The universal quantum invariant and colored ideal triangulations

Sakie Suzuki

RIMS/The Hakubi Center, Kyoto university

2017.11.1
Topological invariants in low dimensional topology @ Shimane University

Introduction

Drinfeld double and Heisenberg double

Universal quantum invariant and its reconstruction

Extension

3-dim. descriptions

Introduction

- Background
- Ideas for reconstruction of quantum invariants
- State sum invariant with weights in a non-commutative ring

Background

1984 Jones polynomial

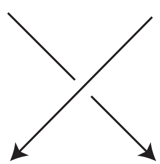
↓

"Quantum invariants"

- Colored Jones polynomial
- Reshetkhin–Turaev invariant
- Universal quantum invariant
- Kontsevich integral

Background

KEY POINT FOR CONSTRUCTIONS

 $\mapsto$ "R-matrix"

RIII move $\mapsto$ "hexagon identity"

Background

- Reshetkhin-Turaev invariant
  $R \in \mathrm{End}(V \otimes V)$, V: fin.dim. linear sp.

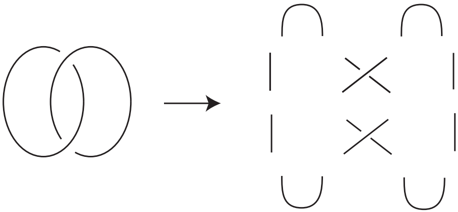
$$(1 \otimes R)(R \otimes 1)(1 \otimes R) = (R \otimes 1)(1 \otimes R)(R \otimes 1)$$

- Universal quantum invariant
  $R \in \mathfrak{R}^{\otimes 2}$, $\mathfrak{R}$: ribbon Hopf algebra

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Background

Definitions are combinatorial and diagrammatic



⇒ It is not easy to see topological properties of links from quantum invariants.

Background

What are "topological properties" of links?

- Properties defined using simple operations or surfaces. e.g. invertible, achiral, Brunnian, ribbon, boundary, etc.
- Properties defined by classical invariants. e.g. genus, homology, fundamental group, bridge number, Milnor invariants, etc.

Background

TASK

Find relationships between quantum invariants and topological properties of links!

Background

METHODS

Link (3-dim. obj.)
(w/ topological properties)

$\downarrow$

Link diagram (2-dim. obj.) $\rightsquigarrow$ Quantum invariants
(w/ planer properties)

Background

METHODS

Link (3-dim. obj.) $\to$ triangulation (3-dim. obj.)
(w/ topological properties)

$\downarrow$ $\rightsquigarrow$

Link diagram (2-dim. obj.) $\rightsquigarrow$ Quantum invariants
(w/ planer properties)

Ideas for reconstruction of quantum invariants

A: a fin-dim Hopf algebra/k

1. Drinfeld double $D(A) \sim_k A^* \otimes A$

   $\Rightarrow R \in D(A)^{\otimes 2}$ s.t.
   $$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in D(A)^{\otimes 3}.$$

2. Heisenberg double $H(A) \sim_k A^* \otimes A$

   $\Rightarrow S \in H(A)^{\otimes 2}$ s.t.
   $$S_{12}S_{13}S_{23} = S_{23}S_{12} \in H(A)^{\otimes 3}.$$

Ideas for reconstruction of quantum invariants

Theorem (Kashaev '97)

There is an algebra embedding

$$\phi\colon\ D(A) \to H(A) \otimes H(A)^{\mathrm{op}},$$

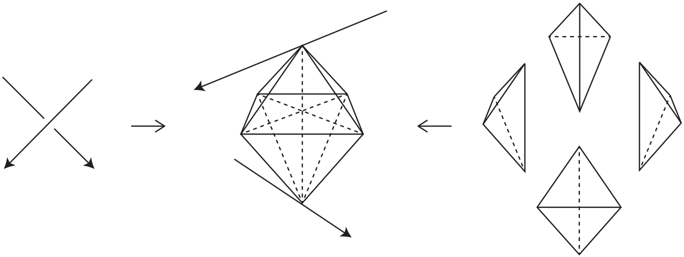
s.t.

