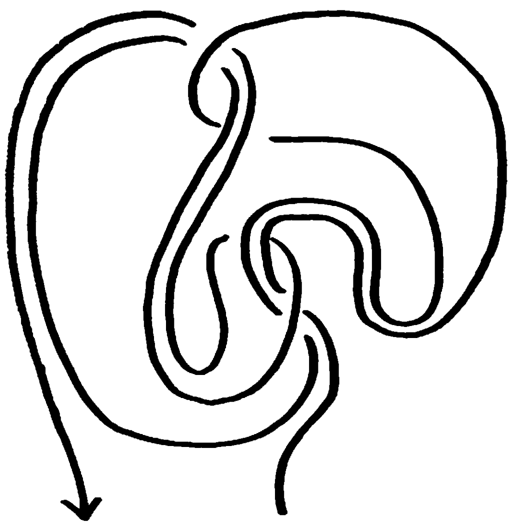
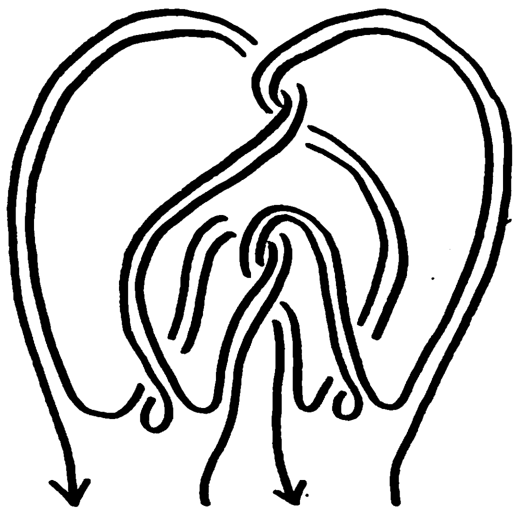
On the universal $\mathfrak{sl}_2$ invariant of bottom tangles

Knots in Poland III

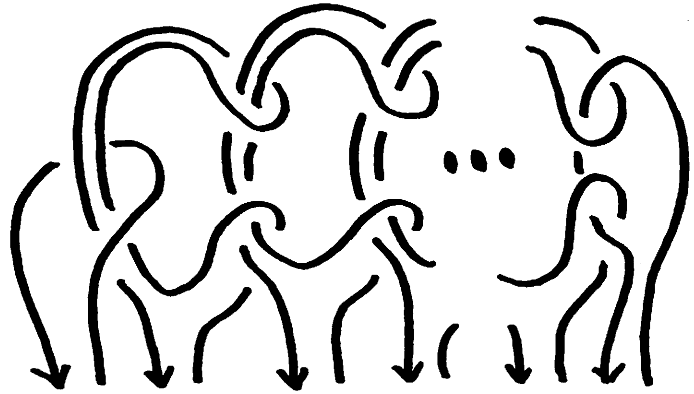
Sakie Suzuki (RIMS)



ribbon



boundary



brunnian

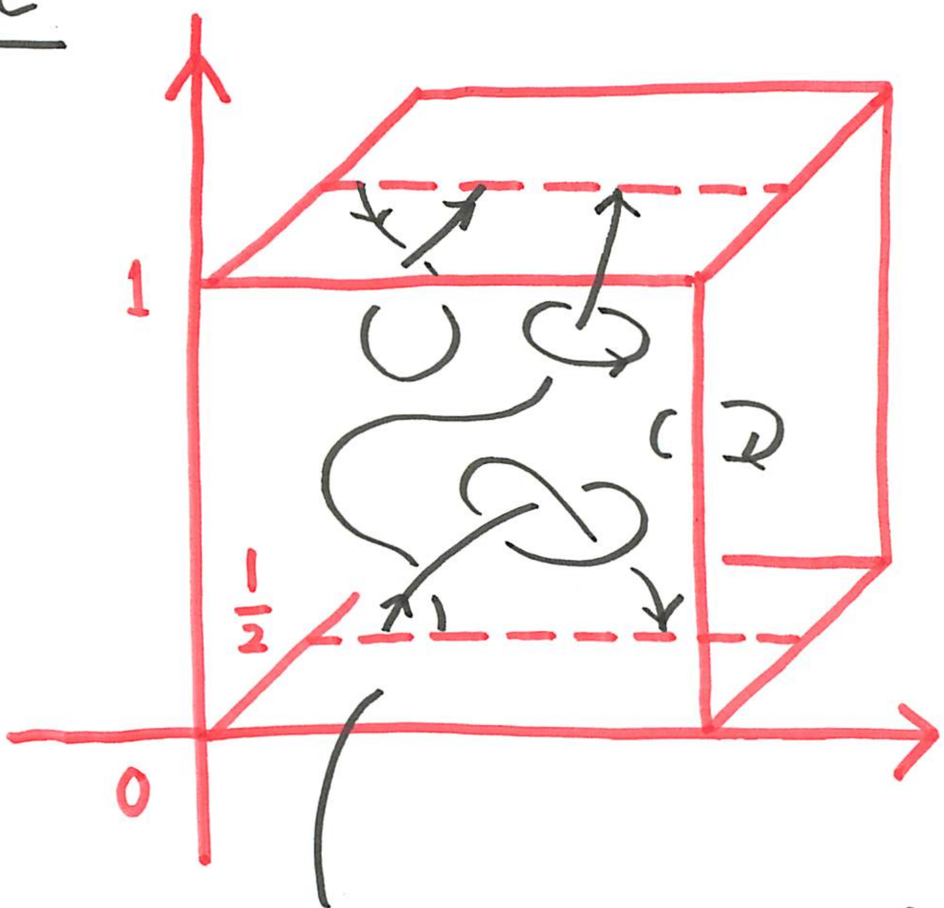
Today's plan

1. Tangles and bottom tangles — { boundary / ribbon / brunnian }
2. Motivation
3. The universal sl_2 invariant
4. Main theorems and applications
5. Conjectures on Milnor invariants
6. Proof

