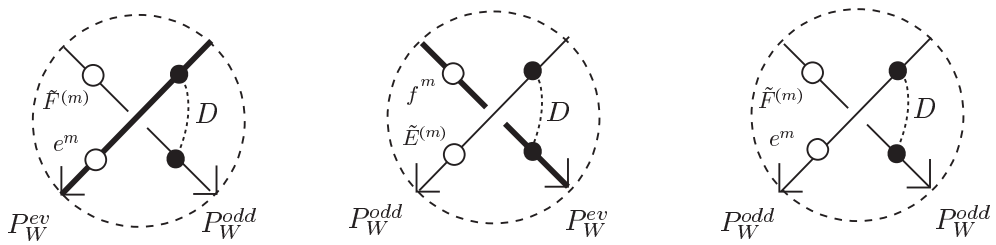

Figure 17: The three types of positive crossings. We work up to multiplication by $\pm q^j, K^{2j} (j \in \mathbb{Z})$.

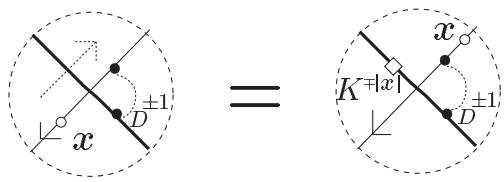

Figure 18: The picture when we slide a homogeneous x through a dot labeled by $D^{\pm 1}$. This is essentially the same with the picture in Figure 6.

(2) Let $(P_W, s)^\bullet$ denote the diagram P_W with the black dots labeled by $D^{\pm 1}$ on crossings of (P_W, s), and dots on $\vec{\cap}$ and $\vec{\cup}$ of (P_W, s).

(3) On (P_W, s), we slide all white dots to the start points of W. When we slide a white dot through a dot on $\vec{\cap}$ or $\vec{\cup}$, a scalar $q^j (j \in \mathbb{Z})$ appears, which we can ignore. When we slide a white dot through a dot labeled by $D^\pm$, a power of K appears, see Figure 18. We attach such element to a new white diamond. For $i = 1, \ldots, 2k$, let $(P_i, s)^\diamondsuit$ denote the diagram P_i with the white diamonds on P_i. Set

$$J(P_W, s)^\diamondsuit = J(P_1, s)^\diamondsuit \otimes \cdots \otimes J(P_{2k}, s)^\diamondsuit.$$

For example, for the decorated diagram (P_W, s) in Figure 19, we have

$$\begin{aligned}
&M(W) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \\
&\tilde{D}^{M(W)} = D^{-2}(v^{H^2/2} K \otimes 1), \\
&J(P_W, s)^\circ \sim f^l \tilde{E}^{(l)} \tilde{F}^{(m)} \tilde{E}^{(n)} \otimes f^n e^m, \\
&J(P_W, s)^\bullet \sim D^{-2}(v^{H^2/2} K \otimes 1), \\
&J(P_1, s)^\diamondsuit \sim K^{-2n} \sim 1, \\
&J(P_2, s)^\diamondsuit \sim K^{-2m} \sim 1.
\end{aligned}$$

We reduce Proposition 6.1 to the following two lemmas.

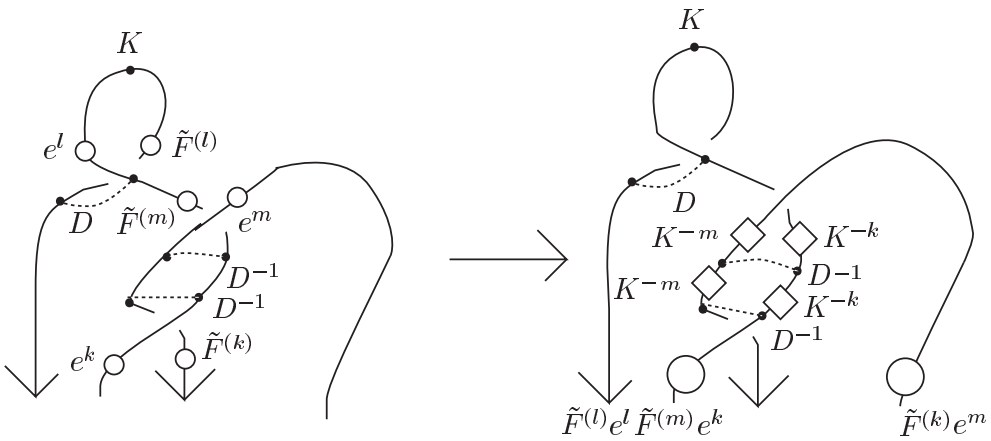

Figure 19: The sliding process for a decorated diagram (P_W, s), where we set $s(c_1) = l, s(c_2) = m$, and $s(c_3) = n$ for the upper, the middle, and the lower crossings c_1, c_2, and c_3, respectively. We work up to multiplication by $\pm q^j, K^{2j} (j \in \mathbb{Z})$.

Lemma 6.4. *We have*

$$J(P_W, s)^\bullet \in \tilde{D}^{M(T)}(\bar{U}_q^{ev0})^{\otimes 2k}.$$

Proof. For each $i = 1, \dots, 2k$, we denote by $\kappa_i \in U^0_{\mathbb{Z},q}$ the product of the $K^{\pm 1}$s on the copies of $\vec{\cap}$ and $\vec{\cup}$ of P_i. Note that $\kappa_{2i} = 1$, for $i = 1, \dots, k$, since P_W is even-trivial. We have

$$\begin{aligned} J(P_W, s)^\bullet =& D^{M(W)}(\kappa_1 \otimes \cdots \otimes \kappa_{2k}) \\ =& \tilde{D}^{M(W)}(K^{-m_{1,1}}\kappa_1 \otimes \cdots \otimes K^{-m_{2k,2k}}\kappa_{2k}). \end{aligned}$$

Since we have $K^{-m_{i,i}}\kappa_i \in U^{ev0}_{\mathbb{Z},q}$ by Lemma 6.3, the right hand side is contained in $\tilde{D}^{M(W)}(\bar{U}_q^{ev0})^{\otimes 2k}$, this completes the proof. □

Lemma 6.5. *For $i = 1, \dots, 2k$, We have*

$$J(P_i, s)^\diamondsuit \sim 1.$$

If we assume Lemma 6.5, then Proposition 6.1 follows from

$$\begin{aligned} J(P_W, s) \sim& J(P_W, s)^\bullet J(P_W, s)^\diamondsuit J(P_W, s)^\circ \\ \in& \tilde{D}^{M(W)}(\bar{U}_q^{ev0})^{\otimes 2k} \cdot (U^{ev}_{\mathbb{Z},q} \otimes \bar{U}_q^{ev})^{\otimes k} \subset \tilde{D}^{M(W)}(U^{ev}_{\mathbb{Z},q} \otimes \bar{U}_q^{ev})^{\otimes k}, \end{aligned}$$

by (35) and Lemma 6.4.