$$\phi^{\otimes 2}(R) = S''_{14} S_{13} \tilde{S}_{24} S'_{23}.$$

$S', S'', \tilde{S}$: modifications of S satisfying pentagon relations

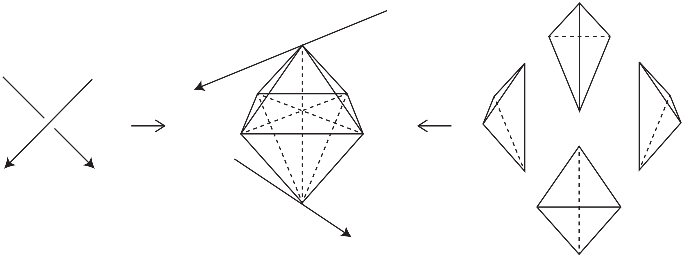
Ideas for reconstruction of quantum invariants

Octahedral triangulations of link complements



Ideas for reconstruction of quantum invariants

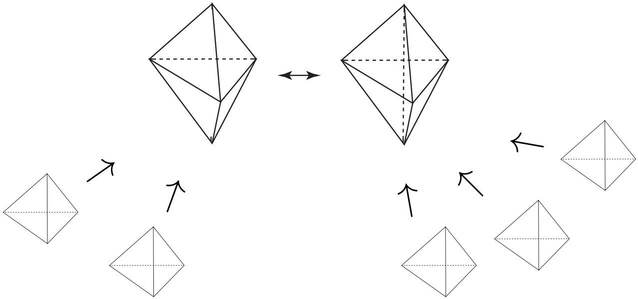
Octahedral triangulations of link complements



$$\phi^{\otimes 2}(R) \quad = \quad S''_{14} S_{13} \tilde{S}_{24} S'_{23}$$

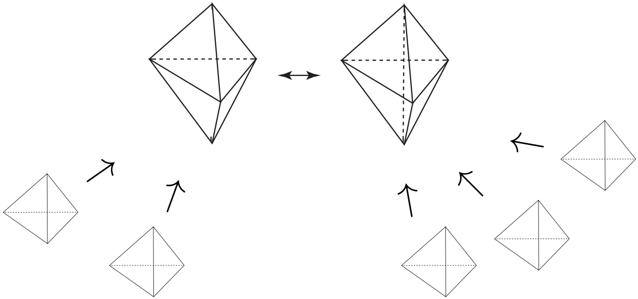
Ideas for reconstruction of quantum invariants

Pachner $(2,3)$ move



Ideas for reconstruction of quantum invariants

Pachner $(2,3)$ move

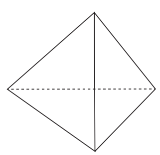


$$S_{23}S_{12} \quad = \quad S_{12}S_{13}S_{23}$$

Ideas for reconstruction of quantum invariants

TO SUM UP...

Idea for the reconstruction

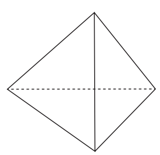
 $\mapsto$ S-tensor

Pachner $(2,3)$ move $\mapsto$ pentagon identity

Ideas for reconstruction of quantum invariants

TO SUM UP...