Tangle in a cube

ex)

$$\coprod^{3} [0,1] \coprod^{2} S^1 \underset{\text{emb}}{\hookrightarrow}$$



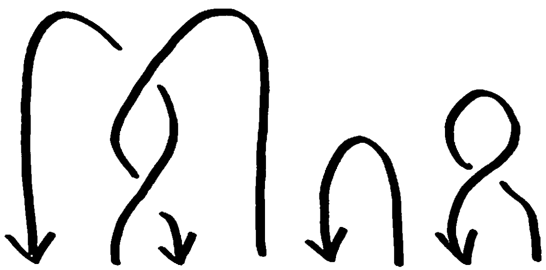
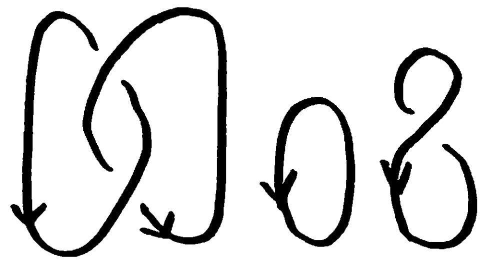
end pts $\subset [0,1] \times \{\frac{1}{2}\} \times \{0,1\}$

- Orientation

- framing

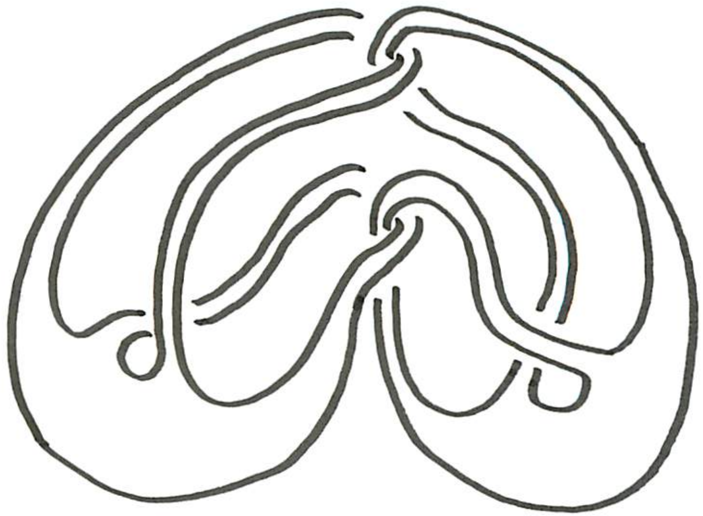
<u>Bottom tangle</u> … tangle in a cube

- only arc components
- end points are on the bottom $[0.1] \times \{\frac{1}{2}\} \times \{0\}$
- two end points of an arc are adjacent
- every arc starts from the right and travels to the left

$T =$  $cl(T) =$ 

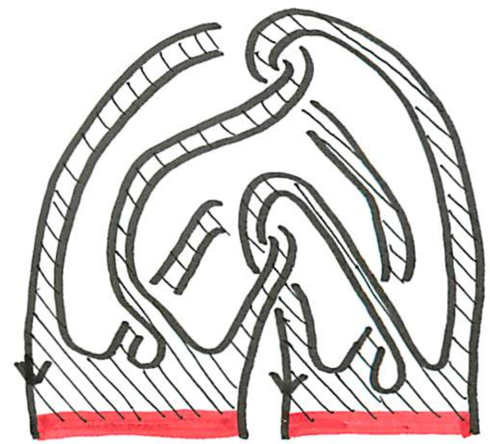
A boundary link

$\underset{\text{def}}{\Longleftrightarrow}$ A link whose components bound mutually disjoint Seifert surfaces in S^3



A boundary bottom tangle T

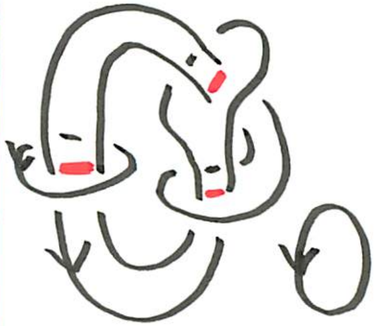
$\underset{\text{def}}{\Longleftrightarrow}$ A bottom tangle s.t. the components of $\overline{T}$ bound mutually disjoint Seifert surfaces in $[0,1]^3$



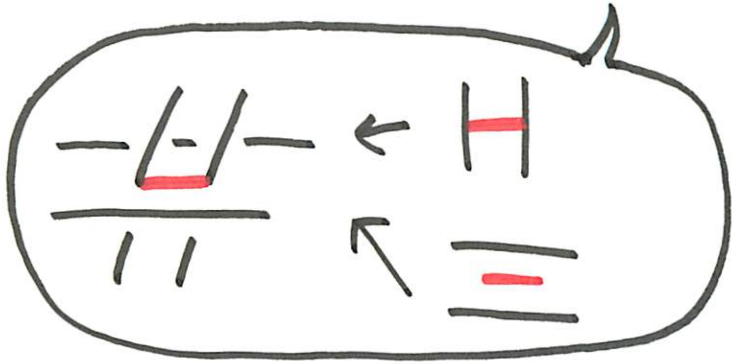
※ $\overline{T}$: the link obtained from T by closing it with the line segments in the bottom.

