Introduction to Garside Calculus

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WORKSHOP ON KNOT THEORY
Zürich, November 21, 2013

Literature

- Patrick DEHORNOY, François DIGNE, Eddy GODELLE, Daan KRAMMER, and Jean MICHEL, Foundations of Garside Theory, book manuscript (703 pages), 2013.
- P. Dehornoy and Volker Gebhardt, Algorithms for Garside calculus, J. Symbol. Comput., to appear (40 pages), 2013.

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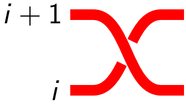

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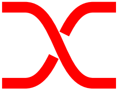

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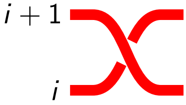

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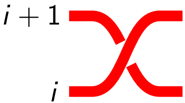

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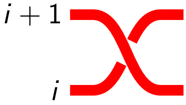

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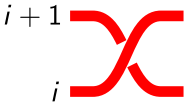

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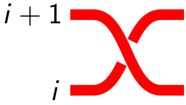

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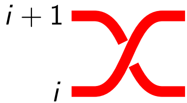

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$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \forall |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \forall i = 1, \ldots, n-2 \end{array} \right\rangle .$$

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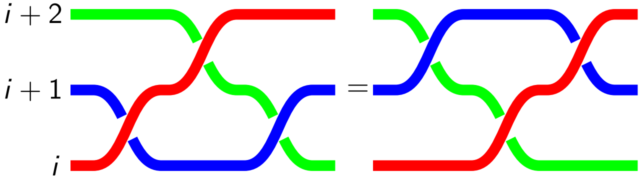

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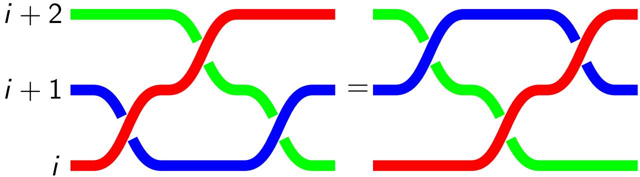

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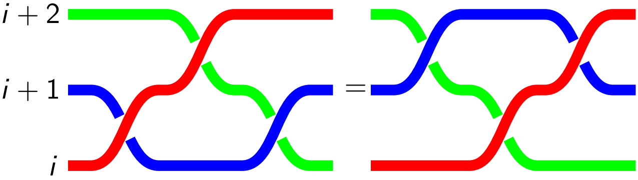

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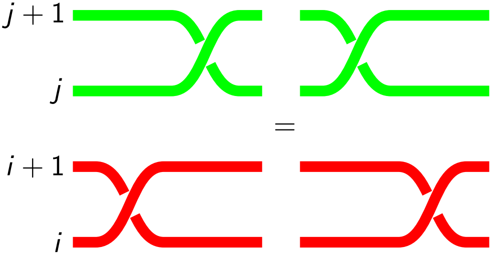
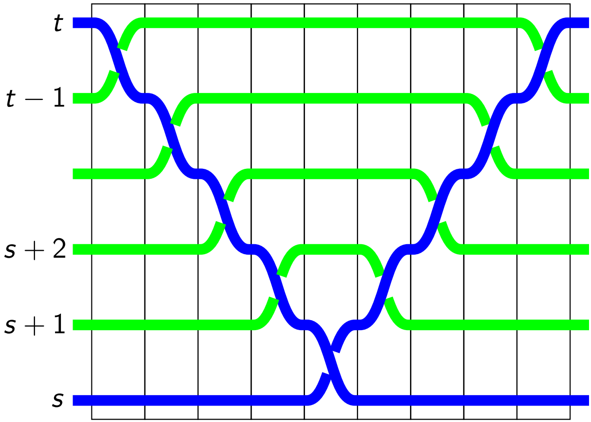


Birman-Ko-Lee generators [BKL98]: a_{ts} $(1 \leq s < t \leq n)$



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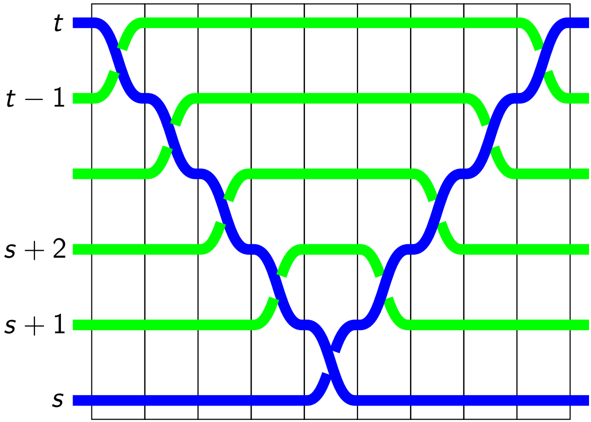

$$a_{ts} = \sigma_{t-1} \cdots \sigma_{s+1} \sigma_s \sigma_{s+1}^{-1} \cdots \sigma_{t-1}^{-1}$$

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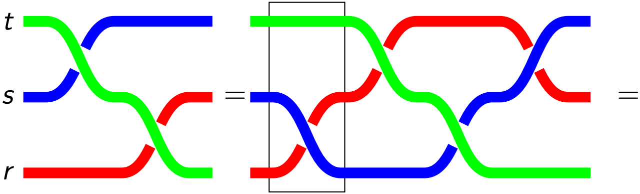

