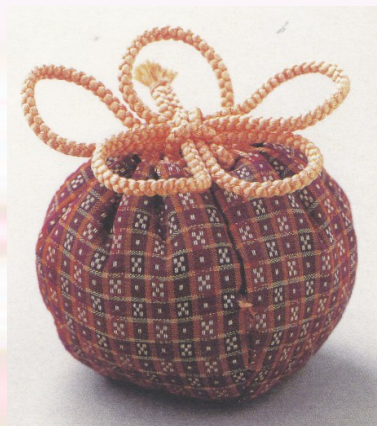


Knots and Quantum Topology

Sakie Suzuki

The Hakubi Center for Advanced Research/RIMS

April 15, 2014, @Hakubi Seminar



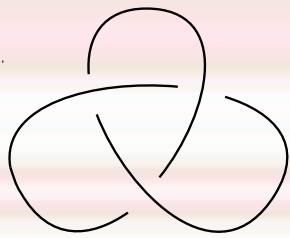
Contents

- 1 . Knots in mathematics
- 2 . Invariants of knots
- 3 . Background and motivation for Q. topology
- 4 . Jones polynomial
- 5 . My research

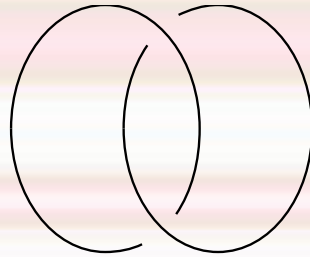
1 . Knots in mathematics

1. Knots

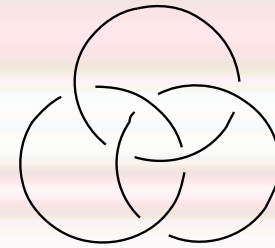
A knot : A circle in the 3-dimensional space



Trefoil knot



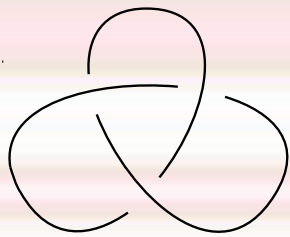
Hopf link



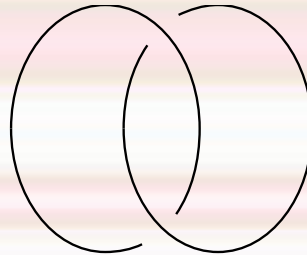
Borromean rings

1. Knots

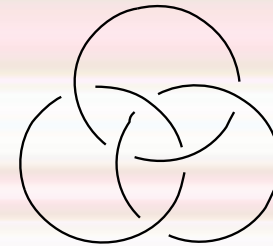
A knot : A circle in the 3-dimensional space



Trefoil knot

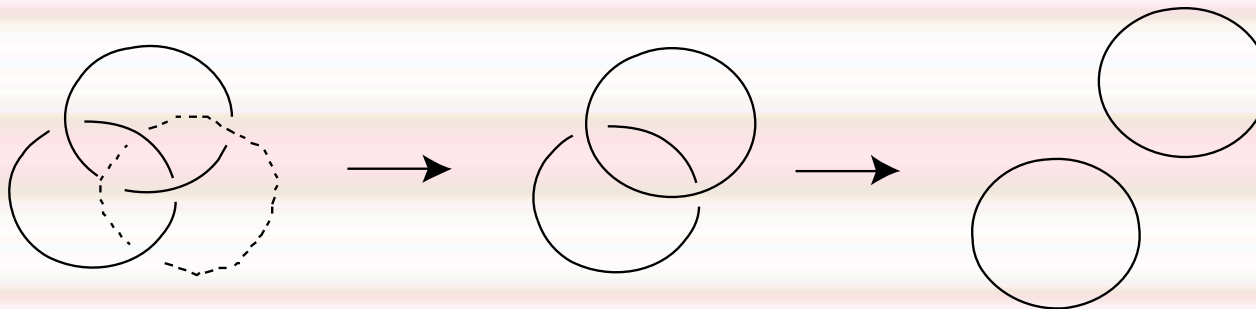


Hopf link



Borromean rings

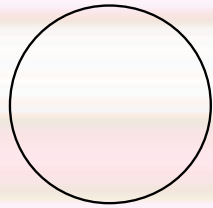
Borromean rings... Any pair of two rings are not linked.
(Three rings together are linked)



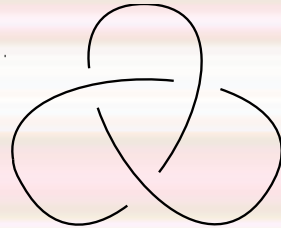
Break

<examples>

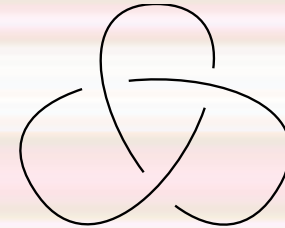
1. Knots



Trivial knot K_0



Trefoil K_{31}



\bar{K}_{31}

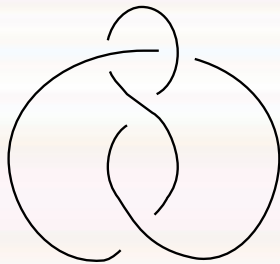
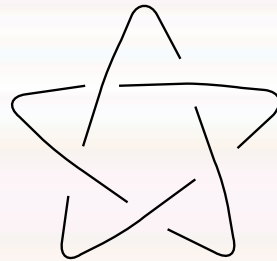
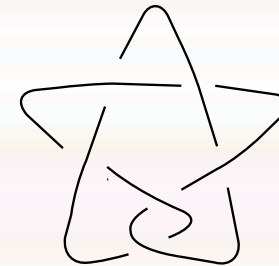