A ribbon link

$\underset{\text{def}}{\Longleftrightarrow}$ A link which bounds the image of an immersion with only ribbon singularities

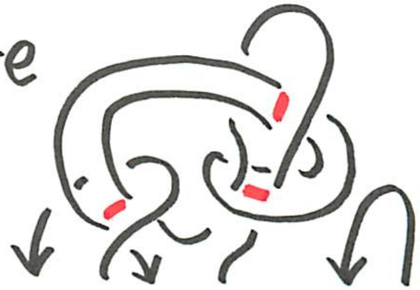


$D \sqcup \cdots \sqcup D \looparrowright S^3$



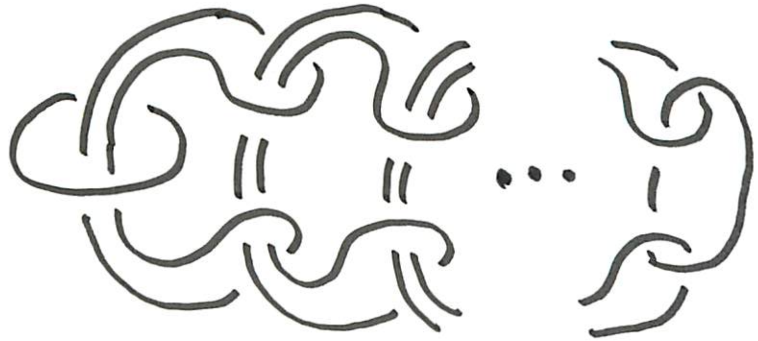
A ribbon bottom tangle

$\underset{\text{def}}{\Longleftrightarrow}$ A bottom tangle whose closure is a ribbon link



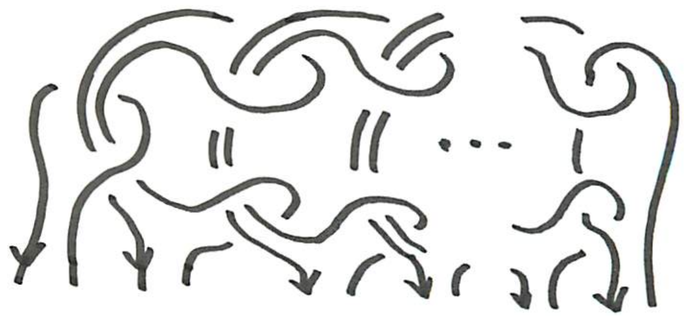
A brunnian link

$\underset{\text{def}}{\Leftrightarrow}$ A link whose proper sublinks are trivial.



A brunnian bottom tangle

$\underset{\text{def}}{\Leftrightarrow}$ A bottom tangle whose proper subtangles are trivial.



※ a trivial bottom tangle $\Leftrightarrow$ 

Motivation: "Good" understanding of the univ. sl_2 inv. of bottom tangles

Univ. sl_2 inv. (Laurence, Ohtsuki)

bottom tangle T — Colored Jones poly.

$$J_T \in U_h(sl_2)^{\hat{\otimes} n} \xrightarrow[\mathrm{tr}_q^{V_1} \otimes \cdots \otimes \mathrm{tr}_q^{V_n}]{\text{Quantum traces}} J(cl(T); V_1, \cdots, V_n)$$

※ $V_1, \cdots, V_n$: finite dimensional representations of $U_h(sl_2)$

- Topological information (boundary, ribbon, brunnian)
- Relationship to the other link inv. (Milnor invariants)

The quantum group $U_h = U_h(sl_2)$ / $\mathbb{Q}[[h]]$

generators : H, E, F,

relations : $HE - EH = 2E$, $HF - FH = -2F$,

$$EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}$$

where $q = \exp h$, $K = \exp \frac{hH}{2}$

※ We can equip U_h with a complete ribbon Hopf algebra structure.

Ribbon Hopf algebra $/\Bbbk$ $U=(U,\mu,\eta,\Delta,\varepsilon,S,R,\theta)$

• Hopf algebra

$\mu : U\otimes U \to U$

$\eta : \Bbbk \to U$

$\Delta : U \to U\otimes U$

$\varepsilon : U \to \Bbbk$

$S : U \to U$

with

• $R \in U\otimes U$: invertible

$R\Delta(x)R^{-1} = \Delta^{op}(x) \quad {}^\forall x \in U$

$(1\otimes\Delta)R = R_{13}R_{12}$

$(\Delta\otimes 1)R = R_{13}R_{23}$

• $\theta \in U$: central, invertible

$\Delta(\theta) = (\theta\otimes\theta)(R_{21}R)^{-1}$

$S(\theta)=\theta, \ \varepsilon(\theta)=1$

Universal sl_2 invariant J_T of a bottom tangle $T = T_1 \cup \cdots \cup T_n$

Step 1. Choose a diagram with 

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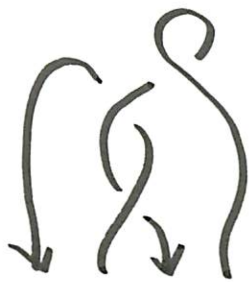
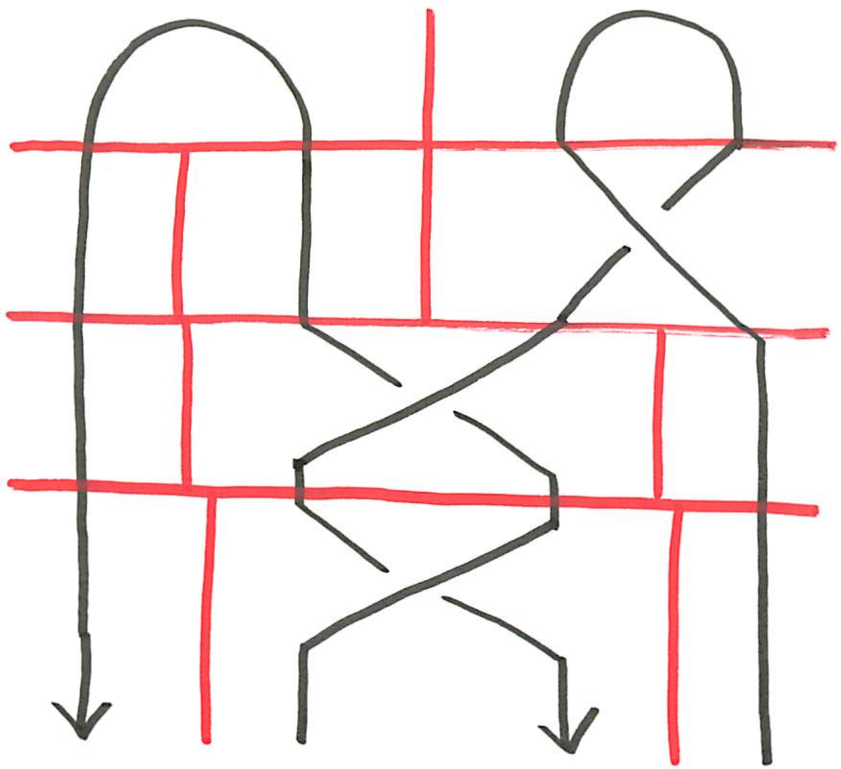
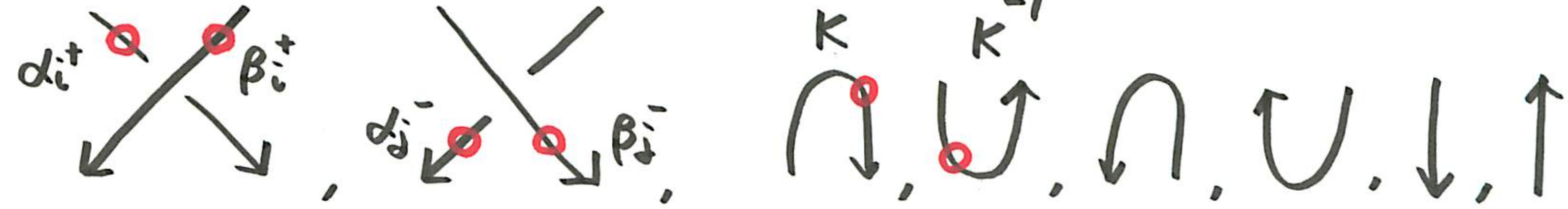
$T =$  $\rightsquigarrow$ 