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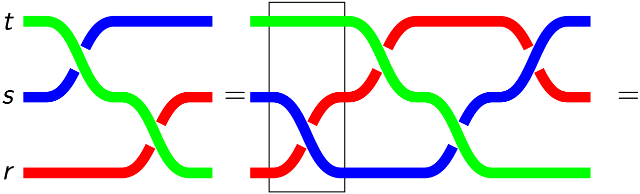

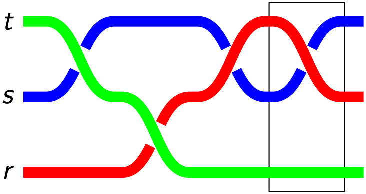

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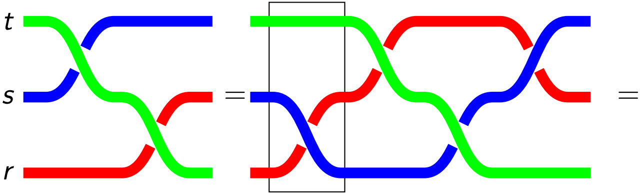
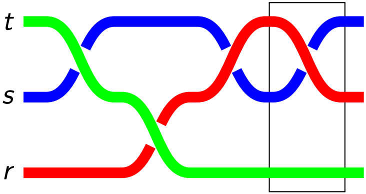
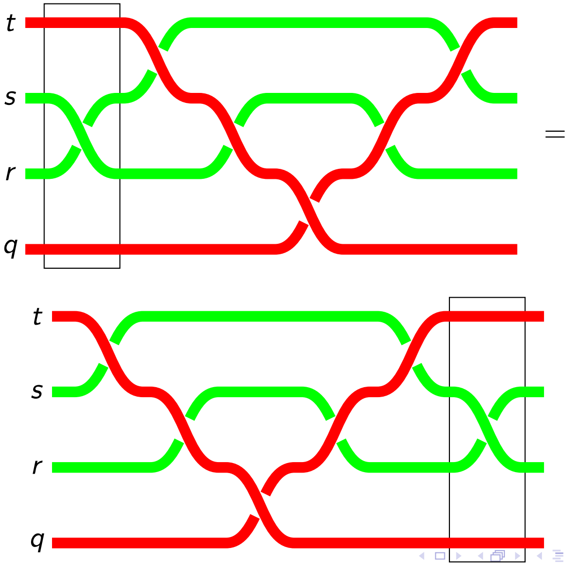




- Usual far commutativity: $a_{ts}a_{rq} = a_{rq}a_{ts}$ for $q < r < s < t$.

BKL Far Commutativity: $a_{sr}a_{tq} = a_{tq}a_{sr}$ $(q < r < s < t)$.



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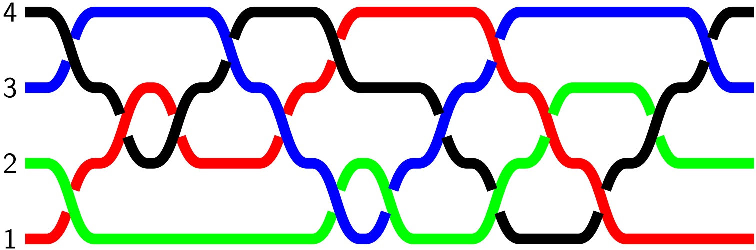
Example: pure braid (uncombed)



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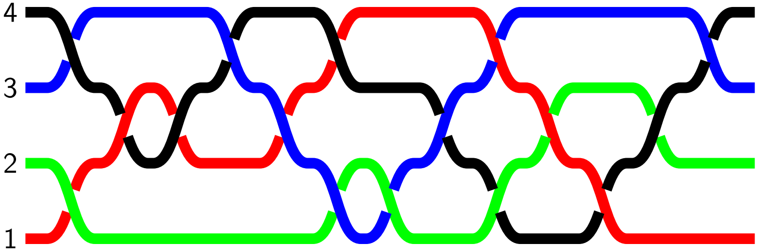

$$\sigma_1\sigma_3\sigma_2^{-2}\sigma_3\sigma_2\sigma_3\sigma_1^2\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_3$$

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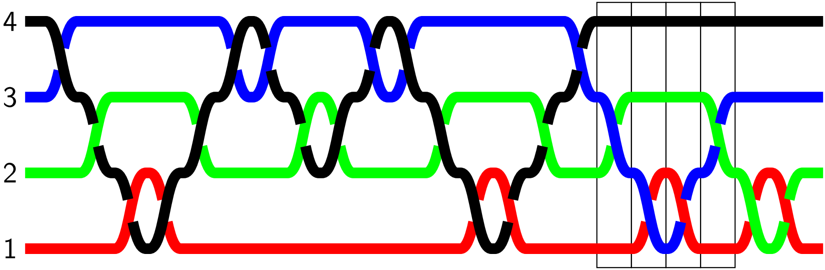

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- Coro [Malcev, Neumann]: $\mathbb{Z}B_n$ has no zero divisors , and $\mathbb{Z}P_n$ embeds in a division algebra.

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Old Definition: Garside monoids and groups

- Let $a, b \in M$ monoid. Denote $a \preceq b$, if $\exists c \in M$ such that $b = ac$.
- If M is a f.g. atomic monoid then $\preceq$ and $\succeq$ are partial orders.
- An element $\Delta \in M$ is balanced if the sets of left and right divisors coincide.

