The universal sl_2 invariant and Milnor invariants

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Introduction

- Quantum Topology
- Result

Quantum Topology

Jones polynomial (1984, Jones)

$\downarrow$ R matrix with respect to $(U_\hbar(\mathfrak{g}), V)$

Quantum $(\mathfrak{g}, V)$ invariant

$\downarrow$ omit V

Universal $\mathfrak{g}$ invariant (1990-, Lawrence, Ohtsuki)

$\downarrow$ KZ-eq. (Kohno, Drinfeld) omit $\mathfrak{g}$

Kontsevich integral (1993, Kontsevich)

Classical invariants and Quantum invariants

Classical invariants

Milnor invariants
Alexander polynomial, ...

- Equivalence Probrem
- Classification Problem
- Property of knots

Quantum invariants

Quantum ($\mathfrak{g}$, V) invariant
Universal $\mathfrak{g}$ invariant
Kontsevich integral

- Structure of <u>the set</u> of tangles
  - Algebraic structures
  - Filtrations
  - Classification by weaker equivalence relations

Classical invariants and Quantum invariants

Classical invariants

Milnor invariants
Alexander polynomial, ...

- Equivalence Probrem
- Classification Problem
- Property of knots

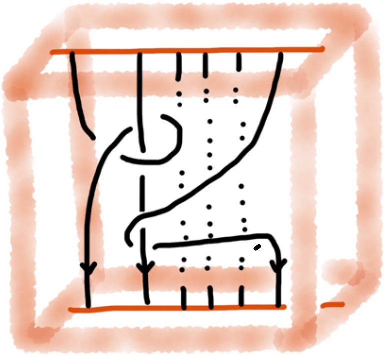
Quantum invariants

Quantum ($\mathfrak{g}$, V) invariant
Universal $\mathfrak{g}$ invariant ($\mathfrak{g} = sl_2$)
Kontsevich integral

- Structure of the set of tangles
  - Algebraic structures
  - Filtrations
  - Classification by weaker equivalence relations

String links

$\bigcup_{i=1}^{l} [0,1]_i \hookrightarrow$



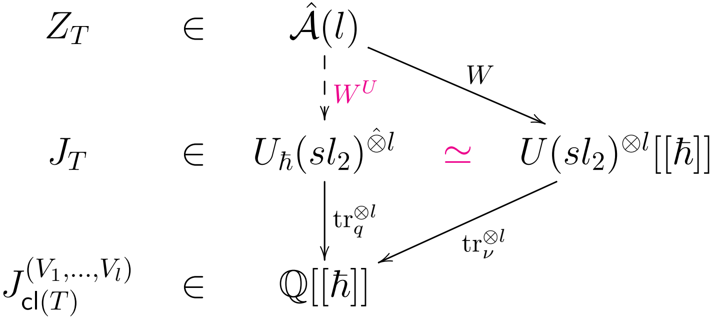
oriented, framed

$$SL(l) := \{l\text{-component string links }\}/\sim$$

Quantum invariants for $T \in SL(l)$

Kontsevich inv.	Z_T	$\in$	$\hat{\mathcal{A}}(l)$
Universal sl_2 inv.	J_T	$\in$	$U_\hbar(sl_2)^{\hat{\otimes} l}$
Colored Jones poly.	$J_{\mathrm{cl}(T)}^{(V_1,\ldots,V_l)}$	$\in$	$\mathbb{Q}[[\hbar]]$

Quantum invariants for $T \in SL(l)$



Quantum invariants and Milnor invariants

[Habegger-Masbaum, 2000]

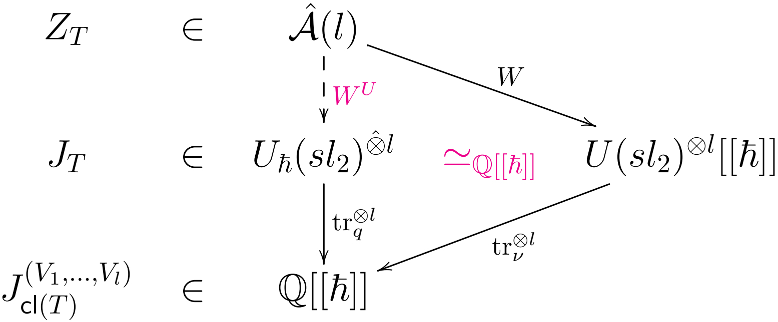
$$Z_T^t = 1 + \mu_m(T) + (\textit{higher})$$