Proof of Lemma 6.5. For a crossing c, we denote by E_c (resp. F_c) the element attached to the white dot on the over (resp. under) strand of c. We slide those white dots to the

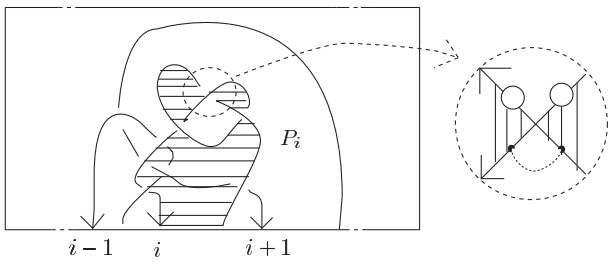

Figure 20: A diagram $P = P_1 \cup \cdots \cup P_n$ colored by chessboard fashion associated to P_i. We depict only the $(i-1)$, i, and $(i+1)$th component.

start point of P_W, and count the powers of K labeled to the white diamonds on each strands.

Note that E_c and F_c are of degree $s(c)$ and $-s(c)$, respectively. Hence each time we exchange the white dot labeled by E_c with one of the two dots connected by dashed line, labeled by $D^{\pm 1}$, a white diamond labeled by $K^{\mp s(c)}$ appears next to the other dot, see Figure 18 again. Similarly, if we exchange the white dot labeled by F_c with one of the two dots labeled by $D^{\pm 1}$, then a white diamond labeled by $K^{\pm s(c)}$ appears next to the other dot.

From now, we identify every dot d on P_W with the element labeled to d. Let $p_i(E_c)$ denotes the number of times E_c traverses the strand P_i during the sliding process described above. Define $p_i(F_c)$ similarly. Then we have $J(P_i, s)^{\diamond} = K^{d_i}$, where

$$d_i \equiv \sum_{c \in C(P_W)} s(c)(p_i(E_c) + p_i(F_c)) \pmod 2.$$

Hence it is enough to prove that $p_i(E_c) + p_i(F_c)$ is even for each crossing c. We prove the assertion with three types of crossings as follows.

(i) Self crossings of P_i.

(ii) Crossings of P_j with P_l for $j \neq i, l \neq i$.

(iii) Crossings of P_i with P_j for $j \neq i$.

Color black or white, in chessboard fashion, the regions of the complements of P_i in the rectangle so that the outermost region is colored white. For example, see Figure 20. Divide the strand P_i into two parts B_i and W_i, each consisting of segments bounded by self crossing points or the boundary points of P_i, such that if one goes along a segment in W_i (resp, B_i) to the start point, then one sees a white region on the left.

Note that the start and the end points of P_l, $i \neq l$, are contained in the white region, and the start and the end point of P_i are contained in W_i.

(i) For a self crossing c of P_i.

Note that when we trace along P_i from the end point to the start point, every time we traverse the self crossing of P_i, B_P and W_P appear one after the other.

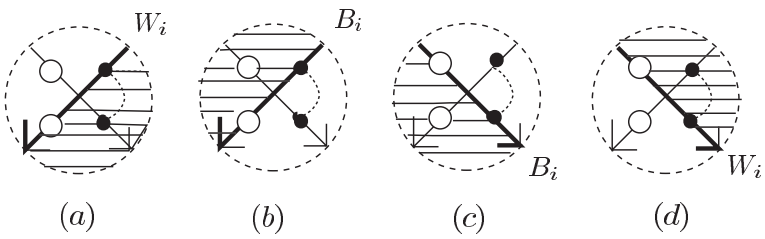

Figure 21: The four types of crossings.

For every self crossing $c \in P_i$, both E_c and F_c are either in B_P or in W_P. Hence if we slide E_c and F_c to the start point, then the parities of $p_i(E_c)$ and $p_i(F_c)$ are the same. Thus, $p_i(E_c) + p_i(F_c)$ is even.

(ii) For a crossing c of P_j and P_l with $j \neq i, l \neq i$.

If the crossing c is in the white region, then both $p_i(E_c)$ and $p_i(F_c)$ are even. If c is in the black region, then both $p_i(E_c)$ and $p_i(F_c)$ are odd. Hence $p_i(E_c)+p_i(F_c)$ is even in both cases.

(iii) For a crossing c of P_i and P_j with $j \neq i$.

See Figure 21. There are four types of crossings such that whether the white dot on P_i is in W_i or in B_i, and whether the white dot on P_j is in the white region or in the black region. We assume P_i is the over strand, i.e., E_c is attached the white dot on P_i. The other case is almost the same. For (a), since E_c starts and ends in W_i, $p_i(E_c)$ is even. Similarly, since F_c starts and ends in the white region, $p_i(F_c)$ is even. Thus, $p_i(E_c) + p_i(F_c)$ is even. For the other three cases, in a similar way, we can prove that the parities of $p_i(E_c)$ and $p_i(F_c)$ are the same. Hence $p_i(E_c) + p_i(F_c)$ is even.

Therefore we have $J(P_i, s)^\diamond \in \bar{U}_q^{ev0} \sim 1$ for $i = 1, \ldots, 2k$, this completes the proof. $\square$

We use two corollaries to the proof of Proposition 6.1, in the proof of Theorem 1.2. For $i \geq 0$, set

$$\delta_i = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

Note that for $X_c \in \{E_c, F_c\}$ as in the proof of Lemma 6.5, we have $X_c \in Q^{\delta_{s(c)}}$. Moreover, if $X_c \in \bar{U}_q^{ev}$, i.e., X_c is on P_W^{ev}, then we have $X_c \in Q_{\delta_{s(c)}}$.

Let $C_1(P_i)$ be the set of self crossings of P_i, and $C_2(P_i)$ the set of crossings of P_i with the other strands. Note that $C_1(P_{2i}) = \emptyset$ since P_W is even-trivial. Let $(P_i, s)^\circ$ denote the diagram P_i with the white dots on P_i of the decorated diagram $(P_W, s)^\circ$. For a crossing $c \in C_1(P_i)$, both E_c and F_c are on the two white dots of $(P_i, s)^\circ$, and for a crossing $c \in C_2(P_i)$, one of E_c and F_c is on the white dot of $(P_i, s)^\circ$. Hence we have the following.

Corollary 6.6. *We have*

$$J(P_{2i-1},s)^\circ \in Q^{d_i}, \quad d_i = \sum_{c\in C_1(P_{2i-1})} 2\delta_{s(c)} + \sum_{c\in C_2(P_{2i-1})} \delta_{s(c)},$$
$$J(P_{2i},s)^\circ \in Q_{b_i}, \quad b_i = \sum_{c\in C_2(P_{2i})} \delta_{s(c)}.$$

Set $C(P_i) = C_1(P_i) \cup C_2(P_i)$. For $s \in S(P_W)$, set $|s|_i = \max\{s(c) \mid c \in C(P_i)\}$. For $p \geq 0$, we denote by $\mathcal{F}_p(U_{\mathbb{Z},q})$ the two-sided ideal of $U_{\mathbb{Z},q}$ generated by $\tilde{E}^{(p')}$ and $\tilde{F}^{(p')}$ for $p' \geq p$. Then $E_c, F_c \in \mathcal{F}_{s(c)}$ implies the following.