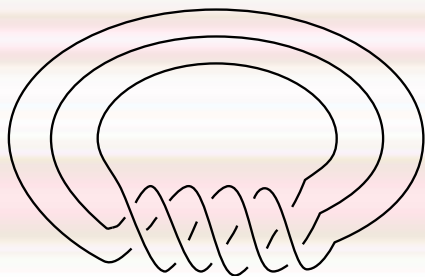
Figure eight K_{41}



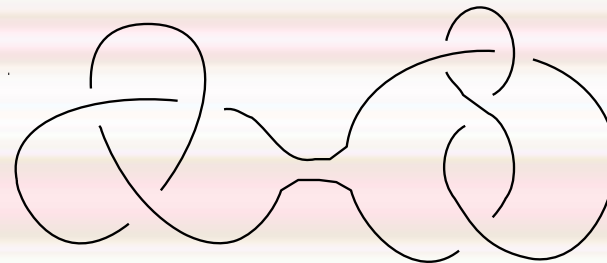
K_{51}



K_{61}



(3,5)-torus knot

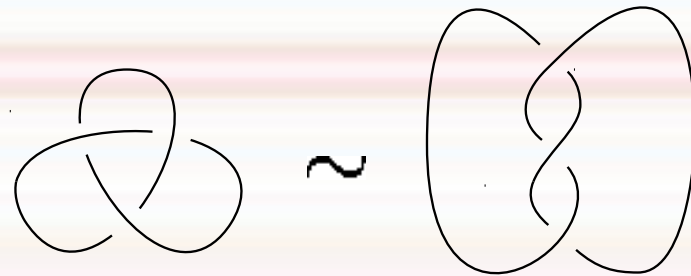


$K_{31} \# K_{41}$

1. Knots

$K \sim K' \Leftrightarrow$ We can transform K to K' continuously

<example>

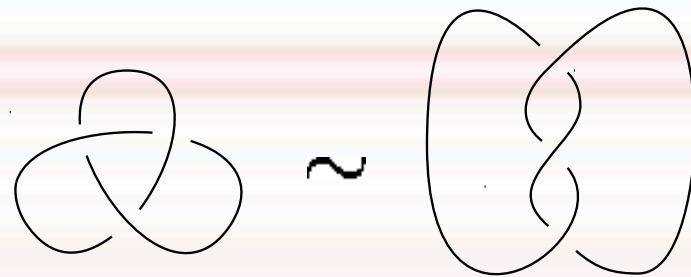


Can you see?

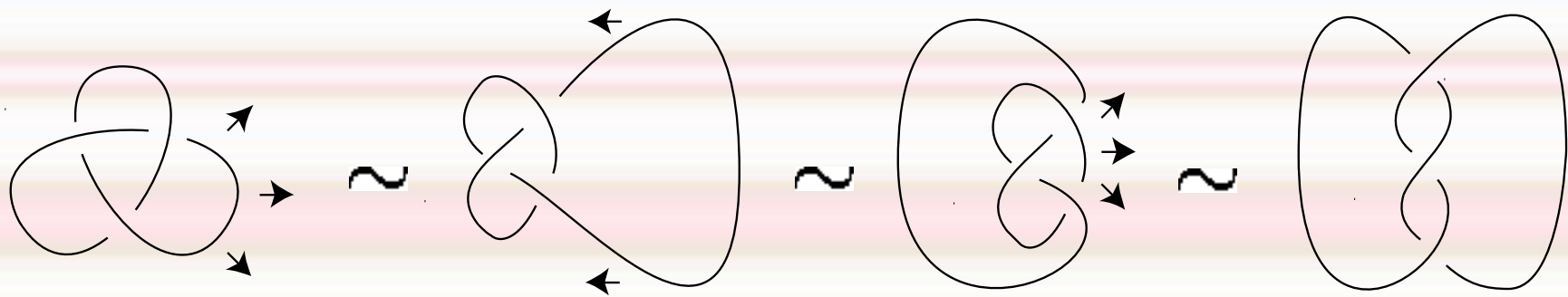
1. Knots

$K \sim K' \Leftrightarrow$ We can transform K to K' continuously

<example>



Can you see?



Interests

1. Equivalence Problem

"Judge whether two given knots are the same or not!"

2. Classification Problem

"Make a table of knots!"

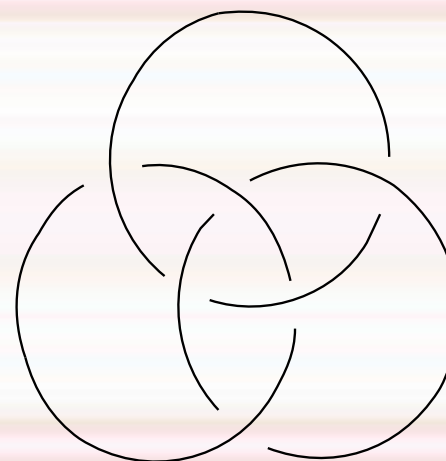
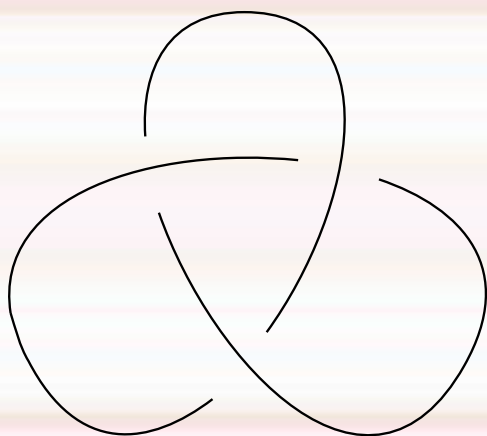
3. Properties of knots

"How is a knot knotted?"

2. Invariants of Knots

2. Invariants

Are these links are the same?



2. Invariants

Number of components... **Invariant** under transformation !

◆ Number of components $s : \{\text{links}\} \rightarrow \{1, 2, 3, 4, \dots\}$

$$s(\text{circle}) = 1,$$

$$s(\text{trefoil knot}) = 1,$$

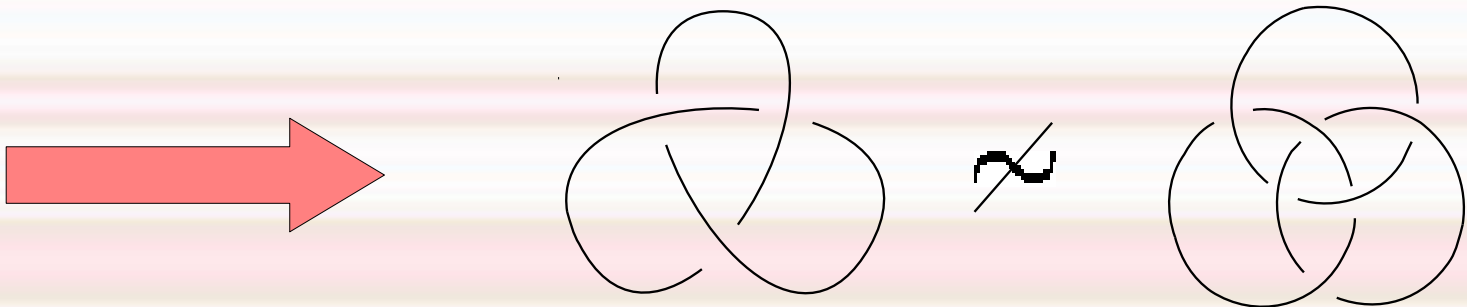
$$s(\text{three circles}) = 3$$

2. Invariants

Number of components... **Invariant** under transformation !