Idea for the reconstruction

 $\mapsto$ S-tensor

Pachner $(2,3)$ move $\mapsto$ pentagon identity

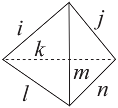
In this talk: w/ universal quantum invariant

State sum invariant with weights in a non-commutative ring

Turaev-Viro's state sum invariant for $(M, \mathcal{T})$:

$$Z(M) = w^{-\#\{\text{verteces}\}} \sum_{\lambda} w_\lambda \prod_{T} W(T;\lambda)$$

- $\mathcal{T}$: a triangulation of M
- λ: a color (giving an integer on each edge)
- T: a tetrahedron in $\mathcal{T}$
- $W(T;\lambda) \in \mathbb{C}$: the weight on T



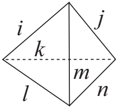
$\mapsto W(T;\lambda)$

State sum invariant with weights in a non-commutative ring

Turaev-Viro's state sum invariant for $(M, \mathcal{T})$:

$$Z(M) = w^{-\#\{\text{verteces}\}} \sum_{\lambda} w_{\lambda} \prod_{T} W(T;\lambda)$$

- $\mathcal{T}$: a triangulation of M
- λ: a color (giving an integer on each edge)
- T: a tetrahedron in $\mathcal{T}$
- $W(T;\lambda) \in \mathbb{C}$: the weight on T satisfying a pentagon identity.



$\mapsto W(T;\lambda)$

State sum invariant with weights in a non-commutative ring

1. [Turaev-Viro]
   (triangulation, quantum $6j$-symbol)

$$\begin{vmatrix} j_1 & j_2 & j_3 \\ i_1 & i_2 & i_3 \end{vmatrix} \begin{vmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{vmatrix} = \sum_n [n]_q \begin{vmatrix} i_1 & i_2 & j_3 \\ k_2 & k_1 & n \end{vmatrix} \begin{vmatrix} i_2 & i_3 & j_1 \\ k_3 & k_2 & n \end{vmatrix} \begin{vmatrix} i_3 & i_1 & j_2 \\ k_1 & k_3 & n \end{vmatrix}$$

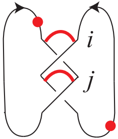
2. [Baseilhac-Benedetti] QHI
   (ideal triangulation, quantum dilogarithm)

$$\Psi(V)\Psi(U) = \Psi(U)\Psi(-UV)\Psi(V)$$

State sum invariant with weights in a non-commutative ring

3. The universal quantum invariant
   (link diagram, the universal R-matrix)

$$R = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in D(A)^{\otimes 2}$$

$J:$ 

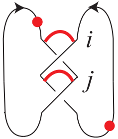
$$\mapsto \quad \sum_{i,j\geq 0} \beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u$$

State sum invariant with weights in a non-commutative ring

3. The universal quantum invariant
   (link diagram, the universal R-matrix)

$$R = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in D(A)^{\otimes 2}$$

$J:$



$$\mapsto \underbrace{\textstyle\sum_{i,j\geq 0}}_{\text{state sum}} \underbrace{\beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u}_{\text{products of weights}}$$

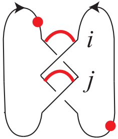
state sum ↗ products of weights ↗

State sum invariant with weights in a non-commutative ring

3. The universal quantum invariant
   (link diagram, the universal R-matrix)

$$R = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in D(A)^{\otimes 2}$$

$J:$

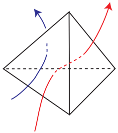


$$\mapsto \underbrace{\sum_{i,j\geq 0}}_{\text{state sum}} \underbrace{\beta'_j \alpha_i u^{-1} \otimes \alpha'_j \beta_i u}_{\text{products of weights}}$$

The orientation of the link $\Rightarrow$ The order of products of weights.

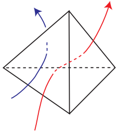
State sum invariant with weights in a non-commutative ring

4. Reconstruction of the universal quantum invariant (colored ideal triangulation, the S-tensor)

J' :  $\mapsto$ $S = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in H(A)^{\otimes 2}$

State sum invariant with weights in a non-commutative ring

4. Reconstruction of the universal quantum invariant (colored ideal triangulation, the S-tensor)

J' :  $\mapsto$ $S = \sum_{i \geq 0} \alpha_i \otimes \beta_i \in H(A)^{\otimes 2}$