Corollary 6.7. *We have*

$$J(P_i,s)^\circ \in \mathcal{F}_{|s|_i}(U_{\mathbb{Z},q}).$$

Remark 6.8. *By the proof of Proposition 6.1, we can obtain a result for the universal sl_2 invariant of an arbitrary, not necessarily even-trivial, bottom tangle as follows. Let $T \in BT_n$ be an n-component bottom tangle, and P_T a diagram of T. As defined in [4], let $\mathcal{U}_q^{ev} \subset U_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra $\mathcal{U}_q^{ev}$ of U_h generated by the elements K^2, k^{-2}, e, and $\tilde{F}^{(n)}$ for $n \geq 1$. Note that*

$$K^m R^+_{m[1]} \otimes R^+_{m[2]}, \quad R^-_{m[1]} \otimes K^m R^-_{m[2]} \in (\mathcal{U}_q^{ev})^{\otimes 2}.$$

Thus, for every state $s \in S(P_T)$, we can attach the elements in $\mathcal{U}_q^{ev}$, instead of the elements in U_q^{ev} and $U_{\mathbb{Z},q}^{ev}$, to the two white dots on every crossing of (P_T,s). Then, by the same argument as in the proof of Proposition 6.1, one can prove

$$J(P_T,s) \in \tilde{D}^{M(T)}(\mathcal{U}_q^{ev})^{\otimes n}.$$

6.3 The element $\tilde{J}(P_W,s)$ and the proof of Lemma 6.2.

In this subsection, we determine the element $\tilde{J}(P_W,s) \in (U_{\mathbb{Z},q}^{ev} \otimes \bar{U}_q^{ev})^{\otimes k}$ satisfying (31), and prove Lemma 6.2.

Lemma 6.9. *For homogeneous elements $x, y \in U_h$, we have*

(i) $\sum (D^\pm_{[1]} \triangleright x) \otimes D^\pm_{[2]} = x \otimes K^{\pm|x|}$,

(ii) $\sum (D^\pm_{[1]} \triangleright x) \otimes (D^\pm_{[2]} \triangleright y) = q^{\pm|x||y|} x \otimes y$, *and*

(iii) $(v^{H^2/2}K)^{\pm 1} \triangleright x = q^{\pm|x|(|x|+1)} x$.

Proof. We prove the formulas associated to D. Then the other formulas associated to the inverse D^{-1} are similar. By the formulas (4)–(6), we have

(i) $\sum (D^+_{[1]} \triangleright x) \otimes D^+_{[2]} = \sum (D^+_{[1]} x D^-_{[1]}) \otimes (D^+_{[2]} D^-_{[2]}) = x \otimes K^{|x|}$.

Using (i), we obtain

(ii) $\sum(D^+_{[1]} \triangleright x) \otimes (D^+_{[2]} \triangleright y) = \sum x \otimes (K^{|x|} \triangleright y) = q^{|x||y|} x \otimes y$, and

(iii) $(v^{H^2/2}K) \triangleright x = \sum(D^+_{[1]} D^+_{[2]} K) \triangleright x = q^{|x|}(D^+_{[1]} D^+_{[2]} \triangleright x) = q^{|x|}(K^{|x|} \triangleright x) = q^{|x|(|x|+1)} x$.

Hence we have the assertion. □

Lemma 6.10. *For $k \geq 0$, let $M = (m_{i,j})_{1 \leq i,j \leq 2k}$ be a symmetric integer matrix of size $2k$, satisfying $m_{2i,2j} = 0$ for $1 \leq i, j \leq k$. Let $X = x_1 \otimes \cdots \otimes x_{2k} \in U_h^{\otimes 2k}$ be a tensor product of homogeneous elements $x_1, \ldots, x_{2k} \in U_h$. We have*

$$\mathrm{ad}^{\otimes k}(\tilde{D}^M X) = q^{N(M,X)} \, \mathrm{ad}^{\otimes k}\left((1 \otimes K^{2a_1(M,X)} \otimes \cdots \otimes 1 \otimes K^{2a_m(M,X)})X\right),$$

where if we set $X_i = x_{2i-1} \triangleright x_{2i}$, then

$$\begin{aligned} a_i(M,X) &= \sum_{1 \leq j \leq k} m_{2i,2j-1} |X_j|, \\ N(M,X) &= \sum_{1 \leq i < j \leq k} 2m_{2i-1,2j-1} |X_i||X_j| + \sum_{1 \leq i \leq k} m_{2i-1,2i-1} |X_i|(|X_i|+1). \end{aligned}$$

Proof. We use induction on $\sum_{1 \leq i,j \leq 2k} |m_{ij}|$. If $\sum_{1 \leq i,j \leq 2k} |m_{ij}| = 0$, i.e., $M = 0$, then the claim is clear. Let us assume $M \neq 0$. Then there is a matrix M' satisfying the assertion, and either

$$\begin{aligned} &M = M' \pm (1_{2i,2j-1} + 1_{2j-1,2i}), \quad \text{for } 1 \leq i \neq j \leq k, \text{ or} \\ &M = M' \pm (1_{2i-1,2j-1} + 1_{2j-1,2i-1}), \quad \text{for } 1 \leq i \neq j \leq k, \text{ or} \\ &M = M' \pm 1_{2i-1,2i-1}, \quad \text{for } 1 \leq i \leq k, \end{aligned}$$

where $1_{i,j}$ is the matrix of size $2k$ such that the (i,j)-component is 1 and the others are 0. Note that

$$\begin{aligned} \tilde{D}^{M' \pm (1_{i,j} + 1_{j,i})} &= \tilde{D}^{M'} D^{\pm 2}_{i,j}, \quad \text{for } 1 \leq i \neq j \leq 2k, \text{ and} \\ \tilde{D}^{M' \pm 1_{ii}} &= \tilde{D}^{M'} (v^{H^2/2}K)^{\pm 1}_i, \quad \text{for } 1 \leq i \leq 2k. \end{aligned}$$

Using the formulas in Lemma 6.9, we have

$$\begin{aligned} \mathrm{ad}^{\otimes k}(D^{\pm 1}_{2i,2j-1} X) &= X_1 \otimes \cdots \otimes (x_{2i-1} \triangleright D^{\pm}_{[1]} x_{2i}) \otimes \cdots \otimes (D^{\pm}_{[2]} \triangleright X_j) \otimes \cdots \otimes X_k \\ &= X_1 \otimes \cdots \otimes (x_{2i-1} \triangleright K^{\pm |X_j|} x_{2i}) \otimes \cdots \otimes X_j \otimes \cdots \otimes X_k, \\ \mathrm{ad}^{\otimes k}(D^{\pm 1}_{2i-1,2j-1} X) &= X_1 \otimes \cdots \otimes (D^{\pm}_{[1]} \triangleright X_i) \otimes \cdots \otimes (D^{\pm}_{[2]} \triangleright X_j) \otimes \cdots \otimes X_k \\ &= q^{\pm |X_i||X_j|} X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_k, \\ \mathrm{ad}^{\otimes k}\left((v^{H^2/2}K)^{\pm 1}_{2i-1} X\right) &= X_1 \otimes \cdots \otimes ((v^{H^2/2}K)^{\pm 1} \triangleright X_i) \otimes \cdots \otimes X_k \\ &= q^{\pm |X_i|(|X_i|+1)} X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_k, \end{aligned}$$

for $1 \leq i \neq j \leq k$. This completes the proof. □

By Proposition 6.1, we have

$$X := (\tilde{D}^{M(W)})^{-1} J(P_W, s) \in (U^{ev}_{\mathbb{Z},q} \otimes \bar{U}^{ev}_q)^{\otimes k}.$$

Since the linking matrix $M(W)$ of W satisfies the assumption of Lemma 6.10, we obtain the element $\tilde{J}(P_W, s) \in (U^{ev}_{\mathbb{Z},q} \otimes \bar{U}^{ev}_q)^{\otimes k}$ satisfying (31), such that

$$\tilde{J}(P_W, s) := q^N (1 \otimes K^{2a_1} \otimes \cdots \otimes 1 \otimes K^{2a_m}) X,$$

where we set

$$N = N(M(W), X),$$

and

$$a_i = a_i(M(W), X).$$

as in Lemma 6.10.