Definition

- A monoid M is an lcm monoid if it is Noetherian, cancellative, and $\forall a, b \in M$ there exist a right and a left lcm.
- Let G be a group and $S \subseteq G$ s.t. $G = \langle S \rangle$. (G, S) is called a Garside system if $G^+ = \langle S \rangle^+$ is an lcm monoid, G is its group of fractions, and $\exists$ a balanced element $\Delta \in G^+$ s.t. $S = \mathrm{Div}(\Delta)$. We call G a Garside group, G^+ a Garside monoid, and Δ a Garside element. The elements of S are called simple elements.

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- (B_n, BKL_n^+, δ_n) with $\delta_n = a_{n,n-1}a_{n-1,n-2}\dots a_{2,1}$,

$$\left\langle \{a_{ts}\}_{s<t} \;\middle|\; \begin{array}{c} a_{ts}a_{rq} = a_{rq}a_{ts}, \;\; (t-r)(t-q)(s-r)(s-q) < 0 \\ a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr}, \;\; t > s > r \end{array} \right\rangle$$

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- Pure braid group $P_3 = \langle a, b, c \mid abc = bca = cab \rangle$,
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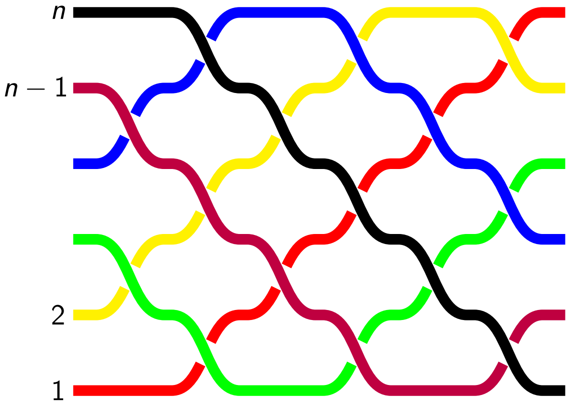
- $G = \langle a, b \mid ababa = b^2\rangle$ with $\Delta = (ab)^3 = (ba)^3 = b^3$ is Garside group with no weighted presentation.
- Many more: torus link groups, complex braid groups, structure

Artin Garside element Δ_n



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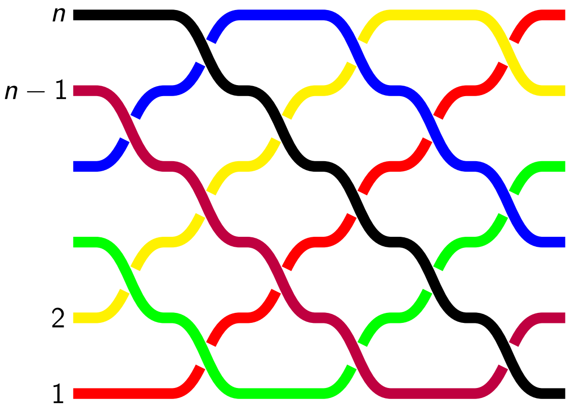
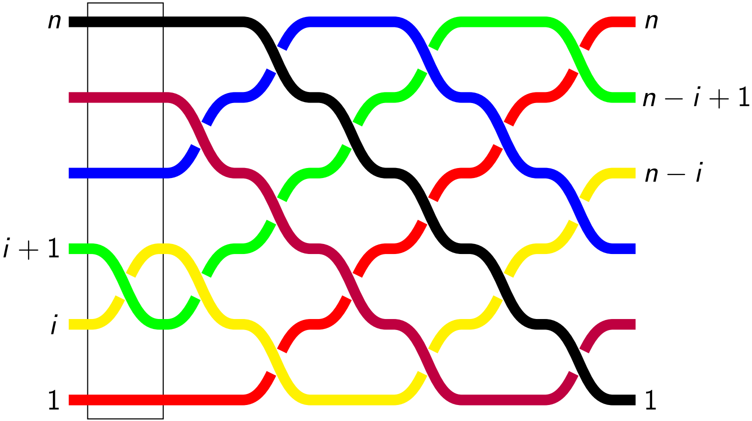


$$\Delta_n = \sigma_1(\sigma_2\sigma_1)\dots(\sigma_{n-1}\sigma_{n-2}\dots\sigma_1)$$

Artin Garside element: induced inner automorphism



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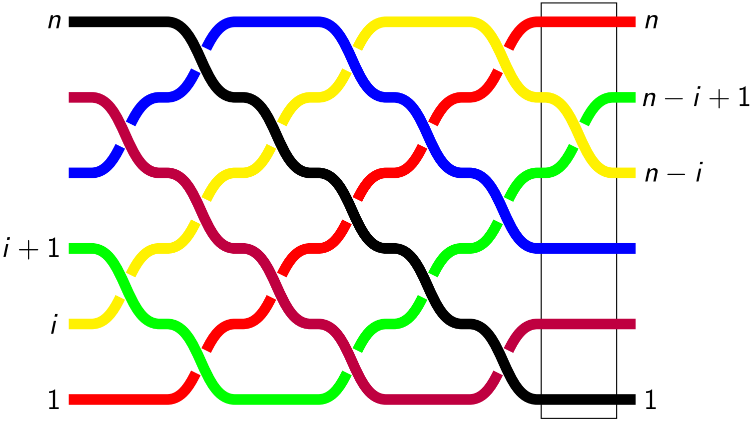