Step2. Put labels. ($R^{\pm 1} = \sum_i \alpha_i^{\pm} \otimes \beta_i^{\pm}$)



(Apply "S" if a strand is oriented upward)

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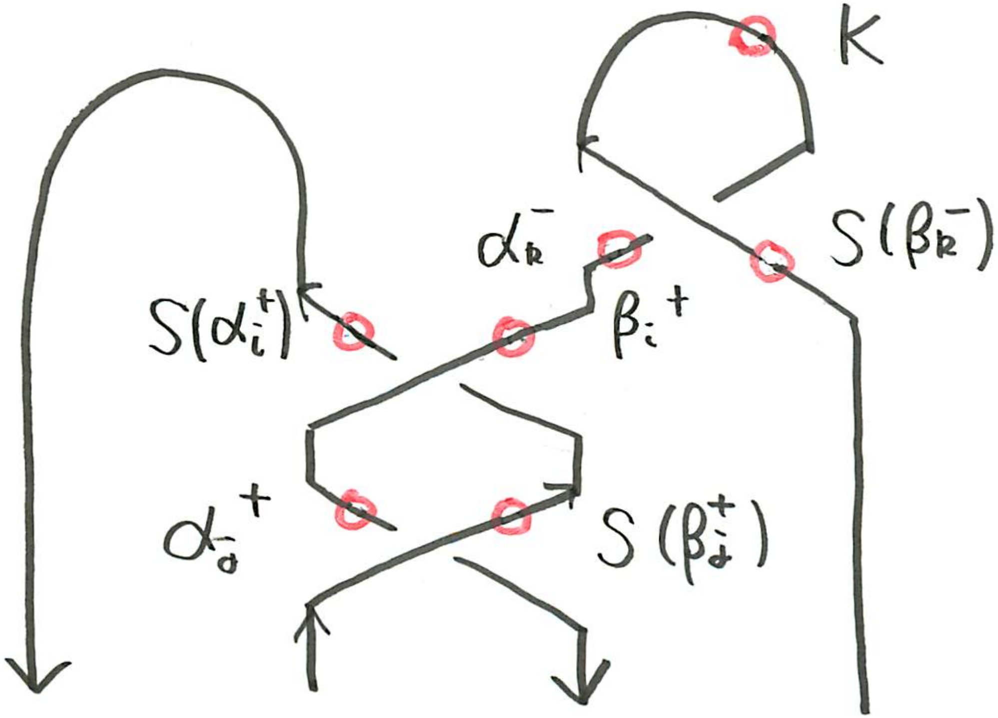

<u>Step3</u> Read the labels, take the tensors, and take the sums over the indices.

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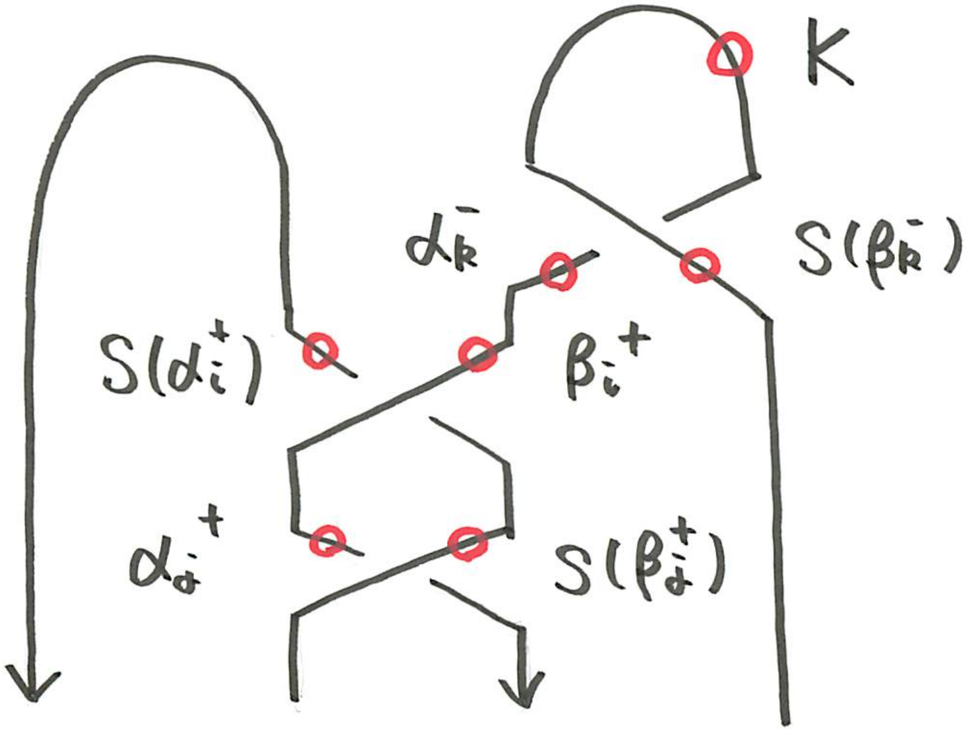

$$J_T = \sum_{i,j,k} S(\alpha_i^+) S(\beta_j^+) \otimes \alpha_j^+ \beta_i^+ \alpha_k^- K\, S(\beta_k^-) \in U_h^{\hat{\otimes} 2}$$