Proof of Lemma 6.2. Recall the elements $J(P_{2i-1}, s)^\circ \in Q^{d_i}$ and $J(P_{2i}, s)^\circ \in Q_{b_i}$, for $i = 1, \ldots, k$, as in Corollary 6.6. Note that

$$\begin{aligned} \tilde{J}(P_W, s) &\sim (\tilde{D}^{M(W)})^{-1} J(P_W, s) \\ &\sim J(P_1, s)^\circ \otimes \cdots \otimes J(P_{2k}, s)^\circ, \end{aligned}$$

where $\sim$ means equality up to multiplication by $\pm q^j, K^{2j} (j \in \mathbb{Z})$ on any tensorands. This means there are elements $w_{2i-1} \in Q^{d_i}$ and $w_{2i} \in Q_{b_i}$, for $i = 1, \ldots k$, satisfying (32). We prove (33). We have

$$\begin{aligned} \sum_{i=1}^k d_i &= \sum_{i=1}^k \Big(\sum_{c \in C_1(P_{2i-1})} 2\delta_{s(c)} + \sum_{c \in C_2(P_{2i-1})} \delta_{s(c)} \Big) \\ &= \sum_{c \in C^{oo}} 2\delta_{s(c)} + \sum_{c \in C^{eo}} \delta_{s(c)}, \\ \sum_{i=1}^k b_i &= \sum_{i=1}^k \sum_{c \in C_2(P_{2i})} \delta_{s(c)} \\ &= \sum_{c \in C^{eo}} \delta_{s(c)}. \end{aligned}$$

Since $\sum_{i=1}^k (b_i - d_i) \leq -\sum_{c \in C^{oo}} 2\delta_{s(c)} \leq 0$, we have the assertion. □

7 The completion $\hat{Q}_0^{(k)}$ of $Q_0^{(k)}$ and proof of Theorem 1.2.

In this section we define the completion $\hat{Q}_0^{(k)}$ of $Q_0^{(k)}$ for $k \geq 0$, and prove Theorem 1.2.

7.1 Filtrations of $\bar{U}_q^{ev}$.

In this subsection, we define six filtrations $\{X_p\}_{p\geq 0}$ of $\bar{U}_q^{ev}$ for $X = A, A', B, B', C, C'$. In the next subsection, we define the completion $\hat{Q}_0^{(k)}$, $k \geq 0$, by using the filtration $\{C_p\}_{p\geq 0}$ in order to prove Theorem 1.2, but in fact, we obtain the same completion by using any other one. In this subsection, we aim to understand the filtration $\{C_p\}_{p\geq 0}$ and the completion $\hat{Q}_0^{(k)}$.

For a subset $X \subset U_q^{ev}$, let $\langle X\rangle_{ideal}$ denote the two-sided ideal of U_q^{ev} generated by X. For $p \geq 0$, set

$$\begin{aligned}
A_p &= \langle U_{\mathbb{Z},q} \triangleright e^p\rangle_{ideal},\\
A'_p &= \langle U_{\mathbb{Z},q} \triangleright f^p\rangle_{ideal},\\
B_p &= \langle K^p(U_{\mathbb{Z},q} \triangleright K^{-p}e^p)\rangle_{ideal},\\
B'_p &= \langle K^p(U_{\mathbb{Z},q} \triangleright f^pK^{-p})\rangle_{ideal},\\
C_p &= \langle \sum_{p'\geq p} (U_{\mathbb{Z},q}\tilde{E}^{(p')} \triangleright \bar{U}_q^{ev})\rangle_{ideal},\\
C'_p &= \langle \sum_{p'\geq p} (U_{\mathbb{Z},q}\tilde{F}^{(p')} \triangleright \bar{U}_q^{ev})\rangle_{ideal}.
\end{aligned}$$

Lemma 7.1. *For $p \geq 0$, we have*

$$A_p = A'_p = B_p = B'_p.$$

Proof. By the formulas

$$f^pK^{-p} = (-1)^p q^{-p^2}\tilde{F}^{(2p)} \triangleright K^{-p}e^p \subset U_{\mathbb{Z},q} \triangleright K^{-p}e^p, \tag{36}$$

$$K^{-p}e^p = (-1)^p q^{p^2}\tilde{E}^{(2p)} \triangleright f^pK^{-p} \subset U_{\mathbb{Z},q} \triangleright f^pK^{-p}, \tag{37}$$

we have $B_p = B'_p$. We prove $A_p = B_p$, then $A'_p = B'_p$ is similar. By Lemma 3.3 and Proposition 3.4, we have

$$\begin{aligned}
K^p(U_{\mathbb{Z},q} \triangleright K^{-p}e^p) &\subset K^p(U_{\mathbb{Z},q} \triangleright K^{-p}) \cdot (U_{\mathbb{Z},q} \triangleright e^p)\\
&\subset \bar{U}_q^{ev}(U_{\mathbb{Z},q} \triangleright e^p) \subset A_p.
\end{aligned}$$

Hence we have $B_p \subset A_p$. Conversely, we have

$$\begin{aligned}
U_{\mathbb{Z},q} \triangleright e^p = U_{\mathbb{Z},q} \triangleright K^pK^{-p}e^p &\subset (U_{\mathbb{Z},q} \triangleright K^p) \cdot (U_{\mathbb{Z},q} \triangleright K^{-p}e^p)\\
&\subset \bar{U}_q^{ev}K^p(U_{\mathbb{Z},q} \triangleright K^{-p}e^p) \subset B_p.
\end{aligned}$$

Hence we have $A_p \subset B_p$, this completes the proof. □

Proposition 7.2. *(i) For $p \geq 0$, we have $C_p = C'_p$.*

(ii) For $p \geq 0$, we have $C_{2p} \subset A_p$.

(iii) If $p \geq 0$ is even, then we have $C_{2p} = A_p$.

Proof. (i) We prove $C_p \subset C'_p$, then $C_p \supset C'_p$ is similar. Using the formula

$$\tilde{E}^{(2p)} \triangleright \tilde{F}^{(p)} K^{-p} = (-1)^p q^{-\frac{1}{2}p(p+1)} K^{-p} \tilde{E}^{(p)},$$

we have

$$\begin{aligned} U_{\mathbb{Z},q} \tilde{E}^{(p)} &\subset U_{\mathbb{Z},q} \big(\tilde{E}^{(2p)} \triangleright \tilde{F}^{(p)} K^{-p} \big) \\ &\subset U_{\mathbb{Z},q} \tilde{F}^{(p)} U_{\mathbb{Z},q}. \end{aligned}$$

Hence we have

$$\begin{aligned} U_{\mathbb{Z},q} \tilde{E}^{(p)} \triangleright \bar{U}_q^{ev} &\subset U_{\mathbb{Z},q} \tilde{F}^{(p)} U_{\mathbb{Z},q} \triangleright \bar{U}_q^{ev} \\ &\subset U_{\mathbb{Z},q} \tilde{F}^{(p)} \triangleright \bar{U}_q^{ev}. \end{aligned}$$

This completes the proof.