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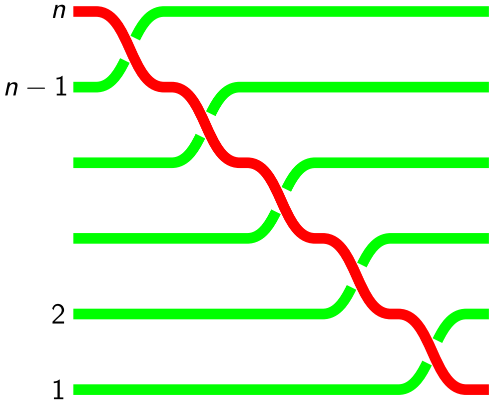
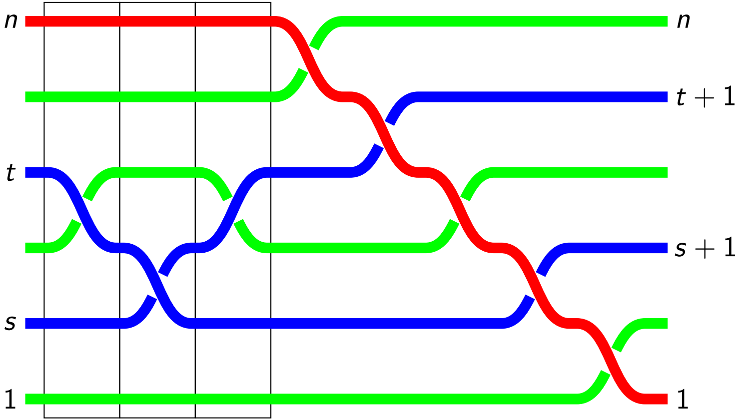


$$\delta_n = a_{n,n-1} a_{n-1,n-2} \cdots a_{21}$$

Dual Garside element δ_n: induced automorphism



$$a_{ts}\delta_n = \delta_n a_{t+1,s+1} \quad \Leftrightarrow \tau(a_{ts}) = a_{t+1,s+1}$$

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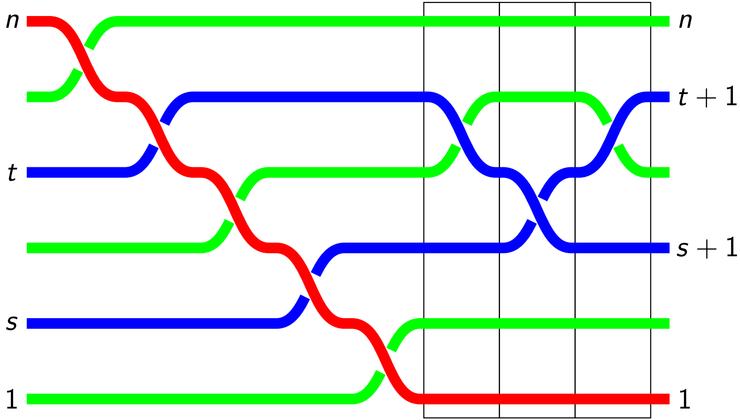

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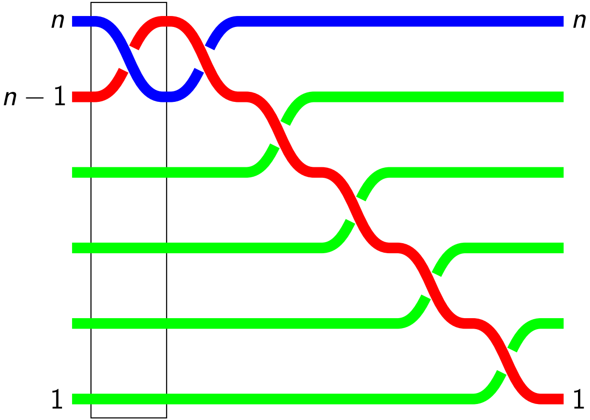

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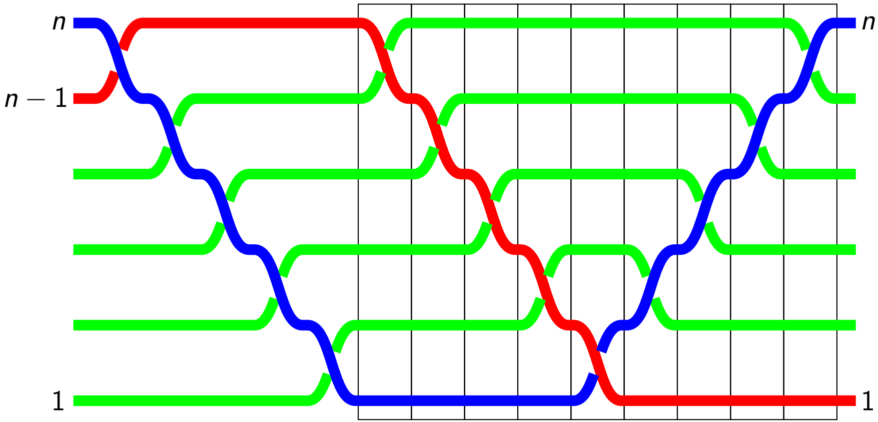