◆ Number of components $s : \{\text{links}\} \rightarrow \{1, 2, 3, 4, \dots\}$

$$s(\text{circle}) = 1, \quad s(\text{trefoil}) = 1, \quad s(\text{three circles}) = 3$$



2. Invariants

An **invariant** is a map $f: \{\text{links}\} \rightarrow G$ (Any set) such that

$$L \sim L' \Rightarrow f(L) = f(L').$$

2. Invariants

An **invariant** is a map $f: \{\text{links}\} \rightarrow G$ (Any set) such that



$$L \sim L' \Rightarrow f(L) = f(L').$$

A language representing "How knots are knotted"

A red speech bubble containing the equation $f(L) \neq f(L')$.
$$f(L) \neq f(L')$$

A large red thought bubble containing the text "We cannot transform L to L'. (These are different knots.)".

We cannot transform L to L'.
(These are different knots.)



2. Invariants

⟨examples⟩

◆ Number of components $s: \{\text{links}\} \rightarrow \{1, 2, 3, 4, \dots\}$

$$s(\text{circle})=1, \quad s(\text{trefoil})=1, \quad s(\text{three circles})=3$$

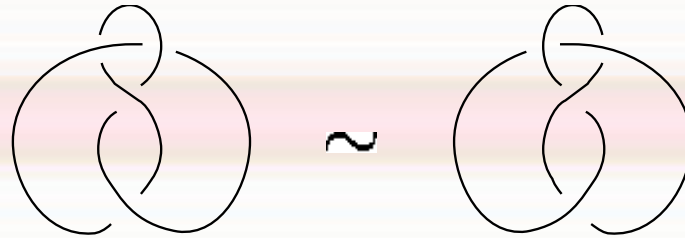
◆ Minimal crossing numbers $m: \{\text{links}\} \rightarrow \{0, 1, 2, 3, \dots\}$

$$m(\text{circle})=0, \quad m(\text{trefoil})=3, \quad m(\text{two circles})=2$$

3. Background and motivation for Q. topology

3. Background and Motivation

- ◆ 1849, Notes by J.B. Listing (Lord Kelvin's Vortex Atom Hypothesis ?)



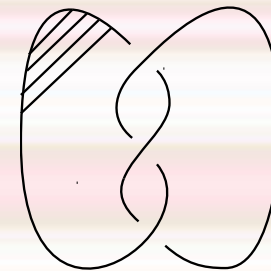
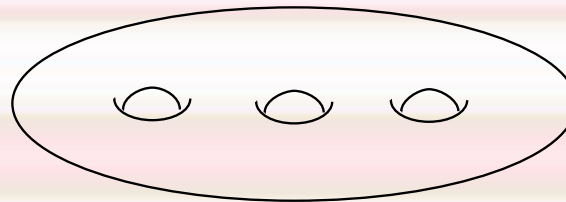
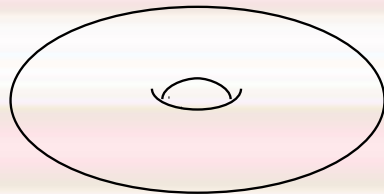
- ◆ 1930's, K. Reidemeister, H. Seifert, J. W. Alexander
- ◆ 1940's~1970's, R. H. Fox, T. Honma, S. Kinoshita, K. Sugimoto, A. Kawauchi... (foundations)
- ◆ 1980's, V. F. R. Jones, T. Ohtsuki, K. Habiro, J. Murakami,...
(Developing Involving Quantum mechanics, Statistic, mathematical physics, etc...)

3. Background and Motivation

Background

Classical invariants

- ◆ Simple invariants
 - ⟨ex⟩ Number of components
- ◆ Invariants described in **topological language**
 - ⟨ex⟩ Fundamental group, Homology, Surfaces,...



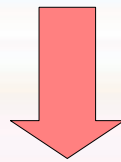
3. Background and Motivation

Background

Quantum invariants

Turning point!

In 1984... The discovery of the Jones polynomial



A lot of invariants

" Quantum invariants "

(Quantum groups + representations)

which are described in **algebraic language**, such as representation theory, mathematical physics, ...

3. Background and Motivation

Motivation

- ◆ \sim 1980's
 - Equivalence Problem
 - Classification Problem
 - Properties of knots
- ◆ 1980's \sim (After the discovery of Quantum invariants)
 - Structure of the set of knots (sociology!)
 - Application to low dimensional topology and others (Rep. theory, Category theory, Math. Physics,...)