(ii) In view of Lemma 3.2, it is enough to prove that

$$\tilde{E}^{(p')} \triangleright f^{i_1} K^{2i_2} e^{i_3} \subset A_p,$$

for $p' \geq 2p$. If $i_1 \geq p' \geq p$, then the assertion follows from Lemma 3.3. If $i_1 < p'$, then we have

$$\begin{aligned} \tilde{E}^{(p')} \triangleright f^{i_1} K^{2i_2} e^{i_3} &\in \langle U_{\mathbb{Z},q} \triangleright f^{i_1} \rangle_{ideal} \cap \langle e^{i_3+p'-i_1} \rangle_{ideal}, \\ &\subset A'_{i_1} \cap A_{i_3+p'-i_1} \\ &\subset A_{\max\{i_1, i_3+p'-i_1\}}, \end{aligned}$$

where the $\in$ follows from the formula (25), and the last $\subset$ follows from Lemma 7.1. Since the assertion follows from

$$\max\{i_1, i_3 + p' - i_1\} \geq \frac{i_3 + p'}{2} \geq p.$$

(iii) If $p \geq 0$ is even, then we have

$$\begin{aligned} K^p (U_{\mathbb{Z},q} \triangleright K^{-p} e^p) &= (-1)^p q^{p^2} K^p (U_{\mathbb{Z},q} \triangleright (\tilde{E}^{(2p)} \triangleright f^p K^{-p})) \\ &\subset \langle U_{\mathbb{Z},q} \tilde{E}^{(2p)} \triangleright \bar{U}_q^{ev} \rangle_{ideal} \subset C_{2p}, \end{aligned}$$

from (36). Hence we have $C_{2p} \supset B_p (= A_p)$, this completes the proof.

□

Corollary 7.3. *For $p \geq 0$, we have*

$$C_{2p} \subset h^p U_h.$$

Proof. Since $e^p \subset h^p U_h$, we have $C_{2p} \subset A_p \subset h^p U_h$ by Proposition 7.2. □

7.2 The completion $\hat{Q}_0^{(k)}$ of $Q_0^{(k)}$.

For $p \geq 0$, set $\mathcal{F}_p(Q_0) = C_p \cap Q_0$. Let $\hat{Q}_0$ denote the completion in U_h of Q_0 with respect to the decreasing filtration $\{\mathcal{F}_p(Q_0)\}_{p\geq 0}$, i.e., $\hat{Q}_0$ is the image of the homomorphism

$$\varprojlim_{p\geq 0} Q_0/\mathcal{F}_p(Q_0) \to U_h.$$

induced by the inclusion $Q_0 \subset U_h$, which is well defined since $\mathcal{F}_{2p}(Q_0) \subset C_{2p} \subset h^p U_h$ for $p \geq 0$. For $k \geq 1$, we define the filtration for $Q_0^{(k)}$, by

$$\mathcal{F}_p\big(Q_0^{(k)}\big) = \sum_{j=1}^{k} \sum_{\substack{i_1,\dots,i_k \in \mathbb{Z} \\ i_1+\cdots+i_k=0}} Q_{i_1} \otimes \cdots \otimes \big(C_p \cap Q_{i_j}\big) \otimes \cdots \otimes Q_{i_k}.$$

Define the completion $\hat{Q}_0^{(k)}$ of $Q_0(k)$ as the image of the homomorphism

$$\varprojlim_{p\geq 0} Q_0^{(k)}/\mathcal{F}_p\big(Q_0^{(k)}\big) \to U_h^{\hat{\otimes} n}.$$

For $k = 0$, it is natural to set

$$\mathcal{F}_p\big(Q_0^{(0)}\big) = \mathcal{F}_p\big(\mathbb{Z}[q,q^{-1}]\big) = \begin{cases} \mathbb{Z}[q,q^{-1}] & \text{if } \ p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$\hat{Q}_0^{(0)} = \mathbb{Z}[q,q^{-1}].$$

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. As in Section 6, let $T \in BT_n$ be a ribbon bottom tangle, and $(W \in BT_{2k}; N_1, \dots, N_n)$ a ribbon data for T. Let P_W be an even-trivial diagram of W, and choose a state $s \in S(P_W)$.

Let $|s| = \max\{s(c) \mid c \in C(P_W)\}$ denote the maximal integer of the image of s. Every crossing of P_W has at least one strand in P_W^{odd}. Thus, we assume $s(c) = |s|$ for a crossing c that has a strand of P_{2j-1}, $1 \leq j \leq k$. For $i = 1, \dots, k$, take the elements $w_{2i-1} \in U_{\mathbb{Z},q}$ and $w_{2i} \in U_q^{ev}$ as in the proof of Lemma 6.2. By Corollary 6.7, we have

$$w_{2j-1} \in \mathcal{F}_{|s|}(U_{\mathbb{Z},q}).$$

By Proposition 3.4, we have

$$\begin{aligned} U_{\mathbb{Z},q}\tilde{E}^{(|s|)}U_{\mathbb{Z},q} \triangleright \bar{U}_q^{ev} &\subset U_{\mathbb{Z},q}\tilde{E}^{(|s|)} \triangleright \bar{U}_q^{ev} \subset C_{|s|}, \\ U_{\mathbb{Z},q}\tilde{F}^{(|s|)}U_{\mathbb{Z},q} \triangleright \bar{U}_q^{ev} &\subset U_{\mathbb{Z},q}\tilde{F}^{(|s|)} \triangleright \bar{U}_q^{ev} \subset C'_{|s|}. \end{aligned}$$

Thus, from Proposition 7.2, we have

$$w_{2j-1} \triangleright w_{2j} \in C_{|s|},$$

and by the fact (34), we have

$$\mathrm{ad}^{\otimes k}(\tilde{J}(P_W, s)) = \mathrm{ad}^{\otimes k}(w_1 \otimes \cdots \otimes w_{2k}) \in \mathcal{F}_{|s|}(Q_0^{(k)}).$$

By Proposition 4.3, we have

$$\begin{aligned} J_T &= \mu^{[N_1,\ldots,N_n]}\,\mathrm{ad}^{\otimes k}(J_W) \\ &= \sum_{l\geq 0}\sum_{s\in S(P_W),|s|=l} \mu^{[N_1,\ldots,N_n]}\,\mathrm{ad}^{\otimes k}(\tilde{J}(P_W, s)) \in \hat{Q}_0^{(n)}, \end{aligned}$$

since $\mu^{[N_1,\ldots,N_n]}\big(\mathcal{F}_{|s|}(Q_0^{(k)})\big) \subset \mathcal{F}_{|s|}(Q_0^{(n)})$. Hence we have the assertion. □