- invariant for "colored" 3-mfds
  ($\exists$ a canonical choice of the color for a link $\Rightarrow$ link inv.)
- invariant for closed 3-mfds if A is involutory

Research topics in front of us

w/ Reconstruction:

- v.s. topological properties of links
- v.s. Volume conjecture
- v.s. Phys?
- "Quantum group theory" for Heisenberg double

Research topics in front of us

w/ Reconstruction:

- v.s. topological properties of links
- v.s. Volume conjecture
- v.s. Phys?
- "Quantum group theory" for Heisenberg double

w/ J' for closed 3-mfds:

- v.s. WRT invariant
- v.s. Turaev-Viro invariant, QHI, and Kuperberg invariant

Drinfeld double and Heisenberg double

Quasi-triangular Hopf algebra

Quasi-triangular Hopf algebra $(\mathfrak{R}, \eta, m, \varepsilon, \Delta, \gamma, R)$: Hopf algebra with the universal R-matrix $R \in \mathfrak{R}^{\otimes 2}$ such that

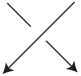
$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1} \quad \text{for } x \in \mathfrak{R},$$
$$(\Delta \otimes 1)(R) = R_{13}R_{23}, \quad (1 \otimes \Delta)(R) = R_{13}R_{12}.$$

Quasi-triangular Hopf algebra

Quasi-triangular Hopf algebra $(\mathfrak{R}, \eta, m, \varepsilon, \Delta, \gamma, R)$: Hopf algebra with the universal R-matrix $R \in \mathfrak{R}^{\otimes 2}$ such that

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1} \quad \text{for } x \in \mathfrak{R},$$
$$(\Delta \otimes 1)(R) = R_{13}R_{23}, \quad (1 \otimes \Delta)(R) = R_{13}R_{12}.$$

$\Rightarrow$ invariant for braids.

 $\mapsto$ R

Ribbon Hopf algebra

Ribbon Hopf algebra $(\mathfrak{R}, \eta, m, \varepsilon, \Delta, \gamma, R, \theta)$: quasi-triangular Hopf algebra with the ribbon element $\theta \in \mathfrak{R}$ such that

$$\theta^2 = u\gamma(u), \quad \gamma(\theta) = \theta, \quad \varepsilon(\theta) = 1, \quad \Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta),$$

where $u = \sum \gamma(\beta)\alpha$ with $R = \sum \alpha \otimes \beta$.

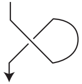
Ribbon Hopf algebra

Ribbon Hopf algebra $(\mathfrak{R}, \eta, m, \varepsilon, \Delta, \gamma, R, \theta)$: quasi-triangular Hopf algebra with the ribbon element $\theta \in \mathfrak{R}$ such that

$$\theta^2 = u\gamma(u), \quad \gamma(\theta) = \theta, \quad \varepsilon(\theta) = 1, \quad \Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta),$$

where $u = \sum \gamma(\beta)\alpha$ with $R = \sum \alpha \otimes \beta$.

$\Rightarrow$ invariant for tangles.

 $\mapsto \quad \theta$

Notation

$A = (A, \eta, m, \varepsilon, \Delta, \gamma)$: a fin-dim Hopf algebra over a field k, with basis $\{e_\alpha\}_\alpha$.

$A^{\mathrm{op}} = (A, \eta, m^{\mathrm{op}}, \varepsilon, \Delta, \gamma^{-1})$: the opposite Hopf algebra of A,

$(A^{\mathrm{op}})^* = (A^*, \varepsilon^*, \Delta^*, \eta^*, (m^{\mathrm{op}})^*, (\gamma^{-1})^*)$: the dual of A^{op}.

