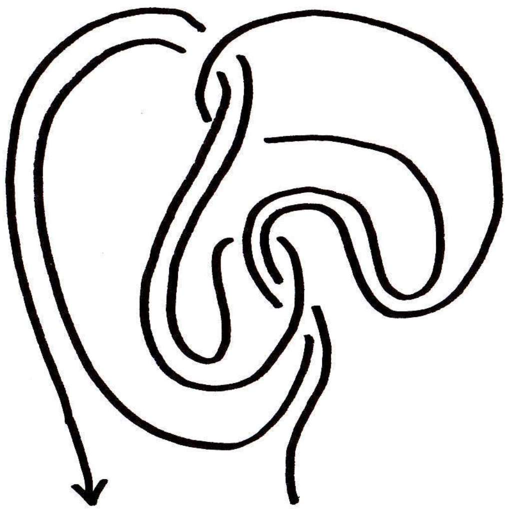
Doctorial Forum of Mathematics between Fudan and Kyoto Universities

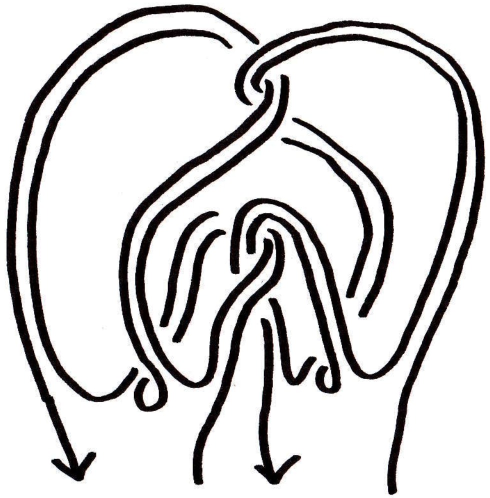
2011. 11. 4 @ Fudan Univ.

On the universal sl_2 invariant of bottom tangles

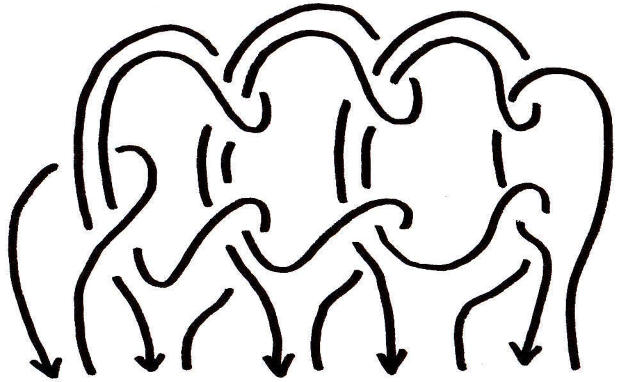
Sakie Suzuki (RIMS)