Notations

- $[i]_q = \dfrac{q^i - 1}{q - 1}$, $[i]_q! = [i]_q [i-1]_q \cdots [1]_q$,
- $e = (q^{1/2} - q^{-1/2})E$, $\tilde{F}^{(i)} = \dfrac{F^i K^i}{[i]_q!}$, $(i \geq 0)$
- $D = q^{\frac{1}{4} H \otimes H} = \exp\left(\dfrac{h}{4} H \otimes H\right)$

Universal R-matrix

$$R = D\left(\sum_{i \geq 0} q^{\frac{1}{2} i(i-1)} \tilde{F}^{(i)} K^{-i} \otimes e^i\right)$$

$$R^{-1} = D^{-1}\left(\sum_{i \geq 0} (-1)^i \tilde{F}^{(i)} \otimes K^{-i} e^i\right)$$

Step3 Read the labels, take the tensors, and take the sums over the indices.

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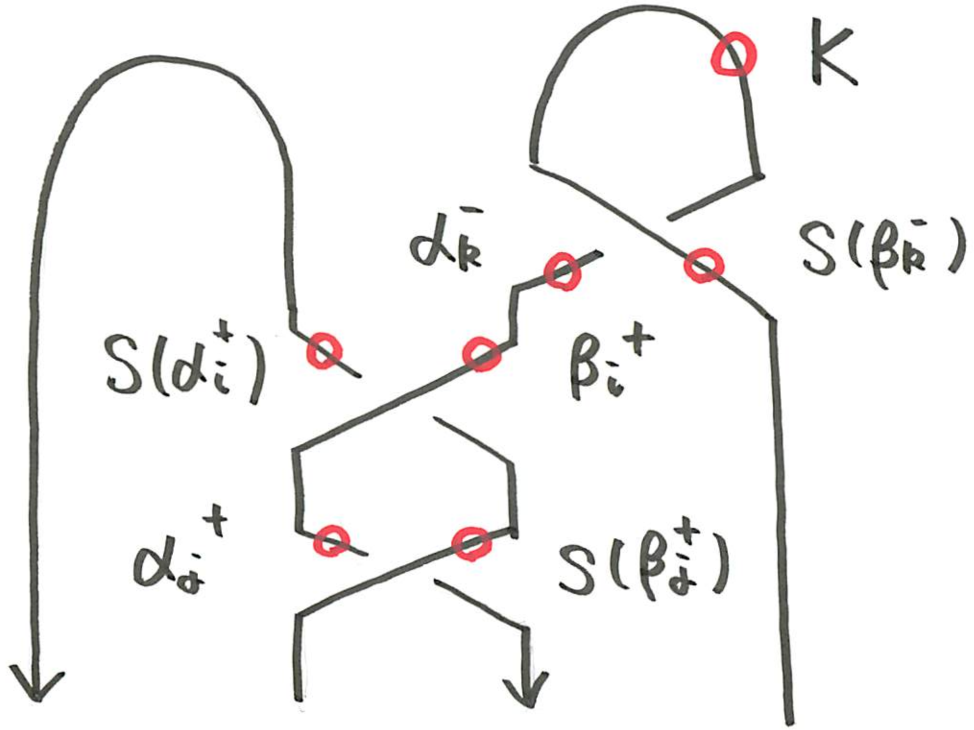
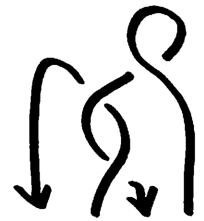


$$J_T = \sum_{i,j,k} S(\alpha_i^+)S(\beta_j^+) \otimes \alpha_j^+ \beta_i^+ \alpha_k^- K\, S(\beta_k^-) \in U_h^{\hat{\otimes} 2}$$

$J($  $)$

$$= \sum_{i,j,k \geq 0} (-1)^{i+j} q^{-\frac{1}{2}k(k-1) - j^2 + 2ij - 3jk - 2ik} \begin{bmatrix} j+k \\ j \end{bmatrix}_q$$

$$D^{-2}\left(1 \otimes q^{\frac{1}{4}H(H+2)}\right)\left(\tilde{F}^{(i)} K^{-2j} e^{j} \otimes \tilde{F}^{(j+k)} K^{2(j-i)} e^{i+k}\right)$$

$$\left(\text{where } \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q [m-1]_q \cdots [m-n+1]_q}{[n]_q!} \in \mathbb{Z}[q, q^{-1}] \quad \text{for } m \in \mathbb{Z},\ n \geq 0 \right)$$

- $f = (q-1)FK \quad \left(= (q-1)\tilde{F}^{(1)} \right)$
- $\bar{U}_q^{ev}$: $\mathbb{Z}[q, q^{-1}]$-subalgebra of U_h generated by $e, f, K^{\pm 2}$.