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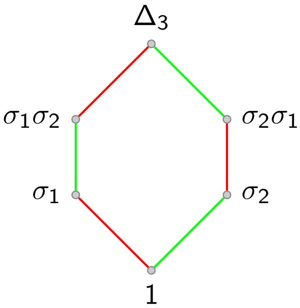
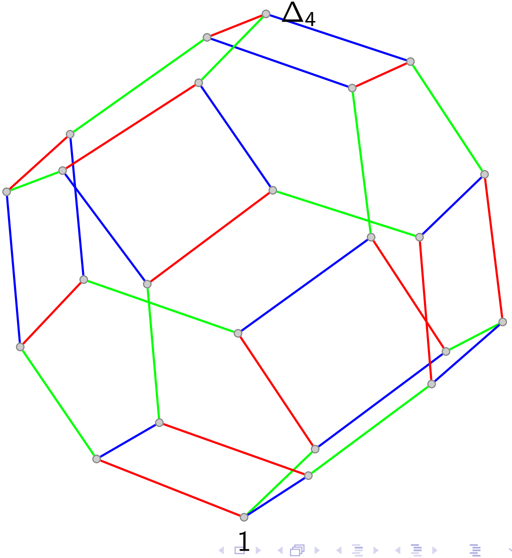
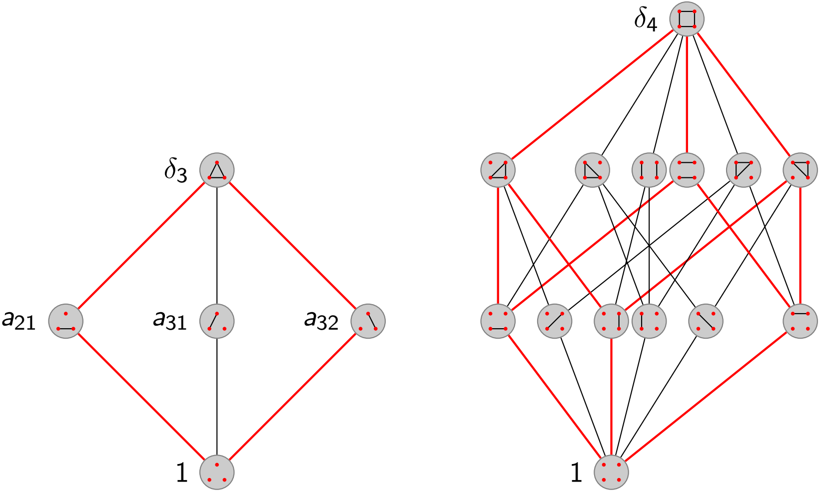
Lattice of simples for Artin-Garside structure ($n = 3, 4$)





Dual lattice of simple elements ($n = 3, 4$)



Artin groups of finite type

Cardinalities of sets of simple elements $S = \mathrm{Div}(\Delta)$.

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Infinite families:

Type	A_n	B_n	D_n	$I_2(m)$
classical	$(n+1)!$	$2^n n!$	$2^{n-1}n!$	$2m$
dual	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\binom{2n}{n} - \binom{2n-2}{n-2}$	$m+2$

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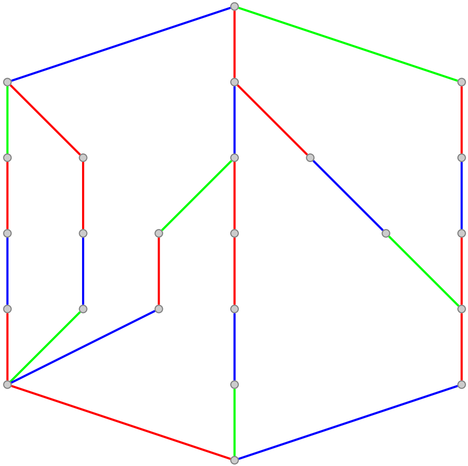
Exceptional cases:

Type	H_3	F_4	H_4	E_6	E_7	E_8
classical	120	1152	14400	51840	2903040	696729600
dual	32	105	280	833	4160	25080

Yet another Garside group [Pi01]



$G = \langle x, y, z \mid xzxy = yzx^2,\ yzx^2z = zxyzx \rangle.$

Properties of Garside groups

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Word problem in Garside groups

- Every $a \in G$ admits unique Δ-normal form $\Delta^p s_1 \cdots s_l$ with *Infimum* $p = \inf(a) = \max\{r \in \mathbb{Z} \mid \Delta^r \preceq a\}$, $s_i \in S \setminus \{1, \Delta\}$ s.t. $s_i = (s_i \cdots s_l) \wedge \Delta$. *Supremum* $\sup(a) = p + l$.
- Every $a \in G$ admits unique fractional normal form $a = b^{-1}c = t_{-p}^{-1} \cdots t_1^{-1} s_1 \cdots s_{l+p}$ with $s_i, t_i \in S \setminus \{1, \Delta\}$ s.t. $s_1 \wedge t_1 = 1$.
- The left-greedy condition $(s_i s_{i+1}) \wedge \Delta = s_i$ is equivalent to:

$$\forall 1 \neq t \preceq s_{i+1} : \quad s_i t \notin S = \mathrm{Div}(\Delta).$$

- The word problem in Garside groups can be solved in $O(l^2)$.

Example

We compute the Δ-LNF and the fractional LNF of the 4-strand braid $b = \sigma_2\sigma_3\sigma_2^{-1}\sigma_1$ for the Garside systems $(B_4, \mathrm{Div}(\Delta_4))$ and $(B_4, \mathrm{Div}(\delta_4))$ are

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&\overset{fLNF}{=} \sigma_3^{-1} \mid\mid \sigma_2\sigma_3\sigma_1, \\
&\overset{LNF*}{=} \delta_4^{-1}\delta_{(421)} \mid \delta_{321} \mid a_{43}, \\
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Computation: see blackboard.