ribbon



boundary



Brunnian

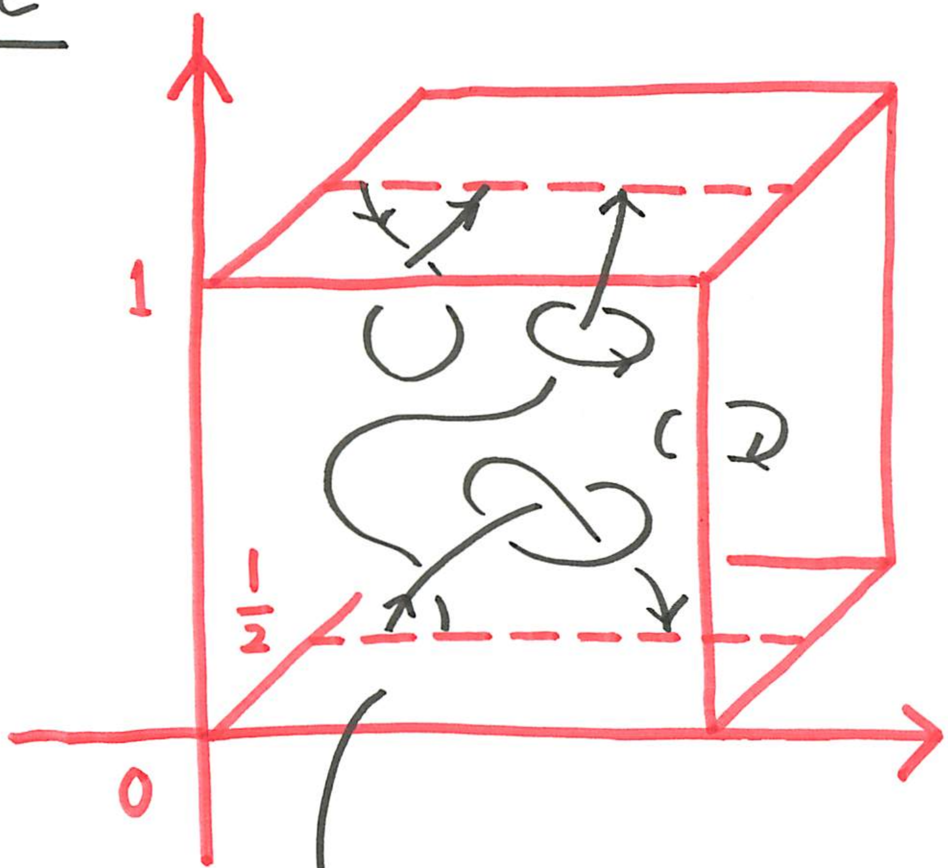
Today's plan

1. Tangles and bottom tangles
2. Motivation
3. The universal sl_2 invariant
4. Main results

Tangle in a cube

ex)

$$\coprod^{3} [0,1] \coprod^{2} S^1 \xrightarrow[\text{emb}]{} $$



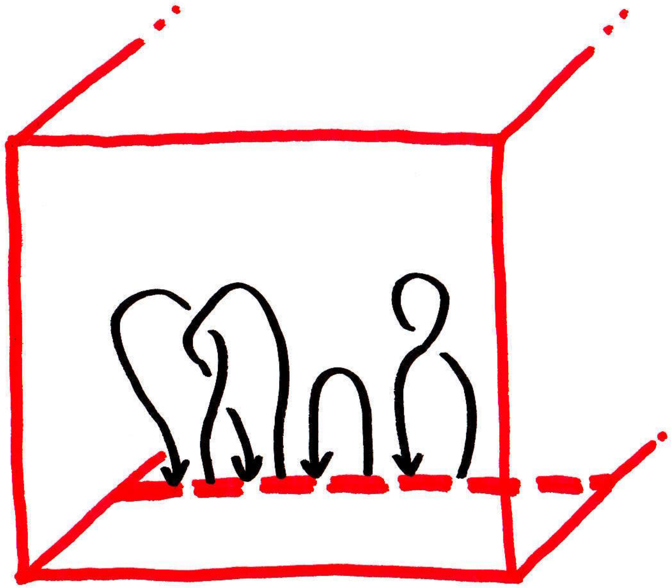
end pts $\subset [0,1] \times \{\frac{1}{2}\} \times \{0,1\}$

- Orientation
- framing

Bottom tangle ··· tangle in a cube s.t.

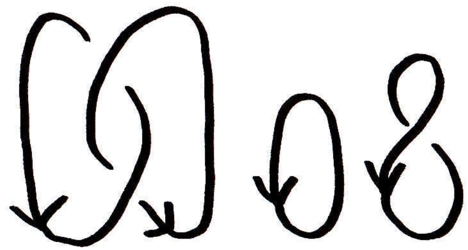
1. Only arcs
   (No cicles)

2. End pts
   $\subset$ bottom line

$T =$ 

4. Orientation is 

3. Two end pts of an arc ··· adjacent 

$\xrightarrow{\text{close the bottom}}$ $cl(T) =$ 

Motivation : "Good" understanding of the univ. $\mathfrak{sl}_2$ inv. of bottom tangles