7.3 The inclusion $Q_0^{(k)} \subset (\bar{U}_q^{ev})^{\tilde{\otimes} k}$.

Here, we recall Habiro's definitions in [4]. Let $\mathcal{U}_q^{ev}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of U_h generated by K^2, K^{-2}, e and $\tilde{F}^{(n)}$ for $n \geq 1$. For $p \geq 0$, let $\mathcal{F}_p(\mathcal{U}_q^{ev})$ denote the two-sided ideal of $\mathcal{U}_q^{ev}$ generated by e^p, and define the completion $\tilde{\mathcal{U}}_q^{ev}$ of $\mathcal{U}_q^{ev}$ respect to the filtration $\{\mathcal{F}_p(\mathcal{U}_q^{ev})\}_{p\geq 0}$. For $k \geq 1$, we define the filtration $\{\mathcal{F}_p((\mathcal{U}_q^{ev})^{\otimes k})\}_{p\geq 0}$ by

$$\mathcal{F}_p((\mathcal{U}_q^{ev})^{\otimes k}) = \sum_{i=1}^{k} (\mathcal{U}_q^{ev})^{\otimes i-1} \otimes \mathcal{F}_p(\mathcal{U}_q^{ev}) \otimes (\mathcal{U}_q^{ev})^{\otimes k-i},$$

and define the completion $(\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} k}$ of $(\mathcal{U}_q^{ev})^{\otimes k}$ respect to the filtration. For $k = 0$, we define $(\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} 0} = \mathbb{Z}[q, q^{-1}]$ in a similar way to the case of $\hat{Q}_0^{(0)}$. Let $(\bar{U}_q^{ev})^{\sim}$ denote the closure in $\tilde{\mathcal{U}}_q^{ev}$ of $\bar{U}_q^{ev}$, i.e., $(\bar{U}_q^{ev})^{\sim}$ is the completion respect to the filtration $\mathcal{F}_p(\bar{U}_q^{ev}) = \mathcal{F}_p(\mathcal{U}_q^{ev}) \cap \bar{U}_q^{ev}$ for $p \geq 0$. For $k \geq 0$, let $(\bar{U}_q^{ev})^{\sim\tilde{\otimes} k}$ denote the closures in $(\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} k}$ of $(\bar{U}_q^{ev})^{\otimes k}$.

Proposition 7.4. *For $k \geq 1$, we have*

$$\hat{Q}_0^{(k)} \subset (\bar{U}_q^{ev})^{\sim\tilde{\otimes} k}.$$

Proof. We prove

$$C_{2p} \subset \mathcal{F}_p(\mathcal{U}_q^{ev}).$$

for every even integer $p \geq 0$. If we assume this inclusion, then the assertion follows from the natural map

$$Q_0^{(k)}/\mathcal{F}_{2p}\big(Q_0^{(k)}\big) \to (\bar{U}_q^{ev})^{\otimes k}/\big(\mathcal{F}_p\big((\mathcal{U}_q^{ev})^{\otimes k}\big) \cap (\bar{U}_q^{ev})^{\otimes k}\big).$$

In view of Proposition 7.2, we have only to prove $A_p \subset \mathcal{F}_p(\mathcal{U}_q^{ev})$. By Lemma 3.1, it is enough to prove

$$\tilde{F}^{(i)} g \tilde{E}^{(j)} \triangleright e^p \subset \mathcal{F}_p(\mathcal{U}_q^{ev}),$$

for $i, j \geq 0$ and $g \in U^0_{\mathbb{Z},q}$. By the formulas (25), and (23), we have

$$\begin{aligned}
&\tilde{E}^{(j)} \triangleright e^p \subset \bar{U}_q^{ev} e^p, \\
&K \triangleright x = q^{|x|} x, \quad \begin{bmatrix} H+k \\ l \end{bmatrix}_q \triangleright x = \begin{bmatrix} 2|x|+k \\ l \end{bmatrix}_q x, \\
&\tilde{F}^{(i)} \triangleright U_q^{ev} e^p \subset \mathcal{F}_p(\mathcal{U}_q^{ev}),
\end{aligned}$$

for $k \in \mathbb{Z}$, $l \geq 0$, and a homogeneous element $x \in U_h$. Hence we have the assertion. □

8 Applications.

In this section, we recall the universality property of the universal sl_2 invariant over the colored Jones polynomials. Then we prove Theorems 1.6 and 1.7.

8.1 The colored Jones polynomials.

In this subsection, we recall the universality property of the universal sl_2 invariant over the colored Jones polynomials. See [4, Section 5] for the details.

If V is a finite dimensional representation of U_h, then the quantum trace $\mathrm{tr}_q^V(x)$ in V of an element $x \in U_h$ is defined by

$$\mathrm{tr}_q^V(x) = \mathrm{tr}^V(\rho_V(K^{-1}x)) \in \mathbb{Q}[[h]],$$

where $\rho_V \colon U_h \to \mathrm{End}(V)$ denotes the left action of U_h on V, and $\mathrm{tr}^V \colon \mathrm{End}(V) \to \mathbb{Q}[[h]]$ denotes the trace in V. For every element $y = \sum_n a_n V_n \in \mathcal{R}_{\mathbb{Q}(v)}$, $a_n \in \mathbb{Q}(v)$, we set

$$\mathrm{tr}_q^y(x) = \sum_n a_n \, \mathrm{tr}_q^{V_n}(x)$$

for $x \in U_h$.

We shall use the following universality property of the universal sl_2 invariant.

Proposition 8.1 (Habiro [4]). *Let $L = L_1 \cup \cdots \cup L_n$ be an n-component, ordered, oriented, framed link in S^3. Choose an n-component bottom tangle $T \in BT_n$ whose closure is isotopic to L. For $y_1, \ldots, y_n \in \mathcal{R}_{\mathbb{Q}(v)}$, the colored Jones polynomial $J_{L;y_1,\ldots,y_n}$ of L can be obtained from J_T by*

$$J_{L;y_1,\ldots,y_n} = (\mathrm{tr}_q^{y_1} \otimes \cdots \otimes \mathrm{tr}_q^{y_n})(J_T).$$

8.2 The irreducible representations of U_h.

For $l \geq 1$, let V_l be the l-dimensional irreducible representation of U_h with a highest weight vector $x_0^{(l)}$ characterized by $Ex_0^{(l)} = 0, Hx_0^{(l)} = (l-1)x_0^{(l)}$, and $U_h x_0^{(l)} = V_l$. We use the following basis vectors mentioned in [4].

$$x_i^{(l)} = \tilde{F}^{(i)} x_0^{(l)}, \quad \text{for } i = 0, \ldots, l-1.$$

Then $U_{\mathbb{Z},q}$ acts on V_l as

$$K^{\pm 1} x_i^{(l)} = v^{l-1-2i} x_i^{(l)}, \tag{38}$$

$$\tilde{E}^{(m)} x_i^{(l)} = \begin{bmatrix} l-1-i+m \\ m \end{bmatrix}_q x_{i-m}^{(l)}, \tag{39}$$

$$\tilde{F}^{(m)} x_i^{(l)} = q^{-mi} \begin{bmatrix} i+m \\ m \end{bmatrix}_q x_{i+m}^{(l)}, \tag{40}$$

for $i = 0, \ldots, l-1$, and $m \geq 1$. (Here we set $x_i^{(l)} = 0$ unless $0 \leq i \leq l-1$.) Set

$$\tilde{V}_l = \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{x_0^{(l)}, \ldots, x_{l-1}^{(l)}\}.$$

By the formulas (38)–(40), $\tilde{V}_l$ is a $U_{\mathbb{Z},q}^{ev}$-submodule of V_l.