3. Background and Motivation

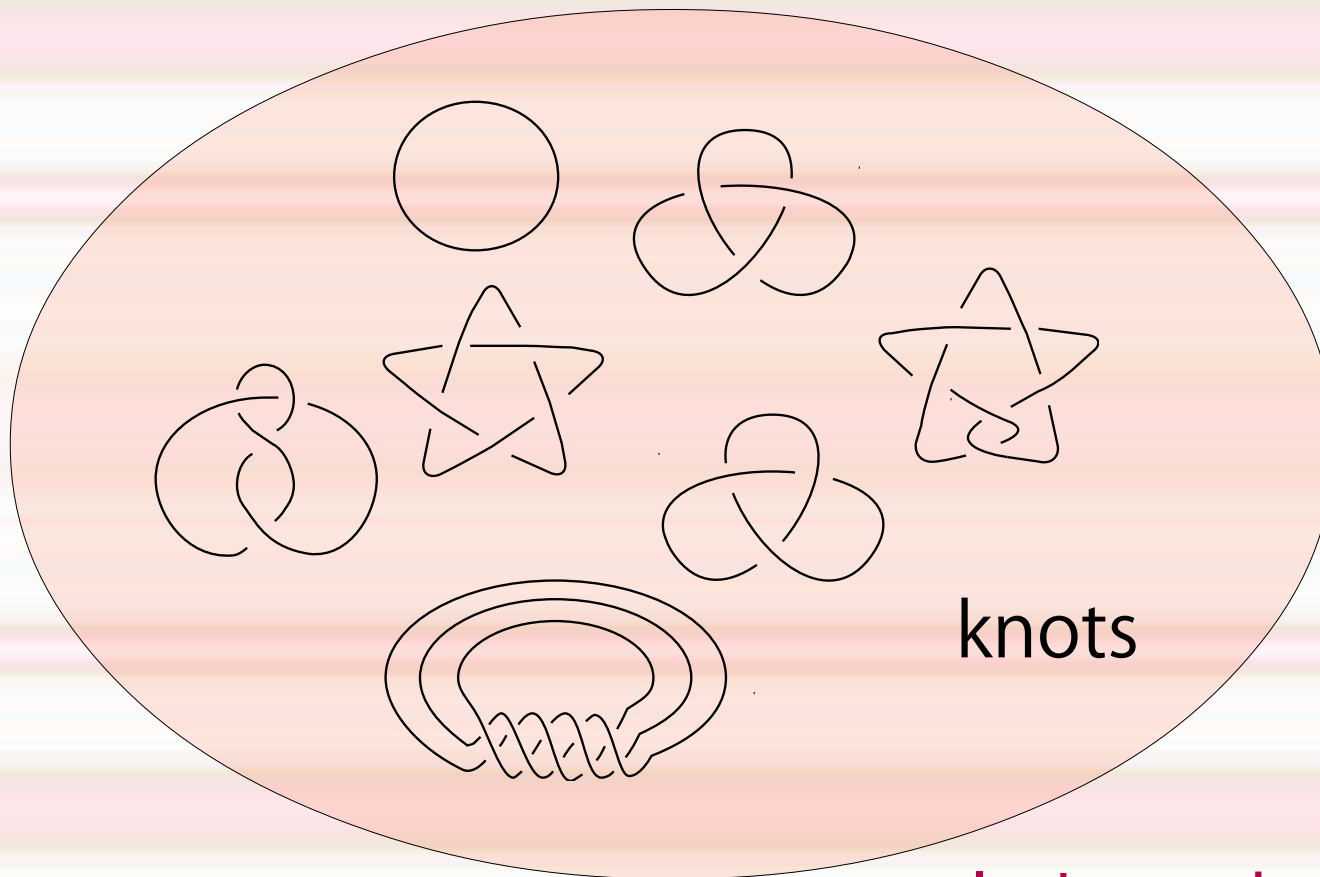
Motivation

- ◆ ~1980's
 - Equivalence Problem
 - Classification Problem
 - Properties of knots
- ◆ 1980's~ (After the discovery of Quantum invariants)
 - **Structure of the set of knots (sociology!)**
 - Application to low dimensional topology and others (Rep. theory, Category theory, Math. Physics,...)

4. Jones polynomial

4. Jones polynomial

Q : What properties does the set of knots has?

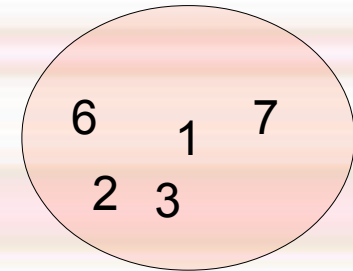


let's study sociology!

4. Jones polynomial

<ex> Natural numbers $N = \{1, 2, 3, \dots\}$

product : $2 \times 3 = 6, 4 \times 7 = 28, \dots$



An operation : input two numbers (a,b)
 \Rightarrow output a number $a \times b$

$(2, 3) \rightarrow 6 = 2 \times 3, (4, 7) \rightarrow 28 = 4 \times 7,$

satisfying

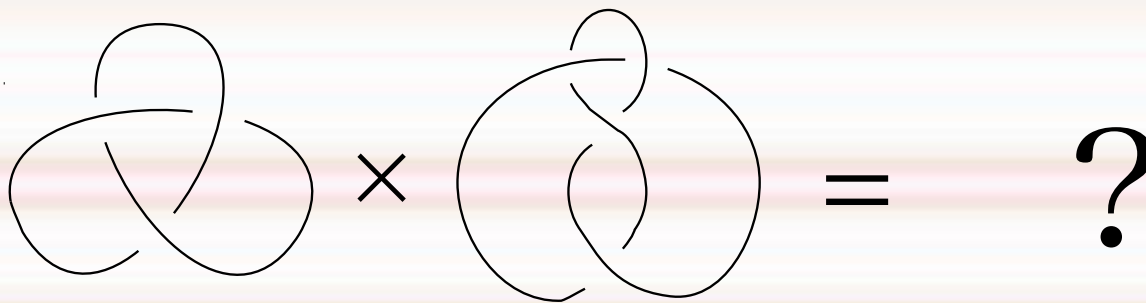
1. Associativity : $(a \times b) \times c = a \times (b \times c)$
2. Unitarity : $1 \times n = n = n \times 1$

4. Jones polynomial

Define a product of knots!

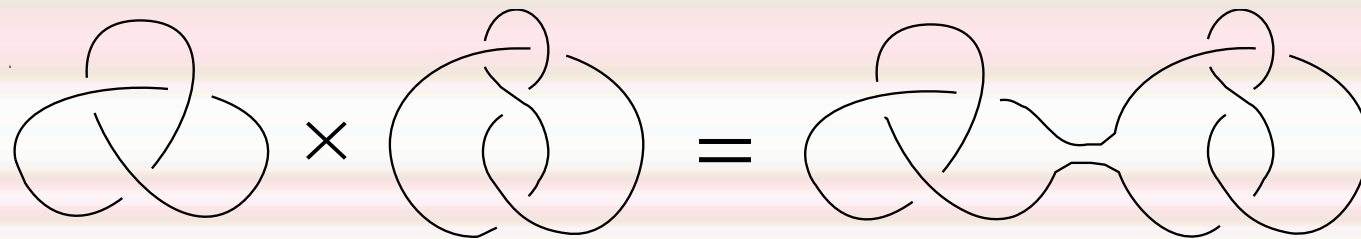
Product : input two numbers (a,b)
 \Rightarrow output a number $a \times b$

+ Associativity, Unitarity ★ 

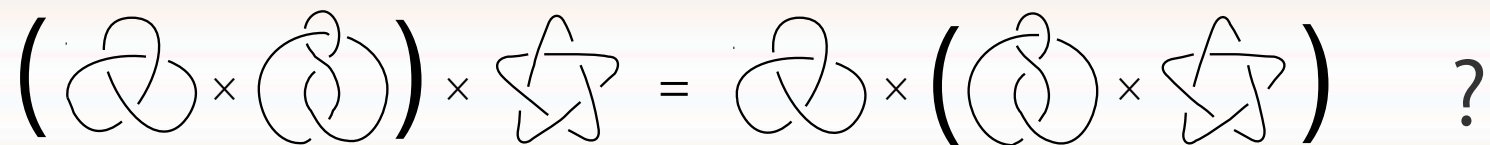


4. Jones polynomial

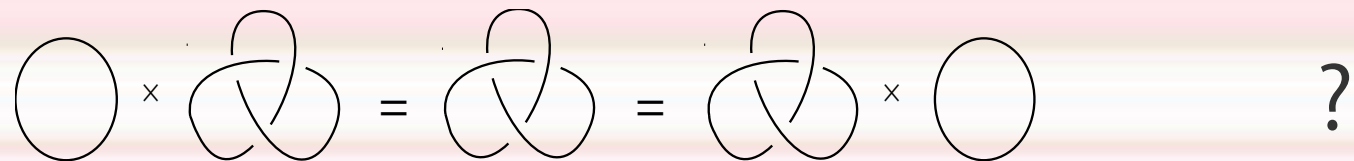
<An example>



Associativity :