Drinfeld double and Heisenberg double

The Drinfeld double (quasi-triangular Hopf algebra):

$$D(A) = ((A^{\mathrm{op}})^* \otimes A, \eta_{D(A)}, m_{D(A)}, \varepsilon_{D(A)}, \Delta_{D(A)}, \gamma_{D(A)}, R)$$

The universal R-matrix $R = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in D(A)^{\otimes 2}$ satisfies

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad \in D(A)^{\otimes 3}.$$

The Heisenberg double (algebra with the S-tensor):

$$H(A) = (A^* \otimes A, \eta_{H(A)}, m_{H(A)})$$

The S-tensor $S = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in H(A)^{\otimes 2}$ satisfies

$$S_{12}S_{13}S_{23} = S_{23}S_{12} \quad \in H(A)^{\otimes 3}.$$

Drinfeld double and Heisenberg double

Set

$$S' = \sum (1 \otimes \tilde{e}_a) \otimes (e^a \otimes 1) \qquad \in H(A)^{\mathrm{op}} \otimes H(A),$$
$$S'' = \sum (1 \otimes e_a) \otimes (\tilde{e}^a \otimes 1) \qquad \in H(A) \otimes H(A)^{\mathrm{op}},$$
$$\tilde{S} = \sum (1 \otimes \tilde{e}_a) \otimes (\tilde{e}^a \otimes 1) \qquad \in H(A)^{\mathrm{op}} \otimes H(A)^{\mathrm{op}},$$

where $\tilde{e}_a = \gamma(e_a)$ and $\tilde{e}^b = (\gamma^*)^{-1}(e^b)$.

Drinfeld double and Heisenberg double

Theorem (Kashaev '97)

We have $\phi\colon\ D(A) \to H(A) \otimes H(A)^{\mathrm{op}}$ *such that*

$$\phi^{\otimes 2}(R) = S''_{14} S_{13} \tilde{S}_{24} S'_{23}.$$

Drinfeld double and Heisenberg double

$D(A)$: Drinfeld double of A.
We have a ribbon Hopf algebra

$$\mathfrak{R} = D(A)[\theta]/\left(\theta^2 - u\gamma(u)\right),$$

where $u = \sum \gamma^*(e^a) \otimes e_a$.
We also consider the algebra

$$\mathcal{H} = \left(H(A) \otimes H(A)^{\mathrm{op}}\right)[\bar{\theta}]/\left(\bar{\theta}^2 - \phi(u\gamma(u))\right),$$

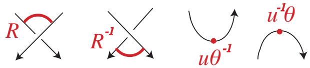
and extend the embedding $\phi\colon D(A) \to H(A) \otimes H(A)^{\mathrm{op}}$ to the map $\bar{\phi}\colon \mathfrak{R} \to \mathcal{H}$ by $\bar{\phi}(\theta) = \bar{\theta}$.

Universal quantum invariant and its reconstruction

Universal quantum invariant for tangles in a cube

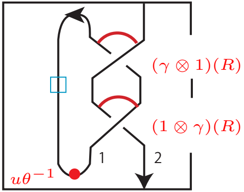
(1) Choose a diagram

(2) Put labels



(3) Read labels

$C =$



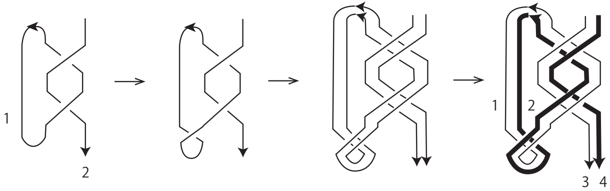
$$J(C) = \sum \gamma(\alpha)\gamma(\beta')u\theta^{-1} \otimes \alpha'\beta \in \bar{\mathfrak{R}} \otimes \mathfrak{R}.$$

$$(R = \sum \alpha \otimes \beta = \sum \alpha' \otimes \beta')$$

Reconstruction of the universal quantum invariant

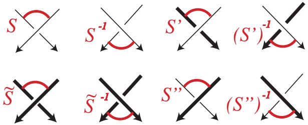
(1) Modify diagram

- Exchange  and  with  and , resp.
- Duplicate stracds
- Thicken the left strands