Thm[S] T : n-component boundary | ribbon bottom tangle with 0-framing

$$\Rightarrow J_T \in \left\{ (\bar{U}_q^{ev})^{\otimes n} \right\}^{\wedge}$$

※) $f^i = q^{-\frac{1}{2}i(i-1)} (q^i - 1)(q^{i-1} - 1) \cdots (q-1) \tilde{F}^{(i)}$

$$
J_T = \sum_{a,M,N,n,k,p_1,p_2,s_1,s_2,z,x,S,\tilde{S}} q^{\left\{\begin{array}{l} p_1^2+p_1(x-k-s_1)+p_2^2+p_2(z-n-s_2)+s_1^2+s_1(N-S-n-x)+s_2^2+s_2(M-\tilde{S}-k-z)+\frac{1}{2}S(S+1) \\ +S(k+x-N-2a)+\frac{1}{2}\tilde{S}(\tilde{S}+1)+\tilde{S}(n+z+M-2N-2a)-k^2+k(6n+2a+M-4N)-n^2+n(M-3N-2a) \\ +\frac{1}{2}N(N+1)+2N^2+N(2a-2z-1)+\frac{1}{2}M(M+1)+M(2a-2x)+x^2+z^2-(x+z)a+\frac{1}{2}a(a-1)\end{array}\right\}}
$$

$$
\begin{bmatrix} k \\ p_1 \end{bmatrix}_q \begin{bmatrix} p_1 \\ s_1 \end{bmatrix}_q \begin{bmatrix} N+2n-2M-p_1+s_1-1 \\ n-p_1 \end{bmatrix}_q \begin{bmatrix} N-p_1 \\ M-x-p_1 \end{bmatrix}_q \begin{bmatrix} M-x-p_1 \\ S-s_1 \end{bmatrix}_q \begin{bmatrix} 2M-x+2n+S+s_1-1 \\ x-k+p_1 \end{bmatrix}_q
$$

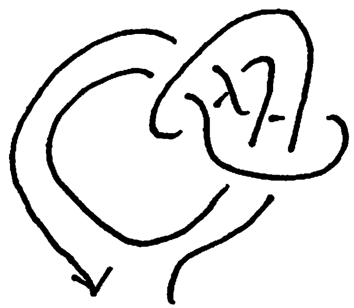
$$
\begin{bmatrix} n \\ p_2 \end{bmatrix}_q \begin{bmatrix} p_2 \\ s_2 \end{bmatrix}_q \begin{bmatrix} M+2k-2N-p_2+s_2-1 \\ k-p_2 \end{bmatrix}_q \begin{bmatrix} M-p_2 \\ N-z-p_2 \end{bmatrix}_q \begin{bmatrix} N-z-p_2 \\ \tilde{S}-s_2 \end{bmatrix}_q \begin{bmatrix} 2N-z+2k+\tilde{S}+s_2+1 \\ z-n+p_2 \end{bmatrix}_q
$$

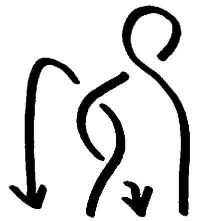
$$
\begin{bmatrix} N-M+x \\ x-z-a \end{bmatrix}_q \begin{bmatrix} z \\ x-z-a \end{bmatrix}_q \{x+z-a\}_q! \quad f^a K^{-2(M+N-n-k-S-\tilde{S})}\{H+z-N+M+3x-2a\}_{q,a-z-x}e^a.
$$

ただし,

$$
\begin{aligned}
&\{i\}_q = q^i - 1, \quad \{m\}_q! = \{m\}_q\{m-1\}_q \cdots \{1\}_q, \\
&\{H+i\}_{q,m} = \{H+i\}_q\{H+i-1\}_q \cdots \{H+i-m+1\}_q, \\
&\{H+j\}_q = q^{H+j} - 1 = q^j K^2 - 1,
\end{aligned}
$$

$i, j \in \mathbb{Z}, m \geq 0.$

T = 

$J\Big($  $\Big)$

$$= \sum_{i,j,k \geq 0} (-1)^{i+j} q^{-\frac{1}{2}k(k-1) - j^2 + 2ij - 3jk - 2ik} \begin{bmatrix} j+k \\ j \end{bmatrix}_q$$

$$D^{-2}\left(1 \otimes q^{\frac{1}{4}H(H+2)}\right)\left(\tilde{F}^{(i)} K^{-2j} e^{j} \otimes \tilde{F}^{(j+k)} K^{2(j-i)} e^{i+k}\right)$$

$$\left(\text{where } \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q [m-1]_q \cdots [m-n+1]_q}{[n]_q!} \in \mathbb{Z}[q, q^{-1}] \quad \text{for } m \in \mathbb{Z},\ n \geq 0 \right)$$

- $f = (q-1)FK \quad \left(= (q-1)\tilde{F}^{(1)} \right)$
- $\bar{U}_q^{ev}$: $\mathbb{Z}[q, q^{-1}]$-subalgebra of U_h generated by $e, f, K^{\pm 2}$.

$$\boxed{\begin{array}{l} \underline{\text{Thm}}\,[\mathrm{S}]\quad T: n\text{-component boundary} \mid \text{ribbon} \\ \qquad\qquad \text{bottom tangle with } 0\text{-framing} \\ \qquad \Rightarrow J_T \in \{(\bar{U}_q^{ev})^{\otimes n}\}^{\wedge} \end{array}}$$

※) $f^i = q^{-\frac{1}{2}i(i-1)}(q^i-1)(q^{i-1}-1)\cdots(q-1)\tilde{F}^{(i)}$

Thm[5] L: n-component boundary | ribbon link with 0-framing

$$\Rightarrow J(L; P_{l_1}, \dots, P_{l_n}) \in \frac{\{2l_j+1\}_q!}{\{1\}_q} I_{l_1} \cdots \widehat{I_{l_j}} \cdots I_{l_n}$$

$P_l = \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}})$, V_2 : 2-dim irr rep.