Unitarity :



4. Jones polynomial

Associativity :

$$\left(\text{trefoil} \times \text{trefoil} \right) \times \text{star} = \text{trefoil} \times \left(\text{trefoil} \times \text{star} \right)$$

4. Jones polynomial

Associativity :

$$\left(\text{loop}_1 \times \text{loop}_2 \right) \times \text{star} = \text{loop}_1 \times \left(\text{loop}_2 \times \text{star} \right)$$

$$\begin{aligned} \left(\text{loop}_1 \times \text{loop}_2 \right) \times \text{star} &= \text{loop}_1 \text{---} \text{loop}_2 \times \text{star} \\ &= \text{loop}_1 \text{---} \text{loop}_2 \text{---} \text{star} \end{aligned}$$

$$\begin{aligned} \text{loop}_1 \times \left(\text{loop}_2 \times \text{star} \right) &= \text{loop}_1 \times \text{loop}_2 \text{---} \text{star} \\ &= \text{loop}_1 \text{---} \text{loop}_2 \text{---} \text{star} \end{aligned}$$

4. Jones polynomial

Unitarity :

$$\bigcirc \times \text{trefoil} = \text{trefoil} = \text{trefoil} \times \bigcirc$$
The diagram illustrates the unitarity property of the Jones polynomial. It consists of a sequence of four diagrams connected by equals signs. The first diagram is a simple circle followed by a trefoil knot, with a small 'x' between them. This is followed by an equals sign, then a single trefoil knot. This is followed by another equals sign, then a trefoil knot followed by a simple circle, with a small 'x' between them. The trefoil knot is drawn with three crossings in a standard orientation.

4. Jones polynomial

Unitarity :

$$\bigcirc \times \text{trefoil} = \text{trefoil} = \text{trefoil} \times \bigcirc$$

$$\begin{aligned} \bigcirc \times \text{trefoil} &= \text{trefoil with a large loop} \\ &= \text{trefoil} \end{aligned}$$

$$\begin{aligned} \text{trefoil} \times \bigcirc &= \text{trefoil with a large loop} \\ &= \text{trefoil} \end{aligned}$$

4. Jones polynomial

We obtained a product of knots!

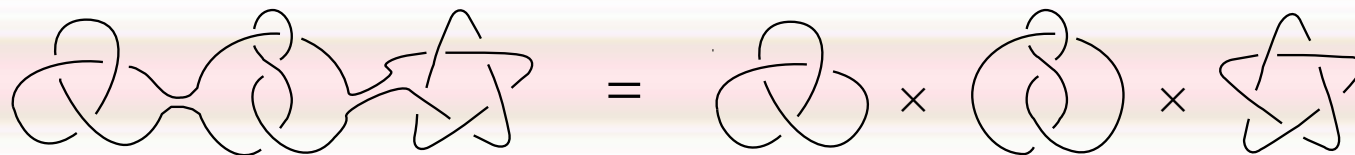
unique-prime-factorization theorem

prime : factors \Rightarrow 1 and itself

prime-factorization : $6=2 \times 3$, $8=2 \times 2 \times 2$, unique

prime knot : factors \Rightarrow \bigcirc and itself

prime-factorization : unique



Break

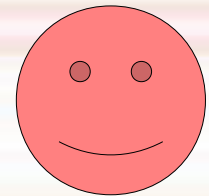
4. Jones polynomial

Let's compute the Jones polynomial!

$$V: \{\text{knots}\} \rightarrow Z[A, A^{-1}]$$

4. Jones polynomial

The Jones polynomial keeps the products!



$$V(\text{product of two knots}) = V(\text{first knot}) V(\text{second knot})$$

- ◆ detect a prime knot
- ◆ study the structure of products

4. Jones polynomial

Jones polynomial $V(K) \in \mathbb{Z}[A, A^{-1}]$

Step 1. Kauffman bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$

Step 2. $V(K) = (-A^3)^{-w(D)} \langle D \rangle$

$$w(D) = \# \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \blacktriangledown \quad \blacktriangledown \end{array} - \# \begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \\ \blacktriangledown \quad \blacktriangledown \end{array}$$

4. Jones polynomial

1. Kauffman bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$

Rule 1 :  $\langle \text{crossing} \rangle = A \langle \text{cup} \rangle + A^{-1} \langle \text{cap} \rangle$

The diagram shows a crossing of two strands within a dashed rectangular box. This is equal to A times the diagram of two strands forming a cup (top-left to top-right, bottom-left to bottom-right) within a dashed box, plus A inverse times the diagram of two strands forming a cap (top-left to top-right, bottom-left to bottom-right) within a dashed box.

Rule 2 : $\langle D \bigcirc \rangle = -(A^2 + A^{-2}) \langle D \rangle$

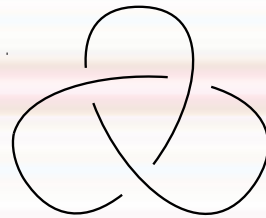
The diagram shows a dashed rectangular box containing a diagram D, with a small circle attached to the right side of the box.

Rule 3 : $\langle \rangle = 1$

The diagram shows an empty dashed rectangular box.

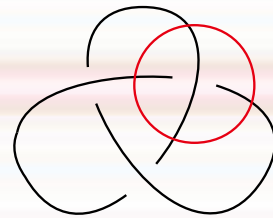
4. Jones polynomial

Rule 1 : $\langle \text{crossing} \rangle = A \langle \text{positive resolution} \rangle + A^{-1} \langle \text{negative resolution} \rangle$



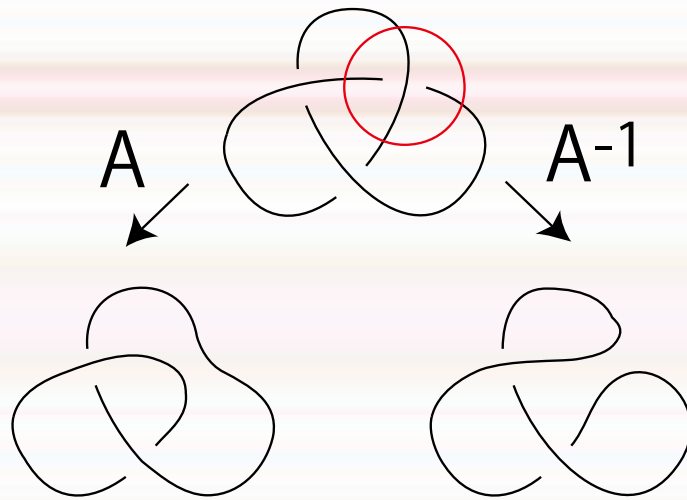
4. Jones polynomial

Rule 1 : $\langle \text{crossing} \rangle = A \langle \text{positive resolution} \rangle + A^{-1} \langle \text{negative resolution} \rangle$