Univ. $\mathfrak{sl}_2$ inv. (Laurence, Ohtsuki)

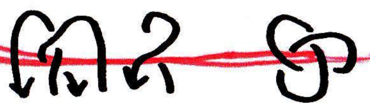
bottom tangle T

$$J_T \in U_h(\mathfrak{sl}_2)^{\hat{\otimes} n} \xrightarrow[\mathrm{tr}_q^{V_1} \otimes \cdots \otimes \mathrm{tr}_q^{V_n}]{\text{Quantum traces}} J(cl(T); V_1, \dots, V_n)$$

Colored Jones poly.

※ $V_1, \dots, V_n$: finite dimensional representations of $U_h(\mathfrak{sl}_2)$

Top. properties $\longleftrightarrow$? $\longleftrightarrow$ Alg. properties



$U_h(\mathfrak{sl}_2)$

The quantum group $U_h = U_h(sl_2)$ $/ \mathbb{Q}[[h]]$

generators : $H, E, F,$

relations : $HE - EH = 2E, \quad HF - FH = -2F,$

$$EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}$$

where $q = \exp h$, $K = \exp \frac{hH}{2}$

※ We can equip U_h with

a complete ribbon Hopf algebra structure.

Ribbon Hopf algebra /$\Bbbk$ $U = (U, \mu, \eta, \Delta, \varepsilon, S, R, \theta)$

• Hopf algebra

$\mu : U \otimes U \to U$

$\eta : \Bbbk \to U$

$\Delta : U \to U \otimes U$

$\varepsilon : U \to \Bbbk$

$S : U \to U$

with

• $R \in U \otimes U$: invertible

$R \Delta(x) R^{-1} = \Delta^{op}(x) \quad {}^{\forall} x \in U$

$(1 \otimes \Delta) R = R_{13} R_{12}$

$(\Delta \otimes 1) R = R_{13} R_{23}$

• $\theta \in U$: central, invertible

$\Delta(\theta) = (\theta \otimes \theta)(R_{21} R)^{-1}$

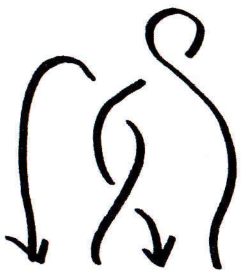
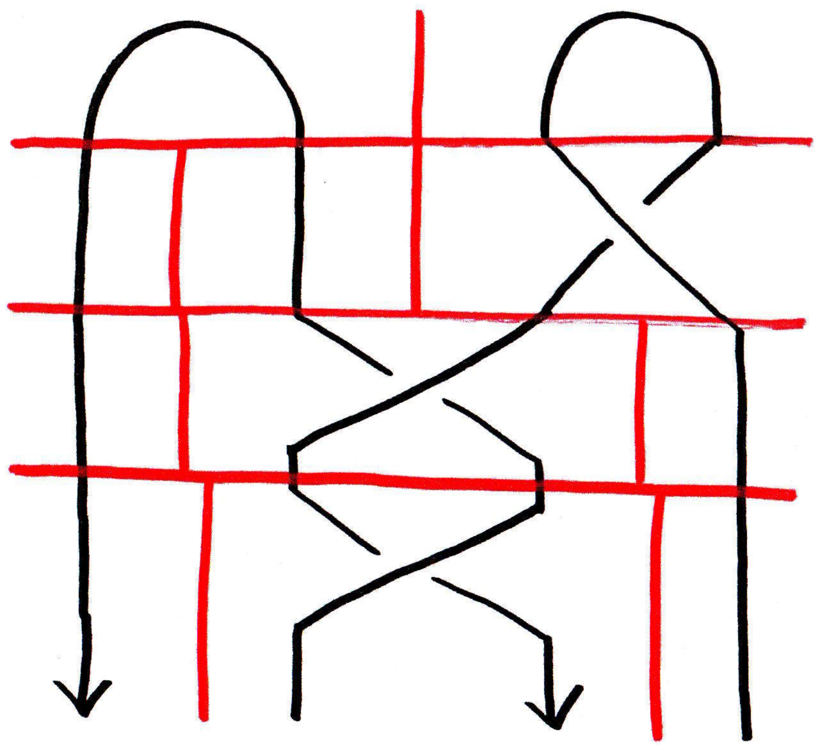
$S(\theta) = \theta, \quad \varepsilon(\theta) = 1$