$I_l = \langle \{k\}_q! \{l-k\}_q! \{l\}_q! \mid 0 \le k \le l \rangle_{\text{ideal in } \mathbb{Z}[q^{1/2}, q^{-1/2}]}$

(where
$\{i\}_q = q^i - 1$, $\{i\}_q! = \{i\}_q \{i-1\}_q \cdots \{1\}_q$ for $i \ge 0$)

⟨ex⟩ L : n-component boundary | ribbon link with 0-framing

$$J(L; P_1, \dots, P_1) \in (q-1)^{2n}(q+1)(q^2+q+1)\mathbb{Z}[q^{\frac12}, q^{-\frac12}]$$

$$J(L; P_2, \dots, P_2) \in (q-1)^{4n}(q+1)^{n+1}(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q+1)\mathbb{Z}[q^{\frac12}, q^{-\frac12}]$$

$$J(L; P_3, \dots, P_3) \in (q-1)^{6n}(q+1)^{2n+1}(q^2+q+1)^{n+1}(q^2+1)(q^4+q^3+q^2+q+1)(q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)\mathbb{Z}[q^{\frac12}, q^{-\frac12}]$$

$J($  $; P_1, P_1) = q^{\frac34}\left\{ q^{-\frac32}(q+1)(q^{\frac12}-1)^2(q+q^{\frac12}+1) \right\} \in \mathbb{Z}[q^{\frac14}, q^{-\frac14}]$

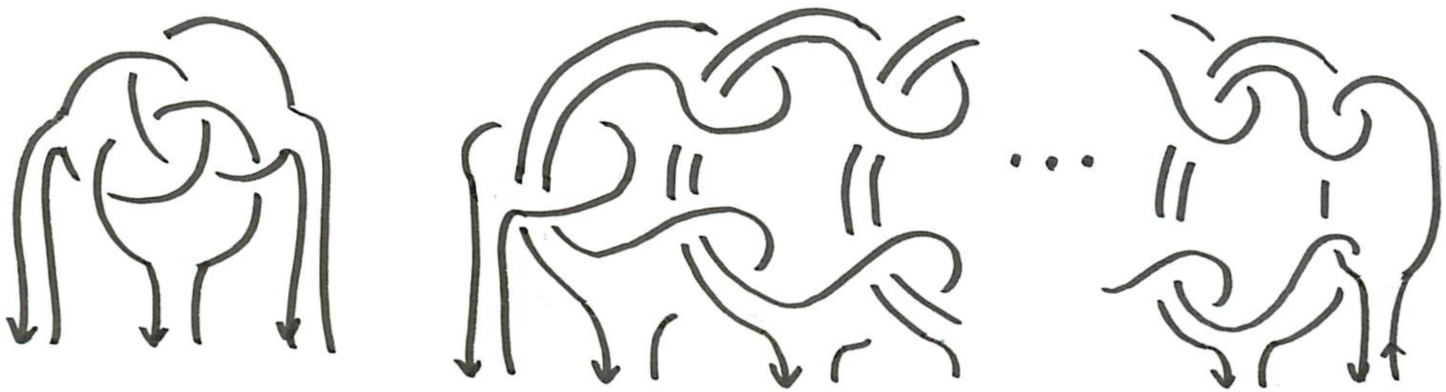
$J($  $; P_1, P_1, P_1) = -q^{-\frac72}(q-1)^4(q+1)(q^2+q+1)$

- $\tilde{E}^{(i)} = (q^{1/2} - q^{-1/2}) E^i / [i]_q!$

- $U^{ev}_{\mathbb{Z},q}$: $\mathbb{Z}[q, q^{-1}]$-subalgebra of U_h generated by $\tilde{E}^{(i)}, \tilde{F}^{(i)}, K^{\pm 2}$ $(i \geq 1)$.

Thm[S] T: n-component brunnian bottom tangle with 0-framing

$$\Rightarrow J_T \in \bigcap^{i} \left\{ (\bar{U}^{ev}_q)^{\otimes i-1} \otimes U^{ev}_{\mathbb{Z}q} \otimes (\bar{U}^{ev}_q)^{\otimes n-i} \right\}^{\wedge}$$



Milnor invariants of a bottom tangle $T = T_1 \cup \cdots \cup T_n$

: a sequence of invariants $\mu_{i_1,\cdots,i_p}(T) \in \mathbb{Z}$, $1 \le i_1,\cdots,i_p \le n$, $p \ge 2$

boundary ribbon	${}^\forall \mu_{i_1,\cdots,i_p}(T) = 0$ $(p \ge 2)$	$J_T \in \{ (\bar{U}_q^{ev})^{\otimes n} \}^\wedge$ $\Rightarrow$ conj I
brunnian	${}^\forall \mu_{i_1,\cdots,i_p}(T) = 0$ $(2 \le p \le n-1)$ ${}^\exists \mu_{i_1,\cdots,i_n}(T) \ne 0$	$J_T \in \bigcap_i \{ (\bar{U}_q^{ev})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{ev} \otimes (\bar{U}_q^{ev})^{\otimes n-i} \}^\wedge$ $\Rightarrow$ conj II

Conj Ⓘ

$T = T_1 \cup \cdots \cup T_n$: bottom tangle s.t.

$^\forall \mu_{i_1, \cdots, i_p}(T) = 0$, $^\forall p \geq 2$

$\Rightarrow J_T \in \{(\bar{U}_q^{ev})^{\otimes n}\}^\wedge$

Milnor invariants of a bottom tangle $T = T_1 \cup \cdots \cup T_n$

: a sequence of invariants $\mu_{i_1,\dots,i_p}(T) \in \mathbb{Z}$, $1 \le i_1,\dots,i_p \le n$, $p \ge 2$

boundary ribbon	$^{\forall}\mu_{i_1,\dots,i_p}(T) = 0$ $(p \ge 2)$	$J_T \in \{(\bar{U}_q^{ev})^{\otimes n}\}^{\wedge}$ $\Rightarrow$ conj (I)
brunnian	$^{\forall}\mu_{i_1,\dots,i_p}(T) = 0$ $(2 \le p \le n-1)$ $^{\exists}\mu_{i_1,\dots,i_n}(T) \ne 0$	$J_T \in \bigcap_i \{(\bar{U}_q^{ev})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{ev} \otimes (\bar{U}_q^{ev})^{\otimes n-i}\}^{\wedge}$ $\Rightarrow$ conj (II)

Thm [S] $T = T_1 \cup \cdots \cup T_n$: bottom tangle s.t.