Theorem (Meilhan-S, 2014)

$$J_T^t = W(\mu_m(T)) + (\textit{higher})$$

Note: (i) These results are essentially independent.
(ii) W is not injective for $m \geq 6$.

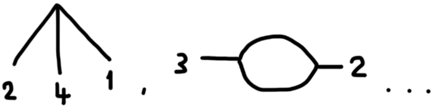
Quantum invariants for $T \in SL(l)$



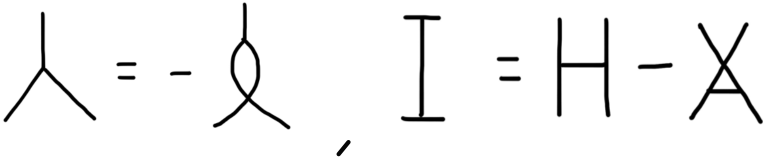
Jacobi diagrams

- The space $\hat{\mathcal{B}}(l)$ of labeled Jacobi diagrams
- The subspace $\mathcal{C}^t(l)$ of tree Jacobi diagrams

The space $\hat{\mathcal{B}}(l)$ of labeled Jacobi diagrams

$\mathcal{B}(l) = \langle$ 

$\rangle_{\mathbb{Q}}/$ AS, IHX

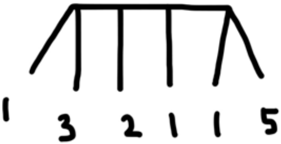


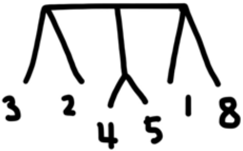
AS IHX

$\deg(D) = \frac{1}{2}\#\{\text{vertices in } D\}$

The subspace $\mathcal{C}^t(l)$ of tree Jacobi diagrams

$$\mathcal{C}^t(l) = \langle \text{ simply connected, connected diagrams } \rangle_{\mathbb{Q}}$$





Milnor invariants

- Artin representation
- Milnor numbers
- Milnor map

Artin representation

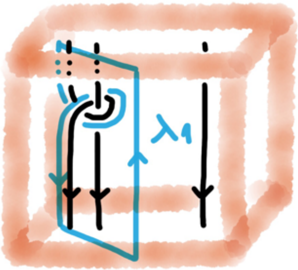
$$
\begin{array}{c}
F_l/(F_l)_{n+1} \\
\downarrow i_1^* \\
\pi_1([0,1]^3 \setminus T)/(\pi_1)_{n+1} \\
\uparrow i_0^* \\
F_l/(F_l)_{n+1}
\end{array}
$$

$$\Rightarrow \quad \mathcal{A}_n : \ SL(l) \to Aut\big(F_l/(F_l)_{n+1}\big)$$

$$x_i \mapsto \lambda_i x_i \lambda_i^{-1}$$

λ_i: the longitude of the ith component.

Milnor numbers



Magnus Expansion:

$$\mu\colon\ F_l \to \mathbb{Z}[[X_1, \ldots, X_l]]$$

$$x_i \mapsto 1 + X_i$$

$$\lambda_i \mapsto \sum \mu_{i_1,\ldots,i_p;i}(T) X_{i_1} \cdots X_{i_p}$$

$$T \in SL_m(l) \overset{\text{def}}{\Leftrightarrow} \forall \mu_{i_1,\ldots,i_p;i}(T) = 0,\ \forall p < m$$

$$\Leftrightarrow \forall \lambda_i \in (F_l)_m$$

Milnor map for $T \in SL_m(l)$

$$\mu_m(T) = \sum_{i=1}^{l} X_i \otimes \bar{\lambda}_i$$

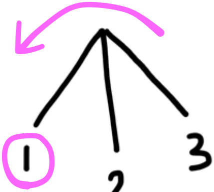
$$\in \mathrm{Ker}\{[-,-]\colon\ \mathrm{Lie}_1(l) \otimes \mathrm{Lie}_m(l) \to \mathrm{Lie}_{m+1}(l)\}$$