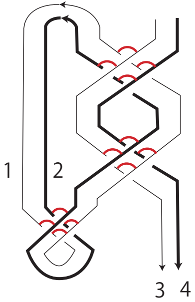


Reconstruction of the universal quantum invariant

(2) Put labels

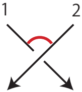


(3) Read the labels

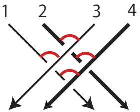


$$J'(C) = (\bar{\theta} \otimes 1)\phi^2(J(C)) \in \bar{\mathcal{H}} \otimes \mathcal{H}.$$

Sketch of proof

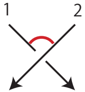
$J:$ 

$\mapsto \quad R$

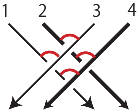
$J':$ 

$\mapsto \quad S''_{14} S_{13} \tilde{S}_{24} S'_{23}$

Sketch of proof

J : 

$\mapsto \quad R$

J' : 

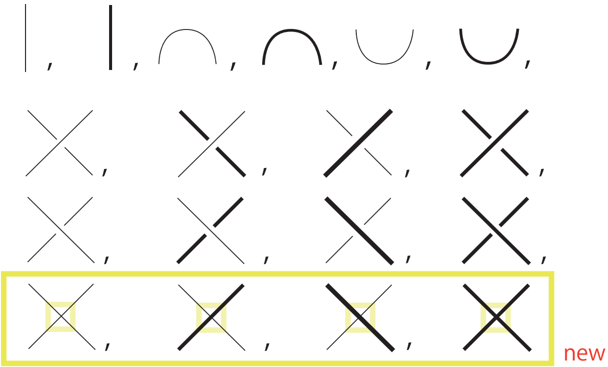
$\mapsto \quad S''_{14} S_{13} \tilde{S}_{24} S'_{23} = \phi^{\otimes 2}(R)$

Extension of the universal quantum invariant

- Colored diagrams
- Colored moves
- Invariance of the universal quantum invariant

Colored diagrams

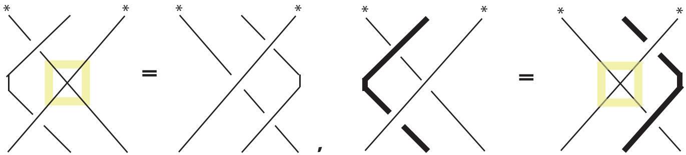
: tangle diagrams obtained from the following parts



We can define the map J' on colored diagrams in a similar way.

Colored moves

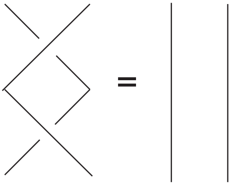
- Colored Pachner $(2,3)$ moves



Here, the orientation of each strand is arbitrary, and the thickness of each strand with $*$-mark is arbitrary.

Colored moves

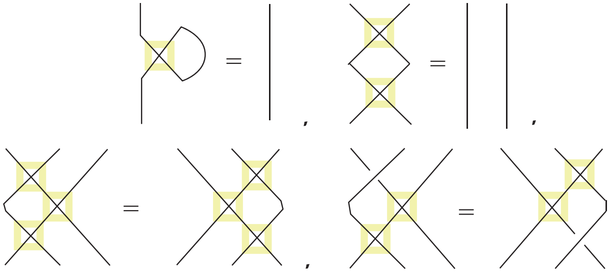
- Colored $(0, 2)$ moves



Here, the orientation and thickness of each strand are arbitrary.

Colored moves

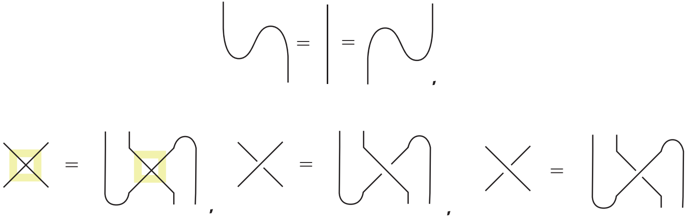
- Colored symmetry moves



Here, the orientation and thickness of each strand are arbitrary.