$\exists j \geq 1$, $T_1 \cup \cdots \cup T_j$: trivial

$\Rightarrow J_T \in \{ (\bar{U}_q^{ev})^{\otimes j} \otimes (U_{\mathbb{Z}q}^{ev})^{\otimes n-j} \}^{\wedge}$

Conj II $T = T_1 \cup \cdots \cup T_n$: bottom tangle s.t.

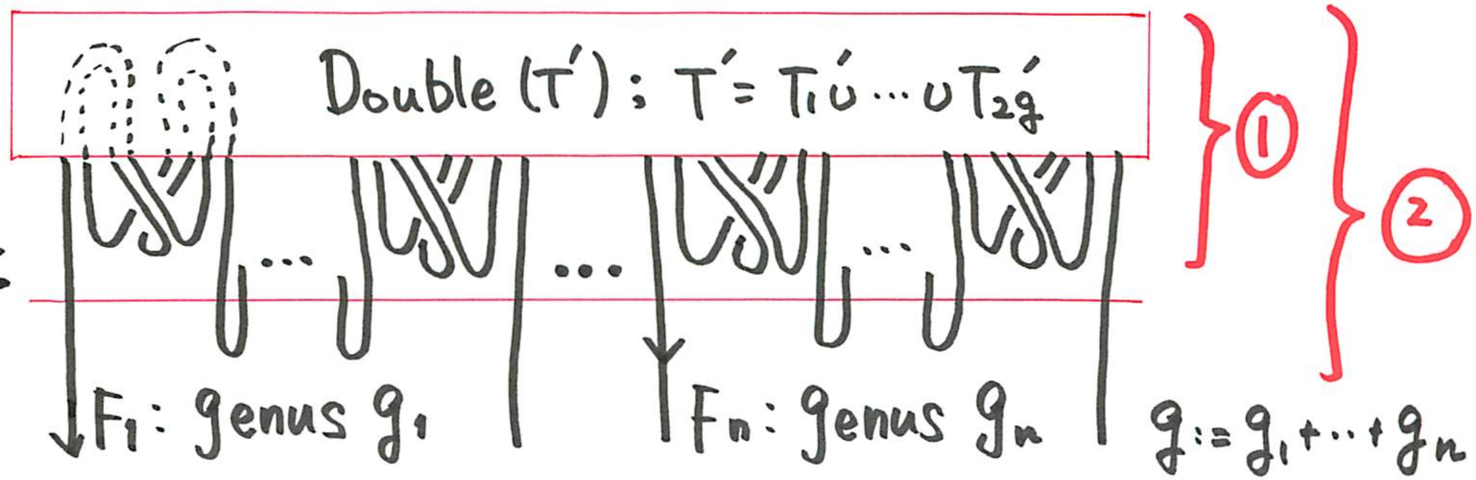
$\exists j \geq 1$, $\forall \mu_{i_1, \cdots, i_p} = 0$ for $1 \leq i_1, \cdots, i_p \leq j$

Milnor invs of $T_1 \cup \cdots \cup T_j$

$\Rightarrow J_T \in \{ (\bar{U}_q^{ev})^{\otimes j} \otimes (U_{\mathbb{Z}q}^{ev})^{\otimes n-j} \}^{\wedge}$

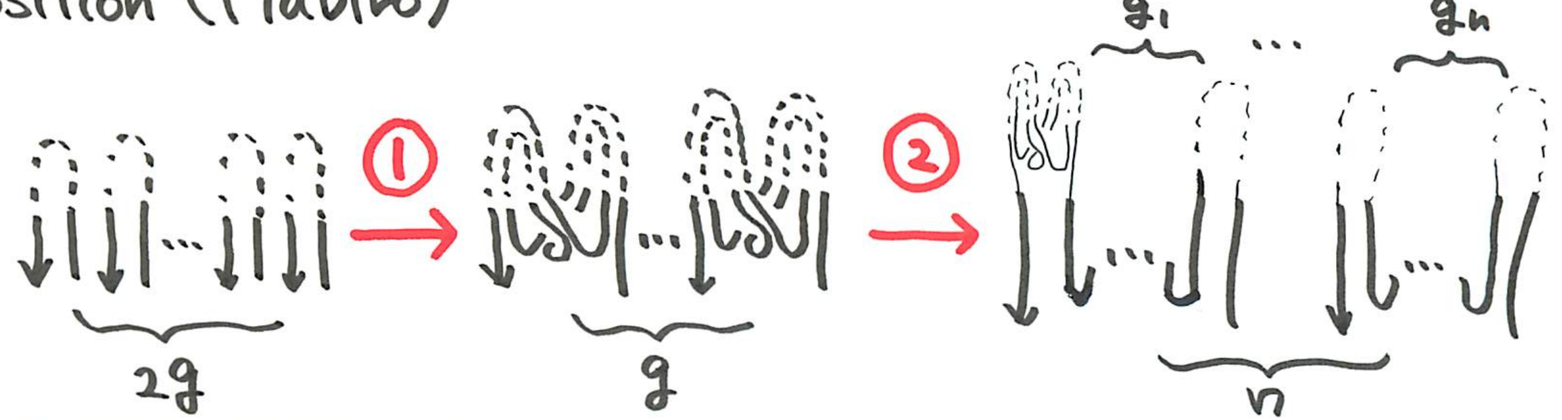
boundary bottom tangle

$T =$



$g := g_1 + \cdots + g_n$

Proposition (Habiro)



$\Downarrow$

$$J_{T'} \xrightarrow{①} Y^{\otimes g}(J_{T'}) \xrightarrow{②} \mu^{[g_1,\cdots,g_n]} \circ Y^{\otimes g}(J_{T'}) = J_T$$

$$\begin{array}{ccccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \hat{U}_h^{\hat{\otimes} 2g} & \xrightarrow[Y^{\otimes g}]{} & \hat{U}_h^{\hat{\otimes} g} & \xrightarrow[\mu^{[g_1,\cdots,g_n]}]{} & \hat{U}_h^{\hat{\otimes} n} \end{array}$$