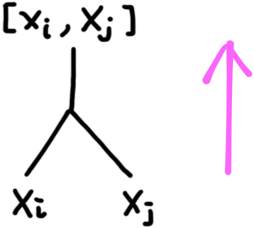
$$\mu_2(T) = X_1 \otimes [X_2, X_3] + X_2 \otimes [X_3, X_1] + X_3 \otimes [X_1, X_2]$$

Milnor map and Jacobi diagrams

$$\mathrm{Ker}\{[-,-]\colon \mathrm{Lie}_1(l)\otimes \mathrm{Lie}_m(l)\to \mathrm{Lie}_{m+1}(l)\}\otimes \mathbb{Q}$$

$$\sim \mathcal{C}^t(l)$$





$$X_1\otimes[X_2,X_3]+X_2\otimes[X_3,X_1]+X_3\otimes[X_1,X_2]$$

Universal sl_2 invariant

- The quantized enveloping algebra $U_\hbar(sl_2)$
- Universal sl_2 invariant

The quantized enveloping algebra $U_\hbar(sl_2)$

: the $\hbar$-adically complete $\mathbb{Q}[[\hbar]]$-algebra

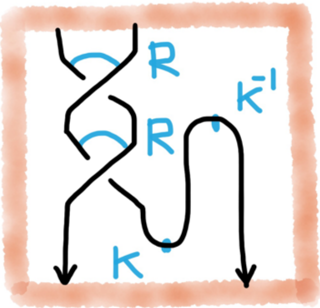
- generaters: $H, E,\ F$
- relations:

$$HE - EH = 2E, \quad HF - FH = -2F,$$
$$EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where $\quad q = \exp \hbar, \quad K = q^{H/2} = \exp \frac{\hbar H}{2}.$

Universal sl_2 invariant

$$R = q^{\frac{H \otimes H}{4}\hbar} \left(\sum_{n \geq 0} q^{\frac{1}{2}n(n-1)} \frac{(q-1)^n}{[n]_q!} F^n \otimes E^n \right)$$

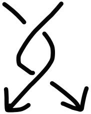


(1) Choose a nice diagram

(2) Put labels

(3) Read the labels

Universal sl_2 invariant

J ()

$$= q^{\frac{H \otimes H}{2} \hbar} \sum_{m,n \geq 0} q^{\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)+m^2} \frac{(q-1)^{m+n}}{[m]_q![n]_q!} F^m K^{-m} E^n \otimes E^m K^m F^n$$

$$= 1 + (\frac{1}{2} H \otimes H + F \otimes E + E \otimes F)\hbar + (\hbar^2)$$

Universal sl_2 weight system

- Universal sl_2 weight system
- Universal sl_2 weight system on $\mathcal{C}^t(l)$

Universal sl_2 weight system

$U = U(sl_2)$: The universal enveloping algebra of sl_2

$S = S(sl_2)$: The symmetric algebra of sl_2

$$\begin{array}{ccc} \hat{\mathcal{A}}(l) & \xrightarrow{W} & U^{\otimes l}[[\hbar]] \\ \Big\downarrow \text{formal PBW} & & \Big\downarrow \text{PBW} \\ \hat{\mathcal{B}}(l) & \xrightarrow{W} & S^{\otimes l}[[\hbar]] \sim_{\mathbb{Q}[[\hbar]]} U_\hbar(sl_2)^{\hat{\otimes} l} \end{array}$$

$$\sum f^i h^j e^k \otimes \cdots \quad \mapsto \quad \sum F^i H^j E^k \otimes \cdots$$