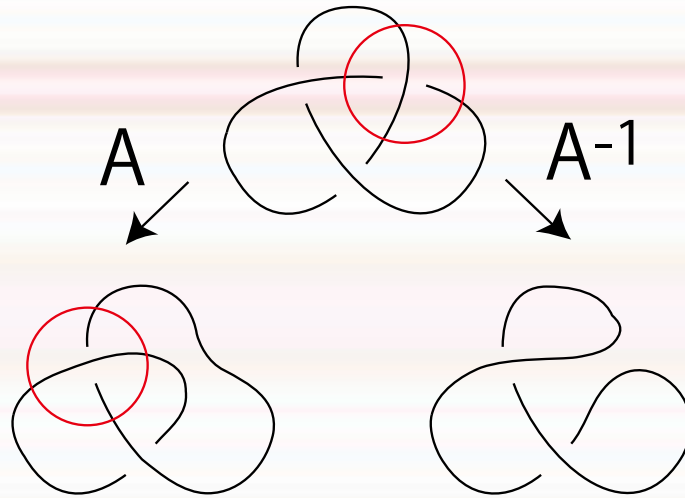
4. Jones polynomial

ルール1 : $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle$



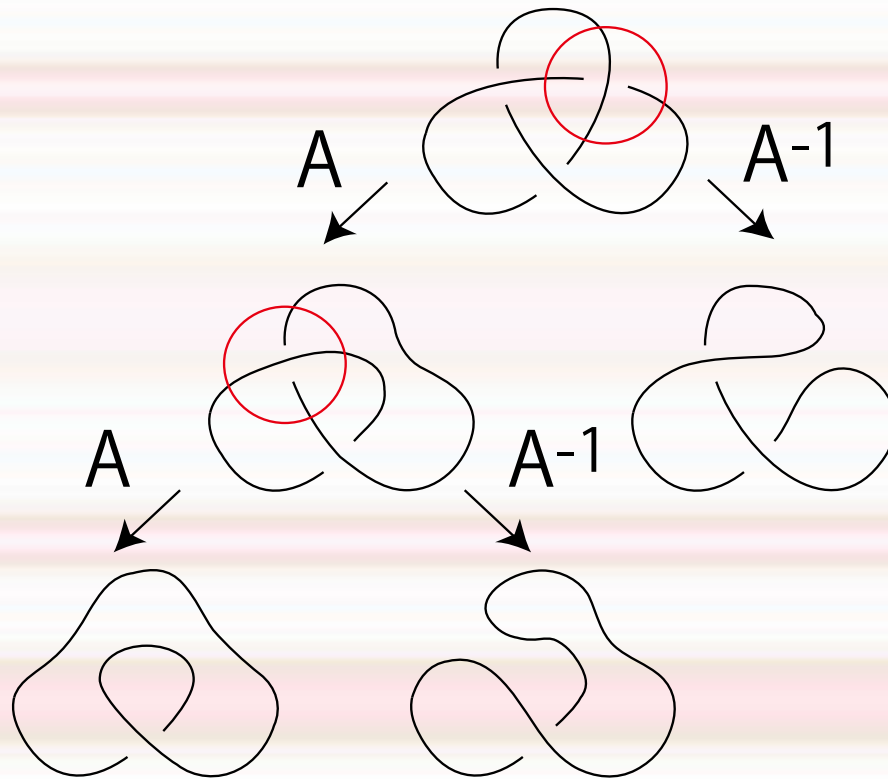
4. Jones polynomial

Rule 1 : $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle$



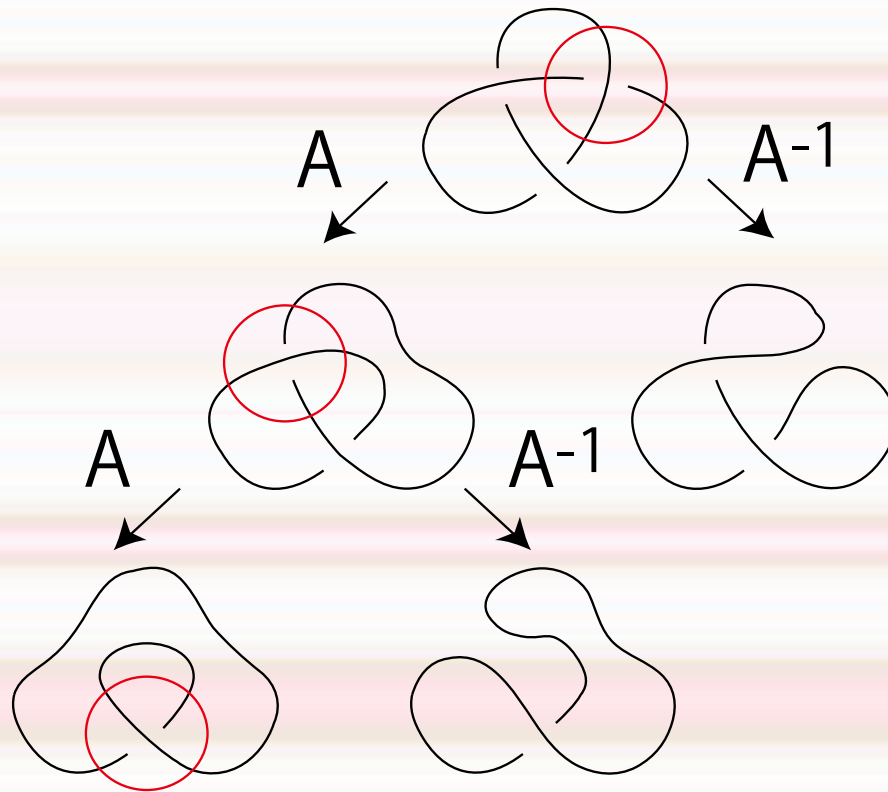
4. Jones polynomial

Rule 1 : $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle$



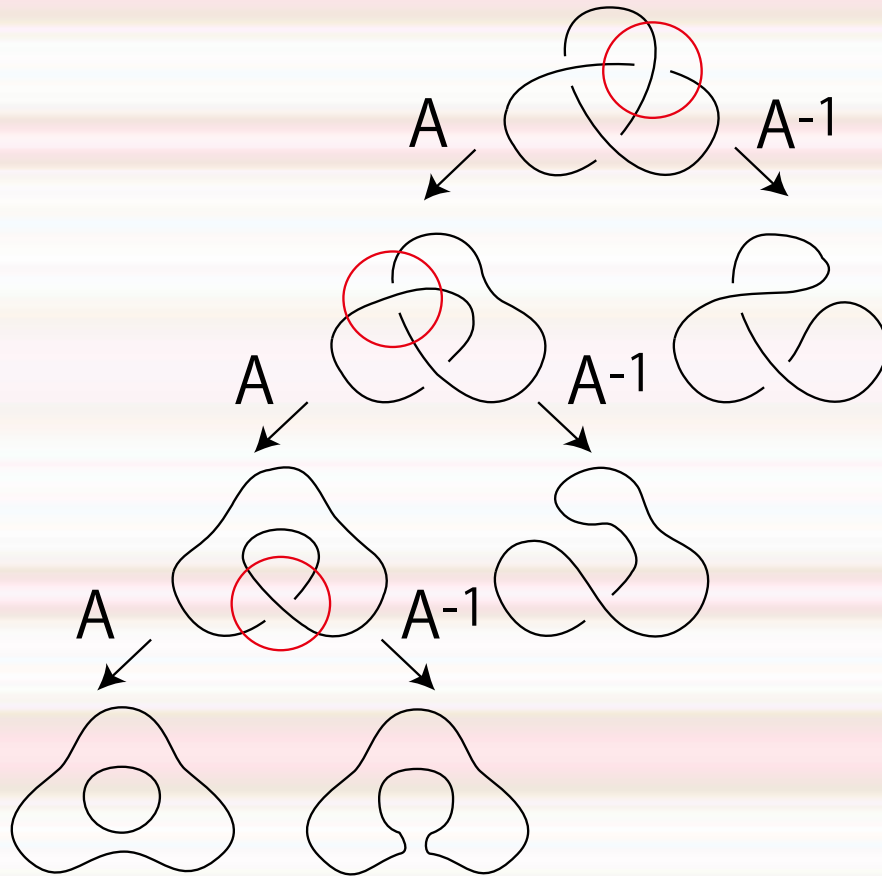
4. Jones polynomial