We use the following lemma.

Lemma 8.2. *(1) If $x \in U_h$ is homogeneous with $|x| \neq 0$, then we have $tr_q^{V_l}(x) = 0$ for $l \geq 1$.*

(2) We have $tr_q^{V_{2l}}(\bar{U}_q^{ev0}) \subset (v+v^{-1})\mathbb{Z}[q,q^{-1}]$ for $l \geq 1$.

Proof. (1) If $|x| \neq 0$, then x acts on each V_l nilpotently. Hence $tr_q^{V_l}(x) = 0$.

(2) We have

$$\begin{aligned} \rho_{V_l}(K) &= \mathrm{diag}(v^{l-1}, v^{l-3}, \ldots, v^{-(l-1)}) \\ &= v^{-(l-1)} \,\mathrm{diag}(q^{l-1}, q^{l-2}, \ldots, 1). \end{aligned}$$

Hence the assertion follows from

$$\begin{aligned} tr_q^{V_{2l}}(K^{2j}) &= tr^V(\rho(K^{2j-1})) \\ &= v^{-(2j-1)(2l-1)}(q^{(2j-1)(2l-1)} + q^{(2j-1)(2l-2)} + \cdots + q^{2j-1} + 1) \\ &= v^{-(2j-1)(2l-1)}(q^{2l}-1)/(q-1) \in (v+v^{-1})\mathbb{Z}[q,q^{-1}], \end{aligned}$$

for $l \geq 1, j \in \mathbb{Z}$.

□

8.3 Proof of Theorem 1.6.

Lemma 8.3. *If $0 \leq 2l \leq m$, then we have $\rho_{V_l}\big(\mathcal{F}_m(Q_0)\big) = \{0\}$.*

Proof. The assertion follows from $\mathcal{F}_m(Q_0) \subset \mathcal{F}_{2l}(Q_0) \subset A_l \subset U_h e^l U_h$, and $\rho_{V_l}(e^l) = 0$ from (39). □

Lemma 8.4. *For $l_1, \ldots, l_k \geq 1$, we have*

$$(\mathrm{tr}_q^{V_{l_1}} \otimes \cdots \otimes \mathrm{tr}_q^{V_{l_k}})(Q_0^{(k)}) = (\mathrm{tr}_q^{V_{l_1}} \otimes \cdots \otimes \mathrm{tr}_q^{V_{l_k}})(Q_0^{\otimes k}).$$

Proof. The ($\supset$) part is obvious. We prove the ($\subset$) part. Take homogeneous elements $w_1, \dots, w_k \in \Psi$ such that $\bar{w}_1, \dots, \bar{w}_k \in \bar{U}_q^{ev}$ are homogeneous and $\bar{w}_1 \otimes \cdots \otimes \bar{w}_k \in Q_0^{(k)}$. If $|\bar{w}_i| \neq 0$, then we have

$$\mathrm{tr}_q^{V_l}(\bar{w}_i) = 0,$$

from Lemma 8.2 (1). If $|\bar{w}_i| = 0$, then we have

$$\deg(w_i) \in 2\mathbb{Z}.$$

This implies that we can renormalize $\bar{w}_1, \dots, \bar{w}_k$, by sharing $(q+1)$, so that every $\bar{w}_i$ is in Q_0 for $i = 1, \dots, k$. For example

$$(q+1)^2 ef \otimes ef^2 e = (q+1)ef \otimes (q+1)ef^2 e.$$

Hence we have the assertion. □

Proof of Theorem 1.6. Let L be an n-component ribbon link, and $T \in BT_n$ a ribbon bottom tangle such that $\mathrm{cl}(T) = L$. Let $(W \in BT_{2k}; N_1, \dots, N_n)$ be a ribbon data for T, and P_W an even-trivial diagram of W. By Proposition 8.1, we have

$$J_{L;V_{2l_1},\dots,V_{2l_n}} = (tr_q^{V_{2l_1}} \otimes \cdots \otimes tr_q^{V_{2l_n}})(J_T).$$

By Theorem 1.2 and Lemma 8.3, We have only to prove

$$(\mathrm{tr}_q^{V_{2l_1}} \otimes \cdots \otimes \mathrm{tr}_q^{V_{2l_k}})(Q_0^{(k)}) \subset (v+v^{-1})^k \mathbb{Z}[q, q^{-1}].$$

By Lemma 8.4, it is enough to prove

$$\mathrm{tr}_q^{V_{2l}}(Q_0) \subset (v+v^{-1})\mathbb{Z}[q, q^{-1}].$$

for $l \geq 1$. Note that

$$Q_0 \subset \bar{U}_q^{ev0} + \langle e^2, f^2 \rangle_{ideal} + \langle (q+1)e, (q+1)f \rangle_{ideal}.$$

By Lemma 8.2 (2), we have

$$\mathrm{tr}_q^{V_{2l}}(\bar{U}_q^{ev0}) \subset (v+v^{-1})\mathbb{Z}[q, q^{-1}].$$

Since $e^2 = (q+1)(q-1)^2 \tilde{E}^{(2)}$ and $f^2 = (q+1)(q-1)^2 q^{-1} \tilde{F}^{(2)}$, we have

$$\rho_{V_{2l}}(e^2), \quad \rho_{V_{2l}}(f^2) \in (q+1)\,\mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\tilde{V}_{2l}).$$

Finally, it is clear that

$$\rho_{V_{2l}}((q+1)e), \quad \rho_{V_{2l}}((q+1)f) \in (q+1)\,\mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\tilde{V}_{2l}).$$

Note that

$$\mathrm{tr}_q^{V_{2l}}(U_{\mathbb{Z},q}^{ev}) = \mathrm{tr}^{V_{2l}}(\rho_{V_{2l}}(K^{-1} U_{\mathbb{Z},q}^{ev})) \subset v\mathbb{Z}[q, q^{-1}],$$

for $l \geq 1$. Thus, we have the assertion. □

8.4 Proof of Theorem 1.7.

Lemma 8.5. *For $k \geq 0$, we have*

$$\Delta^{[k]}(Q_n) \subset Q_n^{(k)}.$$

Proof. It is enough to prove

$$\Delta(K^{\pm 2}) \in Q_0^{(2)}, \tag{41}$$

$$\Delta(e^2) \in Q_0^{(2)}, \quad \Delta(e) \in Q_1^{(2)}, \tag{42}$$

$$\Delta(f^2) \in Q_0^{(2)}, \quad \Delta(f) \in Q_1^{(2)}. \tag{43}$$

(41) follows from $\Delta(K^{\pm 2}) = K^{\pm 2} \otimes K^{\pm 2} \in Q_0^{(2)}$. We prove (42), then (43) is similar. We have

$$\Delta(e^2) = e^2 \otimes 1 + [2]_q eK \otimes e + K^2 \otimes e^2 \in Q_0^{(2)},$$
$$\Delta(e) = e \otimes 1 + K \otimes e \in Q_1^{(2)}.$$

Hence we have the assertion. □

Lemma 8.6. *(1) If $0 \leq 2(l+1) \leq m$, then we have $\mathrm{tr}_q^{P_l}(\mathcal{F}_m(Q_0)) = 0$.*