Universal sl_2 invariant J_T

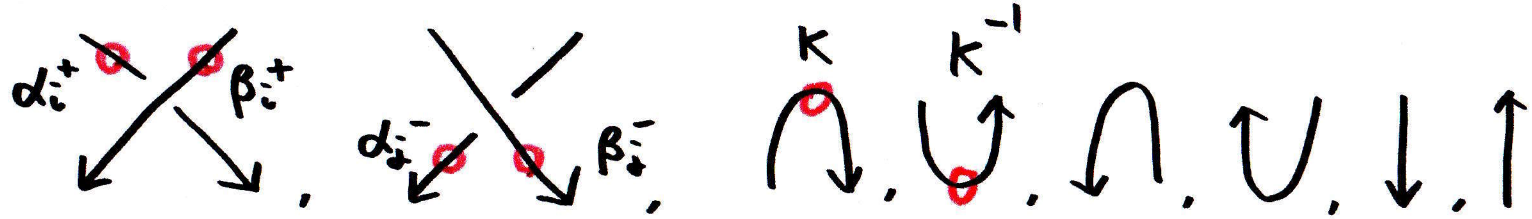
of a bottom tangle $T = T_1 \cup \cdots \cup T_n$

Step 1. Choose a diagram with , , $\cap$, $\cup$, $|$

⟨ex⟩

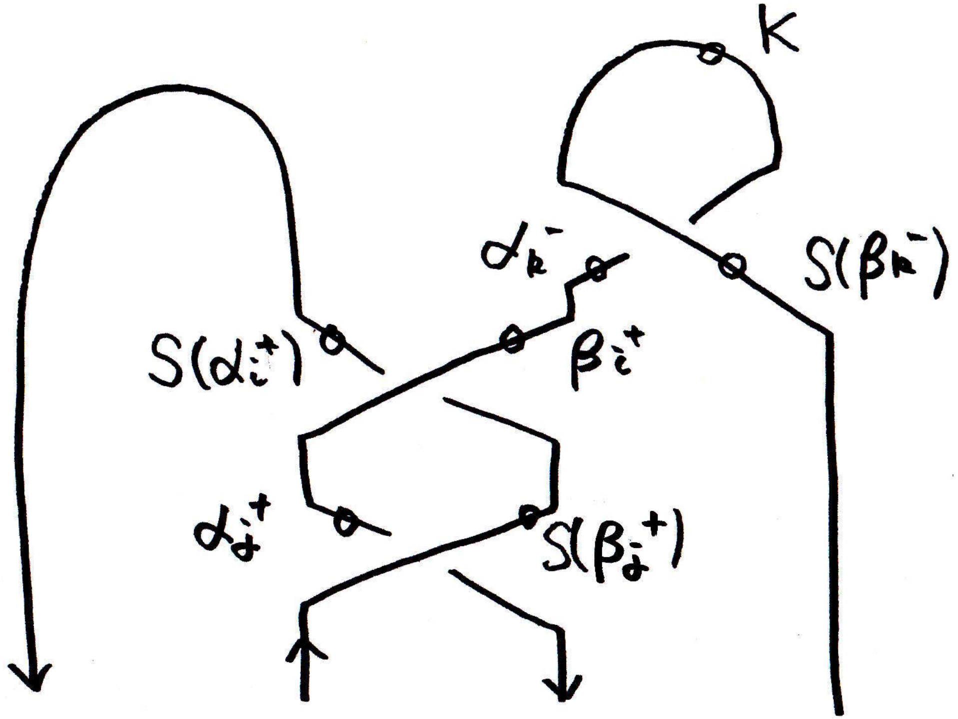
$T =$  $\rightsquigarrow$ 

Step2 Put labels $\left(R^{\pm 1} = \sum_i \alpha_i^{\pm} \otimes \beta_i^{\pm}\right)$