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Let $<$ be left-inv. ordering of B_n s.t. $1 < \Delta_n$. If $b \sim b'$, then $\Delta_n^{2p} \leq b < \Delta_n^{2p+2}$ implies $\Delta_n^{2p-2} \leq b' < \Delta_n^{2p+4}$.

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Conjugacy in Garside groups

Invariant subsets with "convexity property".

$(G, \mathrm{Div}(\Delta))$ Garside system. For any $a \in G$ there exists a finite subset $I(a)$ s.t.
(a) $I(a) = I(b) \Leftrightarrow a \sim b$, and (b) the following "convexity property" holds:
Let and $a, b \in I(a)$ be conjugate. Wlog $b = x^{-1}ax = \tilde{x}\, a\, \tilde{x}^{-1}$ for some $x, \tilde{x} \in G^{+}$. Let $s_1 = x \wedge \Delta$ and $\tilde{s}_1 = x \tilde{\wedge} \Delta$. Then $s_1^{-1} a s_1, \tilde{s}_1 a \tilde{s}_1^{-1} \in I$.

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Corollary.

$(G, \mathrm{Div}(\Delta))$ Garside system. $a, b \in G$ are conjugate iff there exist $l \in \mathbb{N}$, $\tilde{a} = v_0, v_1, \ldots, v_l = \tilde{b} \in I(a)$, and $s_1, \ldots, s_l \in S$ such that

$$\tilde{a} = v_0 \xrightarrow{s_1} v_1 \xrightarrow{s_2} v_2 \xrightarrow{s_3} \ldots \xrightarrow{s_{l-1}} v_{l-1} \xrightarrow{s_l} v_l = \tilde{b}.$$

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- Cyclic sliding operation and sliding circuits SL [GebGM09].
- $SL \subseteq USS \subseteq SSS \subseteq SS$.
- Families of permutation braids with SL of exponential size (in n) known.

New example I: Infinite braids

Consider the braid group

$$B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \quad \forall i,j \in \mathbb{N} : |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \forall i \in \mathbb{N} \end{array} \right\rangle.$$

on infinitely many strands.
Let B_∞^+ the monoid generated by σ_i's only.
Set $S_\infty = \bigcup_{i=1}^\infty \mathrm{Div}(\Delta_n)$.

Normal decomposition

Every braid in B_∞^+ admits a unique decomposition of the form $s_1 \cdots s_p$ with $s_1, \ldots, s_p$ in S_∞ satisfying $s_p \neq 1$, and, for every i,

$$\forall\, t \neq 1 : (t \preceq s_{i+1} \Rightarrow s_i t \notin S_\infty).$$

Note: Monoid not finitely generated. Infinitely many simple elements. No Garside element.

New example II: Klein bottle group

Consider $K = \pi_1(\text{Klein bottle}) = \langle a, b \mid ba = ab^{-1} \rangle$.
Let $K^+ = \langle a, b \mid a = bab \rangle$ be the Klein bottle monoid.

$$a^2 b = ab^{-1} \cdot bab = ba \cdot a = ba^2 \Leftrightarrow [a^2, b] = 1.$$

We conclude: Garside element $\Delta = a^2$ is central.

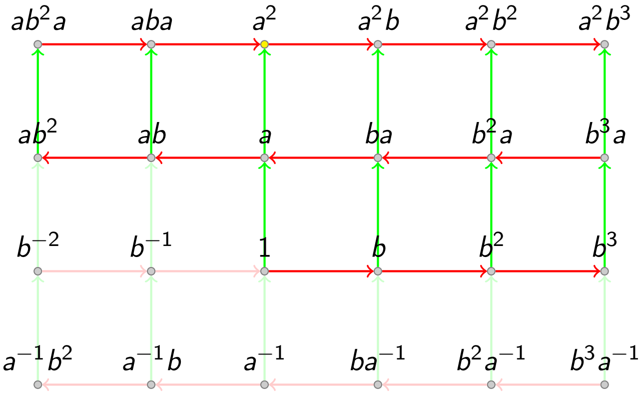
Normal decomposition

Every element of K admits a unique decomposition of the form $\Delta^p s_1 \cdots s_l$ with $p \in \mathbb{Z}$ and $s_1, \ldots, s_l$ in $\mathrm{Div}(\Delta)$ satisfying $s_1 \neq \Delta$, $s_l \neq 1$, and, for every i,

$$\forall g \in K^+ \setminus \{1\} : \quad (g \preceq s_{i+1} \ \Rightarrow s_i g \not\preceq \Delta).$$

Note: Indeed, here we have $l \in \{0, 1\}$.
Monoid NOT Noetherian. Infinitely many divisors of Garside element a^2.

Cayley graph of Klein bottle monoid inside Klein bottle group



New example III: Wreathed free abelian group I

Consider the wreathed free abelian group $G = \mathbb{Z} \wr S_n = \mathbb{Z} \rtimes S_n$ with binary operation given by

$$(v, \pi) * (v', \pi') = (v + (v'\pi^{-1}), \pi\pi').$$

We put $a_i = ((0, \ldots, 0, 1, 0, \ldots, 0), \mathrm{id}_{S_n})$ for all $1 \leq i \leq n$, and $s_i = ((0, \ldots, 0), (i, i+1))$ for all $1 \leq i \leq n-1$.
Further denote $1 = ((0, \ldots, 0), \mathrm{id}_{S_n})$.