Colored moves

- Planer isotopies



Here, the orientation and thickness of each strand are arbitrary.

Invariance of the universal quantum invariant

$\mathcal{CD}$: the set of colored diagrams

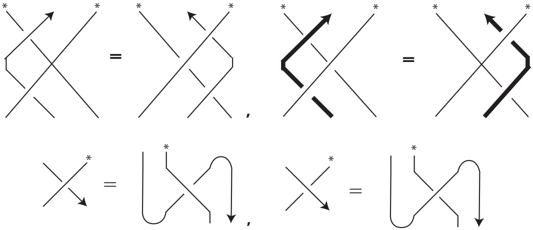
$\sim_c$: the equivalence relation on $\mathcal{CD}$ generated by colored moves.

Theorem (S)

If $\gamma^2 = 1$, then the map J' is an invariant under $\sim_c$.

Invariance of the universal quantum invariant

$\sim'_c$: the equivalence relation on $\mathcal{CD}$ generated by colored moves except for



Theorem (S)

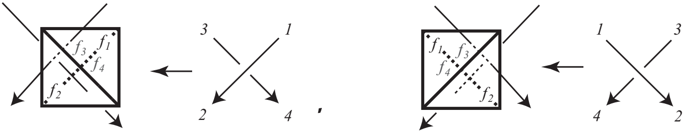
The map J' is an invariant under $\sim'_c$.

3-dimensional descriptions

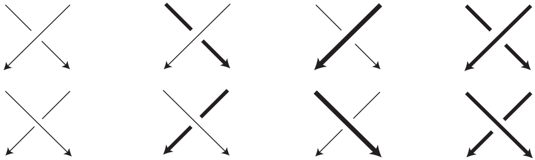
- Colored singular triangulations
- Colored moves
- v.s. link complements

Colored tetrahedron

: a tetrahedron with an ordering f_1, f_2, f_3, f_4 of its faces



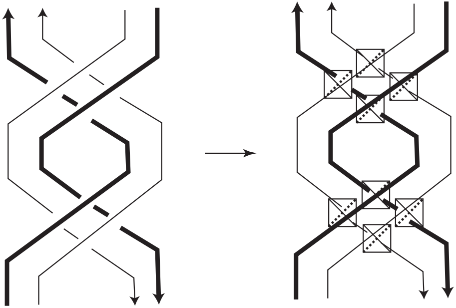
There are eight types of colored tetrahedra:



Colored singular triangulation $\mathcal{C}(Z)$

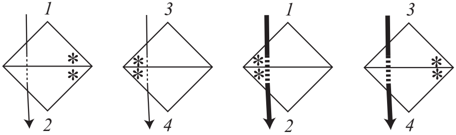
Define $\mathcal{C}(Z)$ for a colored diagram Z as follows.

(1) Place tetrahedra



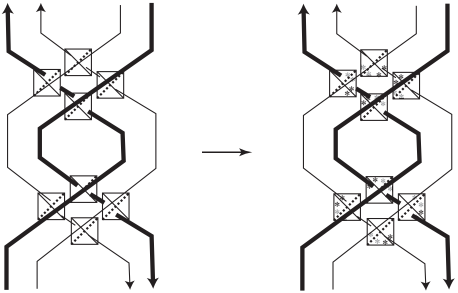
Colored singular triangulation $\mathcal{C}(Z)$