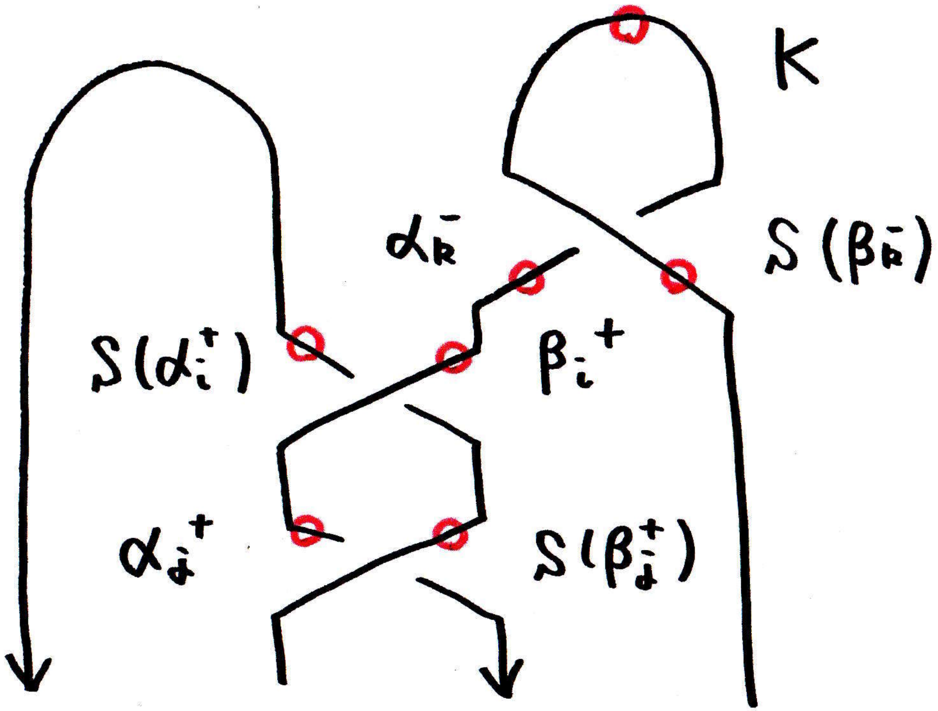
(Apply "S" if a strand is oriented upward)

⟨ex⟩



Step 3 Read the labels, take the tensors, and take the sums over the indices

⟨ex⟩



$$J_T = \sum_{ijk} S(\alpha_i^+)S(\beta_j^+) \otimes \alpha_j^+ \beta_i^+ \alpha_k^- K S(\beta_k^-) \in U_h^{\hat{\otimes} 2}$$

Notations

- $[i]_q = \dfrac{q^i - 1}{q - 1}$, $[i]_q! = [i]_q [i-1]_q \cdots [1]_q$,
- $e = (q^{1/2} - q^{-1/2})E$, $\tilde{F}^{(i)} = \dfrac{F^i K^i}{[i]_q!}$, $(i \geq 0)$
- $D = q^{\frac{1}{4} H \otimes H} = \exp\left(\dfrac{h}{4} H \otimes H\right)$

Universal R-matrix

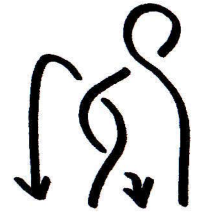
$$R = D \left(\sum_{i \geq 0} q^{\frac{1}{2} i(i-1)} \tilde{F}^{(i)} K^{-i} \otimes e^i \right)$$

$$R^{-1} = D^{-1} \left(\sum_{i \geq 0} (-1)^i \tilde{F}^{(i)} \otimes K^{-i} e^i \right)$$