Presentation of $\mathbb{Z} \wr S_n$

$\mathbb{Z} \wr S_n$ admits a presentation with generators $a_1, \ldots, a_n$, $s_1, \ldots, s_{n-1}$ and relations

$$\begin{array}{ll} [a_i, a_j] = 1 \quad \forall i, j, & [s_i, s_j] = 1 \quad \forall |i-j| \geq 2, \\ s_i s_j s_i = s_j s_i s_j \quad \forall |i-j| = 1, & s_i^2 = 1 \quad \forall i, \\ [s_i, a_j] = 1 \quad \forall j \neq i, i+1, & s_i a_i = a_{i+1} s_i \quad \forall i \leq n-1, \\ a_i s_i = s_i a_{i+1} \quad i \leq n-1. & \end{array}$$

New example III: Wreathed free abelian group II

Consider the monoid $\mathbb{N}^n \wr S_n$ consisting of all pairs (v, π) satisfying $v(k) \geq 0$ for all $k \leq n$. We denote by $\mathcal{S}$ the subset of $\mathbb{N}^n \wr S_n$ consisting of all pairs (v, id) satisfying $v(k) \in \{0, 1\}$ for all $k \leq n$. We put $\Delta_n = ((1, \ldots, 1), \mathrm{id}_{S_n})$.

Normal decomposition

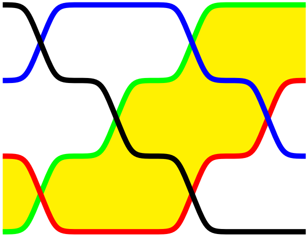
Every element of the group $\mathbb{Z}^n \wr S_n$ admits a unique decomposition of the form $\Delta_n^p s_1 \cdots s_l$ with $p \in \mathbb{Z}$, $s_1, \ldots, s_{l-1} \in \mathcal{S}$, and $s_p \in \mathcal{S} S_n$ satisfying $s_1 \neq \Delta_n$, $s_p \notin S_n$, and, for every i,

$$\forall g \in (\mathbb{N}^n \wr S_n) \setminus \{1\} : \quad (g \preceq s_{i+1} \Rightarrow s_i g \not\preceq \Delta_n).$$

Note: $\mathbb{N}^n \wr S_n$ is NOT a Garside monoid since it has nontrivial invertible elements.

New example IV: Ribbon categories I

For $n > 2$ and $1 \leq i, j < n$, we denote by $BRn(i,j)$ the family of all braids of B_n that contain an (i,j)-ribbon.



Let BR_n be the groupoid of n-strand braid ribbons, whose object set is $\{1, \ldots, n-1\}$ and whose family of morphisms with source i and target j is $BR_n(i,j)$.
Let BR_n^+ be the subcategory of BR_n in which the morphisms are required to lie in B_n^+.

New example IV: Ribbon categories II

For $1 \leq i < n$, we denote by $S_n(i)$ the family of all braids in B_n^+ that leftdivide Δ_n and contain an (i, j)-ribbon for some j. We denote by $\mathcal{S}$ the union of all families $S_n(i)$ for $i = 1, \ldots, n-1$. Observe: Δ_n contains a $(i, n-i)$ ribbon for all $1 \leq i < n$.

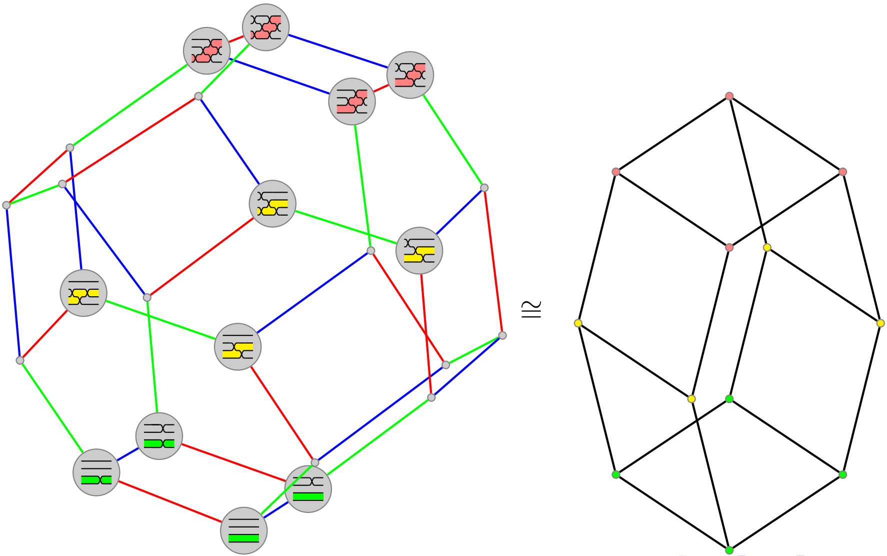
Normal decomposition

Every n-strand braid ribbon admits an unique decomposition of the form $\Delta_n^p s_1 \cdots s_l$ with $p \in \mathbb{Z}$ and $s_1, \ldots, s_l$ morphisms of $\mathcal{S}$ satisfying $s_1 \neq \Delta_n$, $s_l \neq 1$, and, for every i,

$$\forall g \in BR_n^+ \setminus \{1\} : \quad (g \preceq s_{i+1} \Rightarrow s_i g \npreceq \Delta_n).$$

Note: The multiplication is not defined everywhere.

$S_4(1)$



Garside families

- A category $\mathcal{C}$ is called left-cancellative (resp. right-cancellative) if $fg = fg'$ (resp. $gf = g'f$) implies $g = g'$ for all $f, g, g' \in \mathcal{C}$.
- For $f, g \in \mathcal{C}$ left-cancellative category, we denote $f \preceq g$, $\exists g' \in \mathcal{C}$ s.t. $fg' = g$ holds.