(2) For $l_1 \dots l_k \geq 0$, we have

$$(\mathrm{tr}_q^{P_{l_1}} \otimes \cdots \otimes \mathrm{tr}_q^{P_{l_k}})(Q_0^{(k)}) = (\mathrm{tr}_q^{P_{l_1}} \otimes \cdots \otimes \mathrm{tr}_q^{P_{l_k}})(Q_0^{\otimes k}).$$

Proof. For $l \geq 0$, P_l is a linear combination of $V_1, \dots, V_{l+1}$. Then the assertions (1) and (2) follow from Lemmas 8.3 and 8.4, respectively. □

Proof of Theorem 1.7. Let L be an n-component ribbon link, and $T \in BT_n$ a ribbon bottom tangle such that $\mathrm{cl}(T) = L$. Let $(W \in BT_{2k}; N_1, \dots, N_n)$ be a ribbon data for T, and P_W an even-trivial diagram of W. By Proposition 8.1, we have

$$J_{L;P_{k_1},\dots,P_{k_n}} = (\mathrm{tr}_q^{P_{k_1}} \otimes \cdots \otimes \mathrm{tr}_q^{P_{k_n}})(J_T).$$

By Theorem 1.2 and Lemma 8.6 (1), We have only to prove

$$(\mathrm{tr}_q^{P_{l_1}} \otimes \cdots \otimes \mathrm{tr}_q^{P_{l_n}})(Q_0^{(n)}) \subset (v + v^{-1})^{l_1 + \cdots + l_n} \mathbb{Z}[q, q^{-1}].$$

By Lemma 8.6 (2), it is enough to prove $\mathrm{tr}_q^{P_l}(Q_0) \subset (v + v^{-1})^l \mathbb{Z}[q, q^{-1}]$ for $l \geq 0$. For $i \geq 0$, set $p_i = V_2 - (v^{2i+1} + v^{-2i-1})$. Thus, we have $P_l = p_0 \cdots p_{l-1}$. For $z \in U_h$, we have

$$\mathrm{tr}_q^{P_l}(z) = \sum \mathrm{tr}_q^{p_0}(z_{(1)}) \cdots \mathrm{tr}_q^{p_{l-1}}(z_{(l)}),$$

where $\Delta^{[l]}(z) = \sum z_{(1)} \otimes \cdots \otimes z_{(l)}$. By Lemmas 8.5 and 8.6 (2) again, it is enough to prove

$$\mathrm{tr}_q^{p_i}(Q_0) \subset (v + v^{-1}) \mathbb{Z}[q, q^{-1}].$$

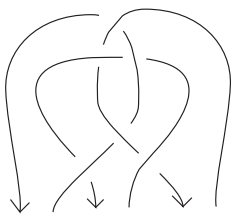

Figure 22: The Borromean tangle $B \in BT_3$.

for $i \geq 0$. Since $p_i = V_2 - v^{-2i-1}(q^{2i+1}+1)$, we have

$$\mathrm{tr}_q^{p_i}(Q(0)) = \mathrm{tr}_q^{V_2}(Q_0) - v^{-2i-1}(q^{2i+1}+1)\varepsilon(Q_0).$$

By Theorem1.6, the right hand side is contained in $(v+v^{-1})\mathbb{Z}[q,q^{-1}]$. Hence we have the assertion. $\square$

Remark 8.7. *One can prove Theorems 1.6 and 1.7 from Theorem 1.5, without using the universal sl_2 invariant as follows. We use the notation $V_l' = V_{l+1}$. The following formula is well known.*

$$V_m' V_l' = V_{m+l}' + V_{m+l-2}' + \cdots + V_{m-l}' \in \mathcal{R}_{\mathbb{Z}[v,v^{-1}]},$$

for $m \geq l \geq 0$. This implies that for every $l \geq 1$, V_{2l} is equal to a linear combination of V_2^i in $\mathcal{R}_{\mathbb{Z}[v,v^{-1}]}$ for $i = 1, \ldots, l$. The colored Jones polynomial of a link $L = L_1 \cup \cdots \cup L_n$, where $V_2^{i_m}$ is attached to a component L_m for $m = 1, \ldots, n$, is equal to the Jones polynomial (with all components colored by V_2) of the link obtained from L by replacing L_m with its i_m parallels $L_{m,1}, \ldots, L_{m,i}$. Then Theorems 1.6 and 1.7 follow from Theorem 1.5 and the fact that the link obtained from a ribbon link by replacing any component with its parallels is a ribbon link.

9 Examples.

The Borromean tangle $B \in BT_3$ is the bottom tangle depicted in Figure 22. Note that B is a 3-component, algebraically-split, 0-framed bottom tangle, and the closure of B is the Borromean rings L_B. It is well known that L_B is not a ribbon link.

The following formula is observed in [4, Section 4.3].

$$J_B = \sum_{m_1,m_2,m_3,n_1,n_2,n_3 \geq 0} q^{m_3+n_3}(-1)^{n_1+n_2+n_3} q^{\sum_{i=1}^3 \left(-\frac{1}{2}m_i(m_i+1)-n_i+m_i m_{i+1}-2m_i n_{i-1}\right)}$$
$$\tilde{F}^{(n_3)} e^{m_1} F^{(m_3)} e^{n_1} K^{-2m_2} \otimes F^{(n_1)} e^{m_2} F^{(m_1)} e^{n_2} K^{-2m_3} \otimes F^{(n_2)} e^{m_3} F^{(m_2)} e^{n_3} K^{-2m_1}, \tag{44}$$

where the index i should be considered modulo 3. The right hand side is not contained in $(\bar{U}_q^{ev})^{\tilde{\otimes} 3}$.

For a 3-component ribbon link L, we have

$$\begin{aligned} V(L) &\in (v+v^{-1})^3\mathbb{Z}[q,q^{-1}], \\ J_{L_B;P_1,P_1,P_1} &\in (v-v^{-1})^8(v+v^{-1})(v^2+1+v^{-2})\mathbb{Z}[q,q^{-1}] \cap (v+v^{-1})^3\mathbb{Z}[q,q^{-1}] \\ &= (v-v^{-1})^8(v+v^{-1})^3(v^2+1+v^{-2})\mathbb{Z}[q,q^{-1}], \end{aligned}$$

by Theorems 1.4, 1.6, and 1.7.

The Jones polynomial $V(L_B)$ of the Borromean rings L_B is

$$\begin{aligned} V(L_B) =& (v+v^{-1})^3 - (v-v^{-1})^4(v+v^{-1})(v^2+1+v^{-2}) \\ &\notin (v+v^{-1})^3\mathbb{Z}[q,q^{-1}]. \end{aligned} \tag{45}$$

The colored Jones polynomial $J_{L_B;P_1,P_1,P_1}$ of L_B is

$$\begin{aligned} J_{L_B;P_1,P_1,P_1} =& -(v-v^{-1})^4(v+v^{-1})(v^2+1+v^{-2}) \\ &\notin (v-v^{-1})^8(v+v^{-1})^3(v^2+1+v^{-2})\mathbb{Z}[q,q^{-1}]. \end{aligned} \tag{46}$$

Thus, each of (44), (45) and (46) implies that the Borromean rings L_B is not a ribbon link.

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