Universal sl_2 weight system

$U = U(sl_2)$: The universal enveloping algebra of sl_2

$S = S(sl_2)$: The symmetric algebra of sl_2

$$\begin{array}{ccc}
\hat{\mathcal{A}}(l) & \xrightarrow{W} & U^{\otimes l}[[\hbar]] \\
\Big\downarrow \text{formal PBW} & & \Big\downarrow \text{PBW} \\
\hat{\mathcal{B}}(l) & \xrightarrow{W} & S^{\otimes l}[[\hbar]] \sim_{\mathbb{Q}[[\hbar]]} U_\hbar(sl_2)^{\hat{\otimes} l}
\end{array}$$

$$\sum f^i h^j e^k \otimes \cdots \quad \mapsto \quad \sum F^i H^j E^k \otimes \cdots$$

Universal sl_2 weight system $W\colon\ \mathcal{B}(l) \to S^{\otimes l}[[\hbar]]$

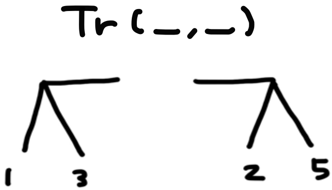
$$c = \tfrac{1}{2} H \otimes H + F \otimes E + E \otimes F \in sl_2^{\otimes 2}$$

$$b = \textstyle\sum_{\sigma \in \mathfrak{S}_3} (-1)^{|\sigma|} \sigma(H \otimes E \otimes F) \in sl_2^{\otimes 3}$$

$c_{i,j}$ $\qquad$ $b_{i,j,k} \quad \in S(sl_2)^{\otimes l}$

Universal sl_2 weight system $W\colon\ \mathcal{B}(l) \to S^{\otimes l}[[\hbar]]$

$w:$ 

$\mapsto$ 

$$= \sum \mathrm{Tr}(b_3 b_1') b_1 \otimes b_2' \otimes b_2 \otimes 1 \otimes b_3'$$

$$D \in \mathcal{B}_m(l) \quad \Rightarrow \quad W(D) = w(D)\hbar^m$$

Universal sl_2 weight system on $\mathcal{C}^t(l)$

Set

$$S_n = \mathrm{Span}_{\mathbb{Q}}\{a_1 a_2 \cdots a_n \mid a_i \in sl_2\} \subset S,$$

$$(S^{\otimes l})_n = \bigoplus_{n_1+\cdots+n_l=n} S_{n_1} \otimes \cdots \otimes S_{n_l}.$$

Proposition

We have $W(\mathcal{C}^t_m(l)) \subset (S^{\otimes l})_{m+1} \hbar^m$.

Results

- Reduction of the universal sl_2 invariant
- Results
- Study on the universal sl_2 weight system

Reduction of the universal sl_2 invariant

Set

$$J^t := p^t \circ J \colon\ SL(l) \to \prod_{m \geq 1} (S^{\otimes l})_{m+1} h^m,$$

where

$$p^t \colon\ U_h^{\hat{\otimes} l} \to \prod_{m \geq 1} (S^{\otimes l})_{m+1} h^m,$$

denotes the projection as $\mathbb{Q}$-modules.

Result

Theorem (Meilhan-S)

For $T \in SL_m(l)$, we have

$$J_T^t = W(\mu_m(T)) + (\textit{higher}).$$

Result

Corollary (Meilhan-S)

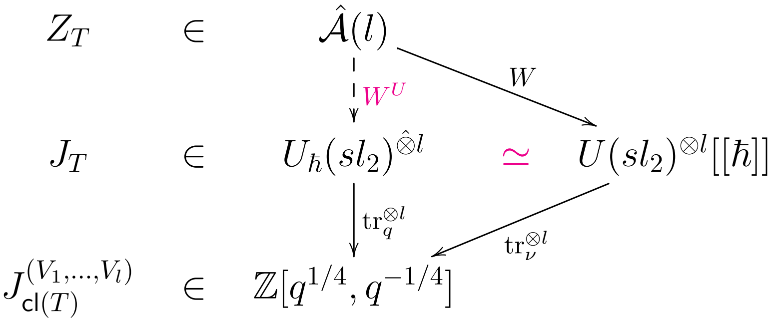
For $T \in SL(l)$ with vanishing all Milnor invariants, we have