$\mathbb{Z}[q, q^{-1}]$- subalgebras of U_h

大	A	$K^{\pm 2}$, $\tilde{F}^{(i)}$, $\tilde{E}^{(i)} = \dfrac{(q^{-\frac{1}{2}}E)^i}{[i]_q!}$, $i \geq 1$,
$\updownarrow$	$\cup$	
	B	$K^{\pm 2}$, $\tilde{F}^{(i)}$, e , $i \geq 1$,
	$\cup$	
小	C	$K^{\pm 2}$, $f = (q-1)FK$, e

※ $A = U_{\mathbb{Z}q}^{ev}$, $B = \mathcal{U}_q^{ev}$, $C = \bar{U}_q^{ev}$

$J\Big($  $\Big)$

$$= \sum_{i,j,k \geq 0} (-1)^{i+j} q^{-\frac{1}{2}k(k-1) - j^2 + 2ij - 3jk - 2ik} \begin{bmatrix} j+k \\ j \end{bmatrix}_q$$

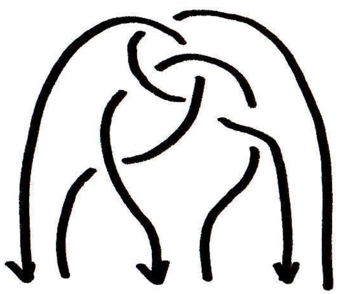
$$D^{-2}\left(1 \otimes q^{\frac{1}{4}H(H+2)}\right)\left(\tilde{F}^{(i)} K^{-2j} e^{j} \otimes \tilde{F}^{(j+k)} K^{2(j-i)} e^{i+k}\right)$$

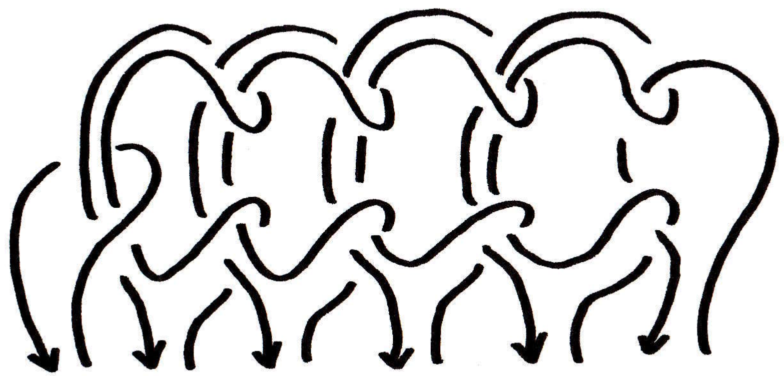
$$\left(\text{where } \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q [m-1]_q \cdots [m-n+1]_q}{[n]_q!} \in \mathbb{Z}[q, q^{-1}] \quad \text{for } m \in \mathbb{Z},\ n \geq 0 \right)$$

Main result

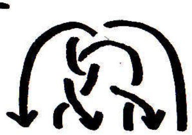
Thm T: $n \geq 3$ component Brunnian bottom tangle

$$\exists J_T \in \bigcap_{i} \left\{ C^{\otimes i-1} \otimes A \otimes C^{\otimes n-i} \right\}^{\wedge}$$

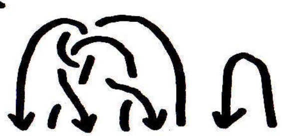




Example

J 

$$\in \{\underline{A} \otimes C \otimes C\}^\wedge \cap \{C \otimes \underline{A} \otimes C\}^\wedge \cap \{C \otimes C \otimes \underline{A}\}^\wedge$$

J 

$$\notin \{C \otimes C \otimes C \otimes \underline{A}\}^\wedge$$