Rule 1 : $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle$



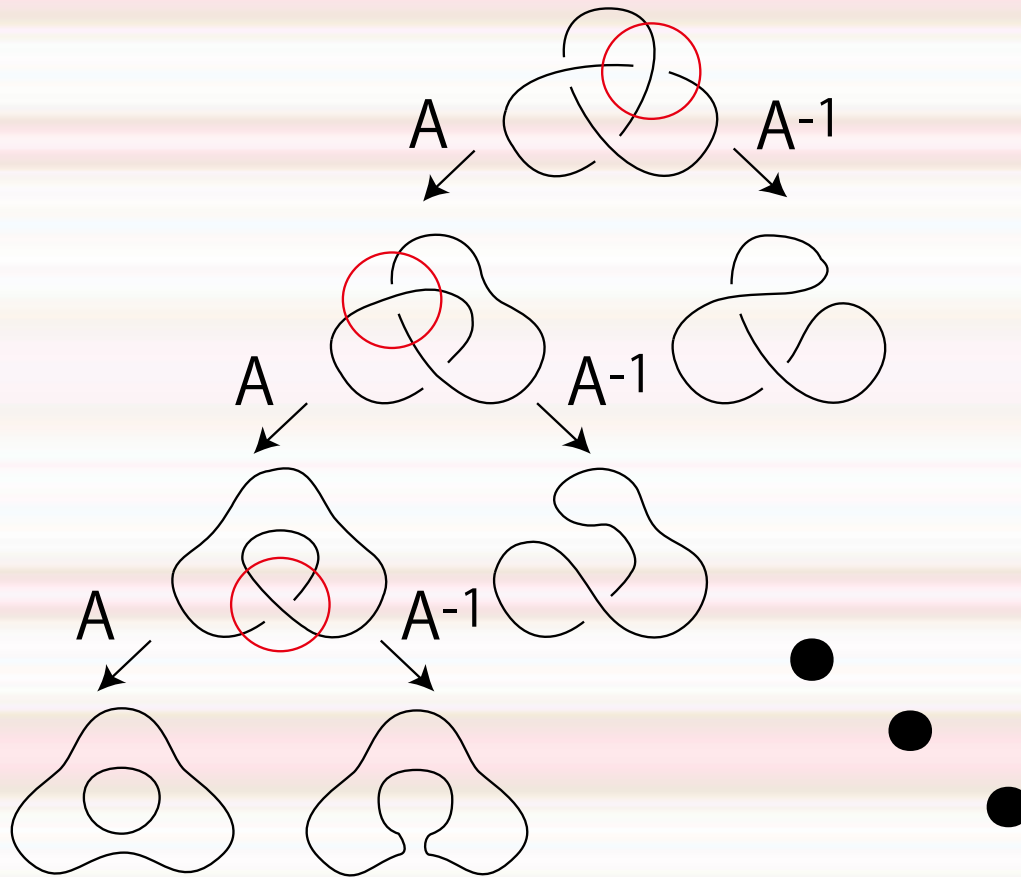
4. Jones polynomial

Rule 1 : $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle$



4. Jones polynomial

Rule 1 : $\langle \text{crossing} \rangle = A \langle \text{positive resolution} \rangle + A^{-1} \langle \text{negative resolution} \rangle$



4. Jones polynomial

$$\text{Rule 2 : } \langle D \bigcirc \rangle = -(A^2 + A^{-2}) \langle D \rangle$$