$$J_T \in \prod_{0 \leq i \leq j} (S^{\otimes l})_i \hbar^j.$$

Future

- Find an integrality property for Milnor's invariant at the level of the universal sl_2 invariant.
- Find the weight system for the universal sl_2 invariant.
- Make an algebraic isomorphism between $U_\hbar(sl_2)$ and $U(sl_2)[[\hbar]]$ which is compatible with the weight systems.

Quantum invariants for $T \in SL(l)$



Study on the universal sl_2 weight system

Subspaces $\mathcal{C}^t(l), \mathcal{C}^h(l)$, and $\mathcal{C}_l$ of $\hat{\mathcal{B}}(l)$

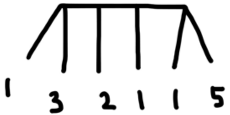
$$\mathcal{C}^t(l) = \langle \text{ simply connected, connected diagrams } \rangle_{\mathbb{Q}}$$

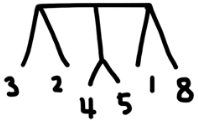
$$\cup$$

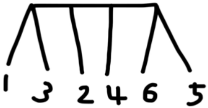
$$\mathcal{C}^h(l) = \langle \text{ non-repeated labeled diagrams } \rangle_{\mathbb{Q}}$$

$$\cup$$

$$\mathcal{C}_l = \langle \text{ each label appears exactly once } \rangle_{\mathbb{Q}}$$







Subspaces $\mathcal{C}^t(l), \mathcal{C}^h(l)$, and $\mathcal{C}_l$ of $\hat{\mathcal{B}}(l)$

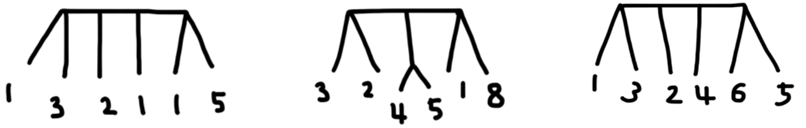
$\mathcal{C}^t(l) = \langle$ simply connected, connected diagrams $\rangle_{\mathbb{Q}}$

$\cup$

$\mathcal{C}^h(l) = \langle$ non-repeated labeled diagrams $\rangle_{\mathbb{Q}}$

$\cup$

$\mathcal{C}_l = \langle$ each label appears exactly once $\rangle_{\mathbb{Q}}$



$\mathcal{C}_l$ is a brick!

① We have

$$\mathcal{C}^h_m(l) = \bigoplus_{1\leq i_1<\cdots<i_{m+1}\leq l} \mathcal{C}^{(i_1,\ldots,i_{m+1})}_{m+1}$$

② The following diagram commutes;

$$\begin{array}{ccc} \mathcal{C}^t_m(l) & \xrightarrow{\quad w \quad} & (S^{\otimes l})_{m+1} \\ \bar{D}^{(p)} \downarrow & \circlearrowleft & \downarrow \bar{\Delta}^{(p)} \\ \mathcal{C}^h_m(pl) & \xrightarrow{w} & \bigoplus \iota^{i_1,\ldots,i_{m+1}}(sl_2^{\otimes m+1}) \subset S^{\otimes pl}_{m+1} \end{array}$$

Universal sl_2 weight system on $\mathcal{C}_l$

Set

$$w_{\mathcal{C}_l} = w|_{\mathcal{C}_l} \colon \mathcal{C}_l \to \mathrm{Inv}(sl_2^{\otimes l}).$$

l	2	3	4	5	6	7	8	9	...	n
$\dim \mathcal{C}_l$	1	1	2	6	24	120	720	5040	...	$(n-2)!$
$\dim \mathrm{Inv}(sl_2^{\otimes l})$	1	1	3	6	15	36	91	232	...	R_n

R_n: Riordan number

Result

Theorem (Meilhan-S, 2014)