Definition

For $\mathcal{S} \subseteq \mathcal{C}$ left-cancellative category, a $\mathcal{C}$-path $g_1 | \cdots | g_p$ is called $\mathcal{S}$-greedy (resp. $\mathcal{S}$-normal) if, for every $i < p$, we have

$$\forall s \in \mathcal{S} \ \forall f \in \mathcal{C} : \quad s \preceq f g_i g_{i+1} \Rightarrow s \preceq f g_i$$

(resp. this and, in addition, every entry g_i lies in $\mathcal{S}^{\#} := \mathcal{S}\mathcal{C}^{\times} \cup \mathcal{C}^{\times}$.

$\mathcal{S} \subseteq \mathcal{C}$ ($\mathcal{C}$ left-cancellative category) is called a Garside family if every element of $\mathcal{C}$ admits an $\mathcal{S}$-normal decomposition.

Garside germs I

Definition

A germ is a triple $(\mathcal{S}, 1_\mathcal{S}, \bullet)$ where $\mathcal{S}$ is a precategory, $1_\mathcal{S}$ is a subfamily of $\mathcal{S}$ consisting of an element 1_x with source and target x for each object x, and $\bullet$ is a partial map of $\mathcal{S}^{[2]}$ into $\mathcal{S}$ that satisfies

- if $s \bullet t$ is defined, its source is the source of s and its target is the target of t,
- $1_x \bullet s = s = s \bullet 1_y$ hold or each s in $\mathcal{S}(x, y)$,
- if $r \bullet s$ and $s \bullet t$ are defined, then $(r \bullet s) \bullet t$ is defined iff $r \bullet (s \bullet t)$ is, in which case they are equal.

The germ is called left-associative if, for all $r, s, t \in \mathcal{S}$, it satisfies: if $(r \bullet s) \bullet t$ is defined, then $s \bullet t$ is defined,
and it is called left-cancellative if, for all $s, t, t' \in \mathcal{S}$, it satisfies if $s \bullet t$ and $s \bullet t'$ are defined and equal, then $t = t'$ holds.

Garside germs II

Defintion

If $\underline{\mathcal{S}}$ is a germ, we denote by $Cat(\underline{\mathcal{S}})$ the category $\langle \mathcal{S} \mid \mathcal{R}_\bullet \rangle$, where $\mathcal{R}_\bullet$ is the family of all relations $s \mid t = s \bullet t$ with $s, t \in \mathcal{S}$ and $s \bullet t$ defined.

$\bullet$	1	a	b	ab	ba	Δ
1	1	a	b	ab	ba	Δ
a	a		ab		Δ	
b	b	ba		Δ		
ab	ab	Δ				
ba	ba		Δ			
Δ	Δ					

Example: Germ $\underline{\mathcal{S}}$ of B_3^+.

Definition

A germ $\underline{\mathcal{S}}$ is said to be a Garside germ if $\underline{\mathcal{S}}$ embeds in $Cat(\underline{\mathcal{S}})$, the latter is left-cancellative, and $\underline{\mathcal{S}}$ is a Garside family in that category.

Example: Not a Garside family

Consider $M = \langle a, b \mid ab = ba,\ a^2 = b^2 \rangle$. Let $\mathcal{S} = \{1, a, b, ab, a^2\}$.

The germ $\underline{\mathcal{S}}$ induced by $\mathcal{S}$.

$\bullet$	1	a	b	a^2	ab
1	1	a	b	a^2	ab
a	a	a^2	ab		
b	b	ab	a^2		
a^2	a^2				
ab	ab				

The category (here the monoid) $Cat(\underline{\mathcal{S}})$ is (isomorphic to) M, as the relations $a|a = a^2 = b|b$ and $a|b = ab = b|a$ belong to the family $\mathcal{R}_\bullet$. However $\mathcal{S}$ is not a Garside family in M, as a^3 admits no $\mathcal{S}$-normal decomposition:
$a^2|a$ is not $\mathcal{S}$-greedy as ab left-divides a^3 but not a^2, and $ab|b$ is not $\mathcal{S}$-greedy as a^2 left-divides a^3 but not ab.

Recognizing Garside families

Definition

Assume that $\underline{\mathcal{S}}$ is a germ.
(i) We define the local left-divisibility relation $\preceq_{\mathcal{S}}$ of $\mathcal{S}$ by saying that $s \preceq_{\mathcal{S}} t$ holds if and only if there exists t' in $\mathcal{S}$ satisfying $t = st'$.
(ii) For $s_1|s_2$ in $\mathcal{S}^{[2]}$, we put

$$\mathcal{J}(s_1, s_2) = \{t \in \mathcal{S} \mid s_1 \bullet t \text{ defined and } t \preceq_{\mathcal{S}} s_2\}.$$

Proposition [DDGKT13]

A germ $\underline{\mathcal{S}}$ is a Garside germ if and only if it is left-associative, left-cancellative, and if, for any s_1, s_2 in $\mathcal{S}$ there exists a $\preceq_{\mathcal{S}}$-greatest element in $\mathcal{J}(s_1, s_2)$ (that is, an element r in $\mathcal{J}(s_1, s_2)$ such that $t \preceq_{\mathcal{S}} r$ holds for all $t \in \mathcal{J}(s_1, s_2)$).

Thank you!!