$$\langle \bigcirc \rangle = 1$$

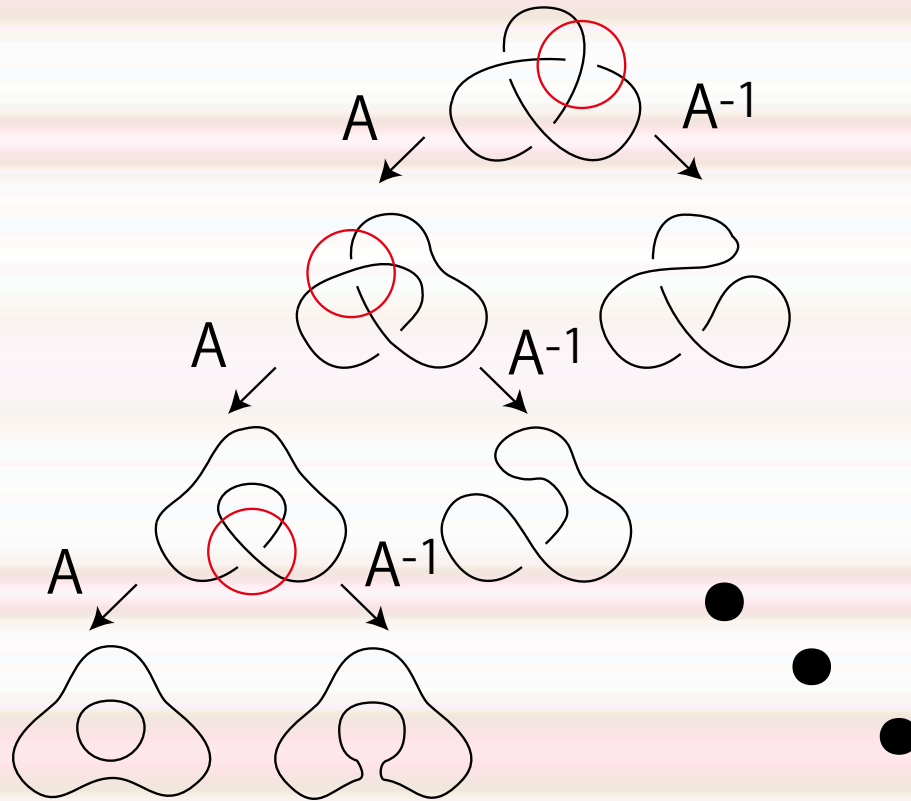
$$\langle \bigcirc \bigcirc \rangle = -(A^2 + A^{-2}) \langle \bigcirc \rangle = -(A^2 + A^{-2})$$

$$\langle \bigcirc \bigcirc \bigcirc \rangle = -(A^2 + A^{-2}) \langle \bigcirc \bigcirc \rangle = (A^2 + A^{-2})^2$$

$$\langle \underbrace{\bigcirc \bigcirc \cdots \bigcirc}_n \rangle = (-1)^{n-1} (A^2 + A^{-2})^{n-1}$$

4. Jones polynomial

Rule 1 : $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle$



Rule 2 : $-A^3 (A^2 + A^{-2}) + A + \dots$

4. Jones polynomial

Jones polynomial $V(K) \in \mathbb{Z}[A, A^{-1}]$

Step 1. Kauffman bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$

Step 2. $V(K) = (-A^3)^{-w(D)} \langle D \rangle$

$$w(D) = \# \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \blacktriangledown \quad \blacktriangledown \end{array} - \# \begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \\ \blacktriangledown \quad \blacktriangledown \end{array}$$

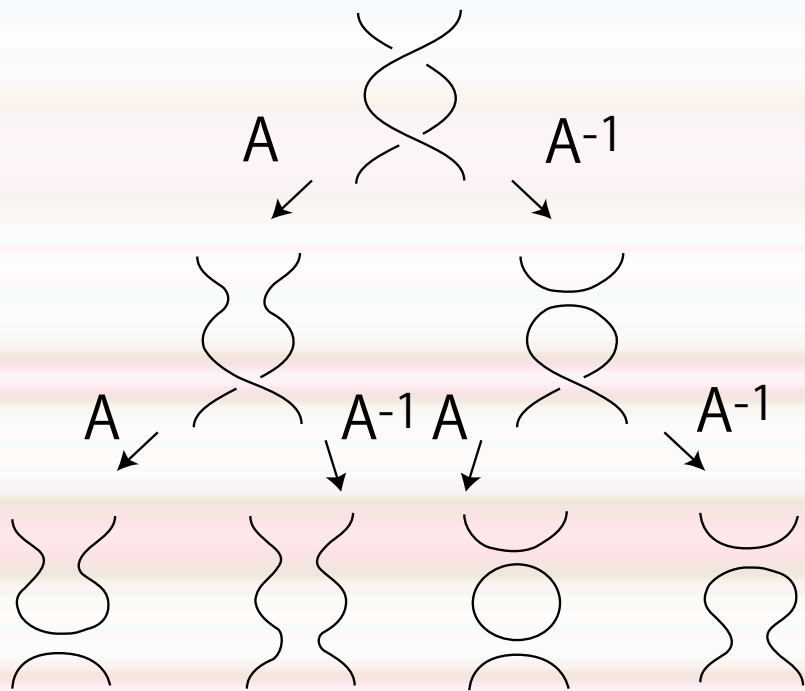
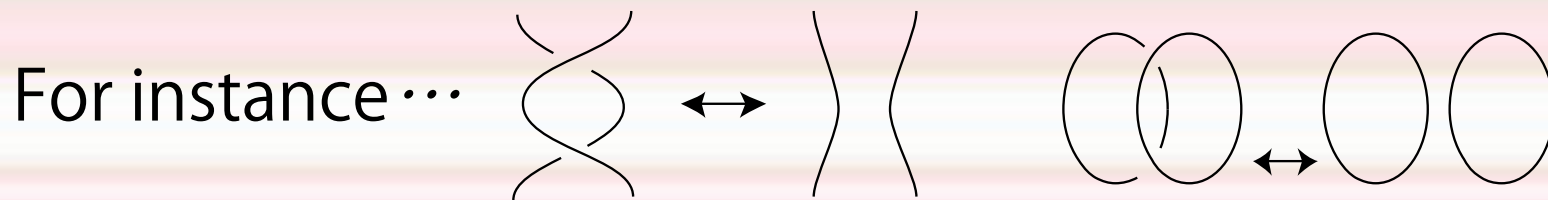
4. Jones polynomial

$$w \left(\text{trefoil} \right) = 3$$

$$\begin{aligned} V \left(\text{trefoil} \right) &= (-A)^{-3} \langle \text{trefoil} \rangle \\ &= (-A)^{-3} (-A^5 - A^{-3} + A^{-7}) \\ &\neq V \left(\text{circle} \right) \end{aligned}$$

4. Jones polynomial

Jones polynomial is invariant ?



Coefficient $\begin{matrix} \cup \\ \cup \end{matrix} =$

$$A^2 - (A^2 + A^{-2}) - A^{-2} = 0$$

Coefficient $\begin{matrix} \rangle \\ \langle \end{matrix} = 1$

5. My research

5. My research

Quantum invariants

Kontsevich invariant

$$Z(L) \in A(I)$$

Hopf algebra structure

Complete invariant ?

Universal sl_2 invariant

$$J(L) \in U_h(sl_2)^{\otimes I}$$

Hopf algebra structure

Studies of quantum groups

Colored Jones polynomial

$$V_n(L) \in Z[q^{1/4}, q^{-1/4}]$$

(including Jones polynomial $V=V_2$)

Thank you 🐱