(i) *For* $l = 2$ *or* $l \geq 3$ *odd,* $w_{\mathcal{C}_l}$ *is surjective.*

(ii) *For* $l \geq 4$ *even,* $\mathrm{coker}(w_{\mathcal{C}_l})$ *is spanned by* $\overline{c^{\otimes \frac{l}{2}}}$.

l	2	3	4	5	6	7	8	9
$\dim \mathcal{C}_l$	1	1	2	6	24	120	720	5040
$\dim \mathrm{Inv}(sl_2^{\otimes l})$	1	1	3	6	15	36	91	232
$\dim \mathrm{coker}(w_{\mathcal{C}_l})$	0	0	1	0	1	0	1	0
$\dim \ker(w_{\mathcal{C}_l})$	0	0	0	0	10	84	630	4808

$\mathfrak{S}_l$-module structure

Proposition (Kontsevich)

As a $\mathfrak{S}_l$-module, the character of C_l is

$$\chi(1^l) = (l-2)!, \quad \chi(1^1 a^b) = (b-1)! a^{b-1}\mu(a), \quad \chi(a^b) = -(b-1)! a^{b-1}\mu(a),$$

and $\chi_{\mathcal{C}_l}(*) = 0$ for other conjugacy classes.

Lemma

$$\mathrm{Inv}(sl_2^{\otimes l}) \simeq \bigoplus V_\lambda$$

with the summation over partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of l s.t. each λ_i is odd or each λ_i is even, and $n \leq 3$.

$\mathfrak{S}_l$-module structure

Corollary (Meilhan-S, 2014)

(i) For $l = 2$ or $l \geq 3$ odd, we have

$$\chi_{\ker(w_{\mathcal{C}_l})} = \chi_{\mathcal{C}_l} - \chi_{\mathrm{Inv}(sl_2^{\otimes l})},$$

$$\chi_{\mathrm{Im}(w_{\mathcal{C}_l})} = \chi_{\mathrm{Inv}(sl_2^{\otimes l})}.$$

(ii) For $l \geq 4$ even, we have

$$\chi_{\ker(w_{\mathcal{C}_l})} = \chi_{\mathcal{C}_l} - \chi_{\mathrm{Inv}(sl_2^{\otimes l})} + \chi_{(l)},$$

$$\chi_{\mathrm{Im}(w_{\mathcal{C}_l})} = \chi_{\mathrm{Inv}(sl_2^{\otimes l})} - \chi_{(l)}.$$

$\mathfrak{S}_l$-module type of C_l with $l \leq 8$

Red components are in kernel

$l = 2, 3 :$

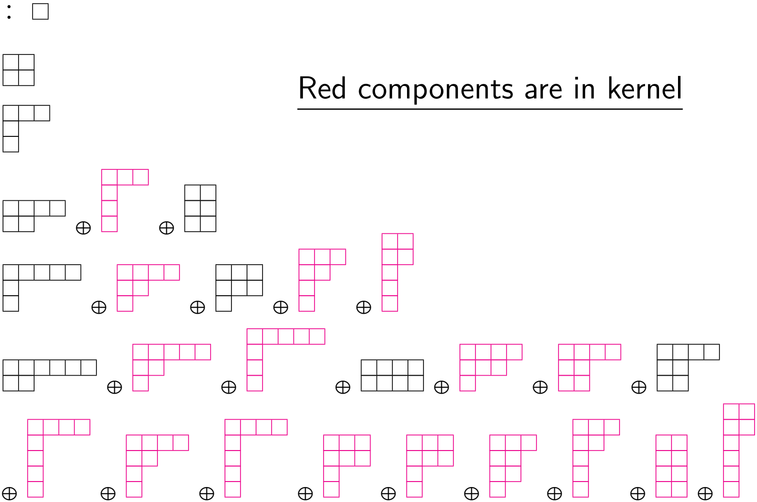
$l = 4 :$

$l = 5 :$

$l = 6 :$

$l = 7 :$

$l = 8 :$



Thank you !