(2) Define star-vertices

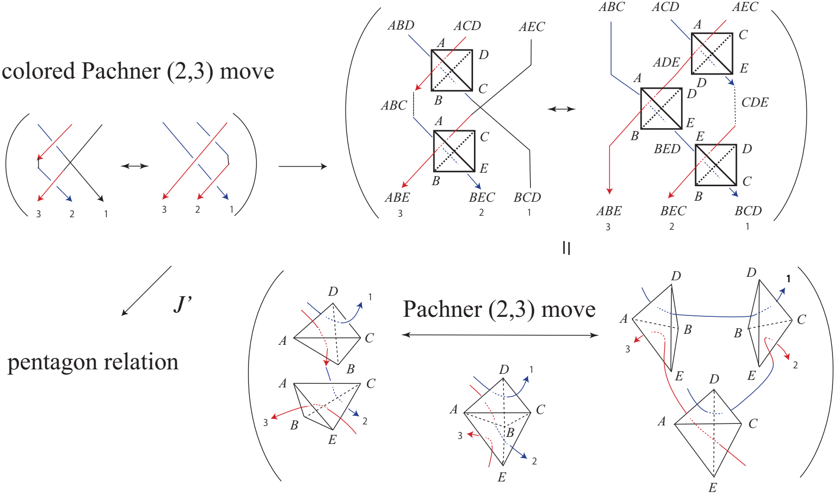


Colored singular triangulation $\mathcal{C}(Z)$

(2) Attach the tetrahedra



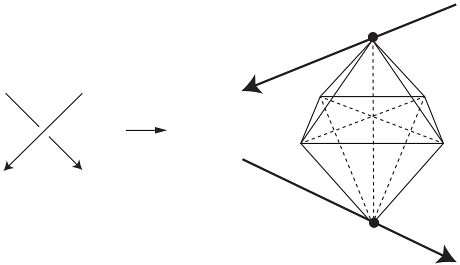
Colored moves



v.s. link complements in $S^3 \setminus \{\pm\infty\}$

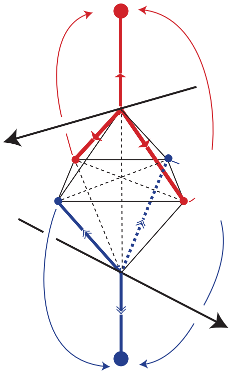
The octahedral decomposition $\mathcal{O}(D)$:

(1) Place an octahedron at each crossing

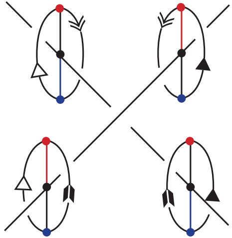


v.s. link complements in $S^3 \setminus \{\pm\infty\}$

(2) Attach the octahedra



$\xrightarrow{\partial}$



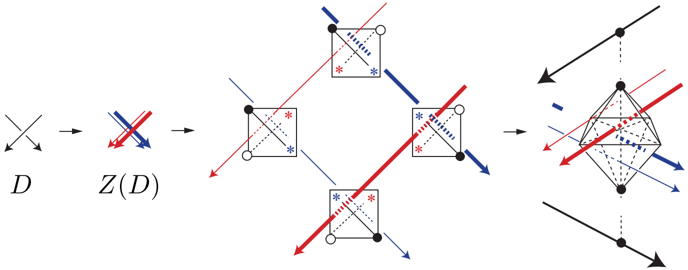
the boundary of the octahedron

v.s. link complements in $S^3 \setminus \{\pm\infty\}$

Theorem (S)

The octahedral triangulation $\mathcal{O}(D)$ admits a colored ideal triangulation $\mathcal{C}(Z(D))$.

Sketch of the proof



Remarks

- $\gamma^2 = 1 \Rightarrow J'$ is an inv. of closed 3-mfd.
- (Conj) $\gamma^2 \neq 1 \Rightarrow J'$ is an inv. of framed 3-mfd.
- The colored diagrams form a strict monoidal category and J' is formulated as a functor.
- Hoping to get TQFT if we take $L^2(\mathbb{R})$ as a module of $H(B_q(sl_2))$, which may give $\mathrm{Vol}(M) + i\mathrm{CS}(M)$.