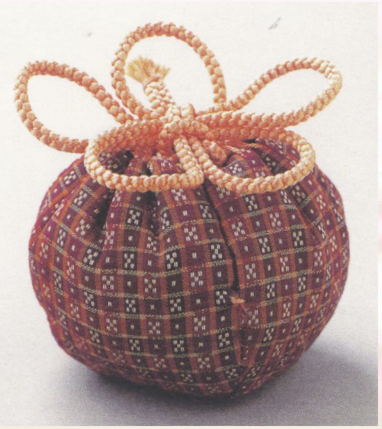
Knots and Quantum Topology

Sakie Suzuki
The Hakubi Center for Advanced Research/RIMS

April 15, 2014, @Hakubi Seminar



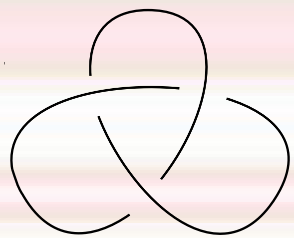



Contents

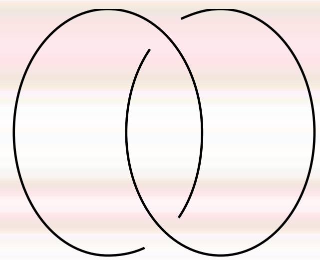
1． Knots in mathematics

1. Knots

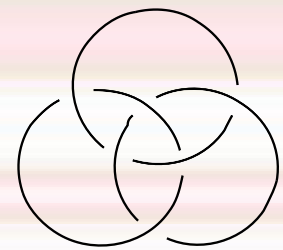
A knot：A circle in the 3-dimensional space



Trefoil knot

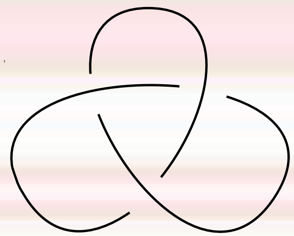


Hopf link

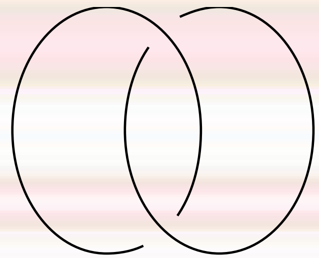


Borromean rings

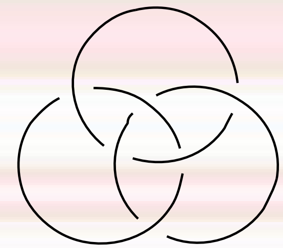
A knot：A circle in the 3-dimensional space



Trefoil knot

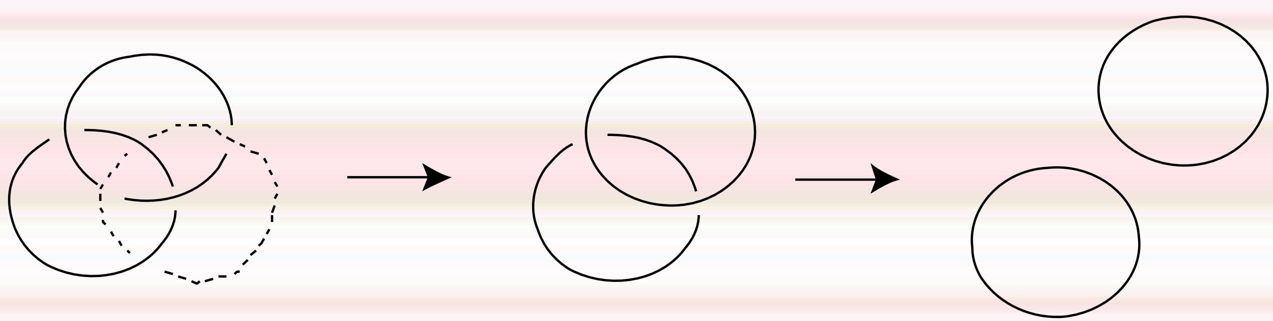


Hopf link



Borromean rings

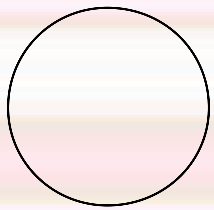
Borromean rings･･･Any pair of two rings are not linked.
(Three rings together are linked)



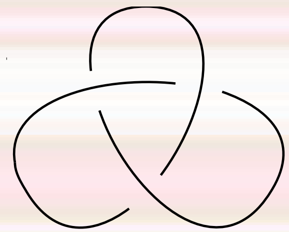
Break

1. Knots

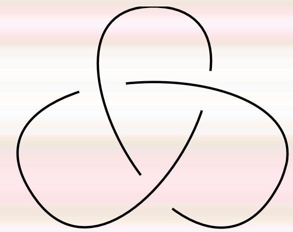
<examples>



Trivial knot K_0



Trefoil K_{31}



$\bar{K}_{31}$

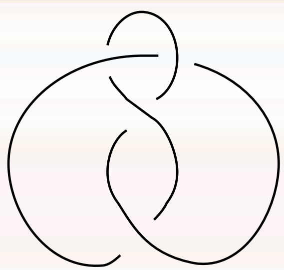
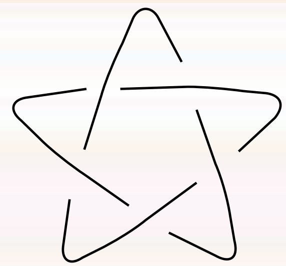
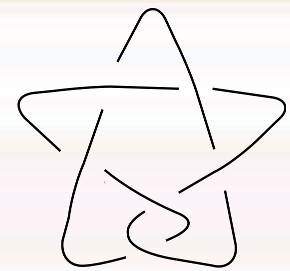


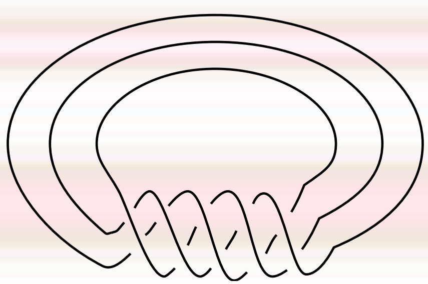
Figure eight K_{41}



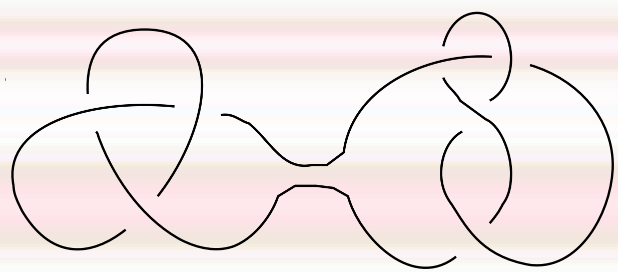
K_{51}



K_{61}



(3,5)-torus knot

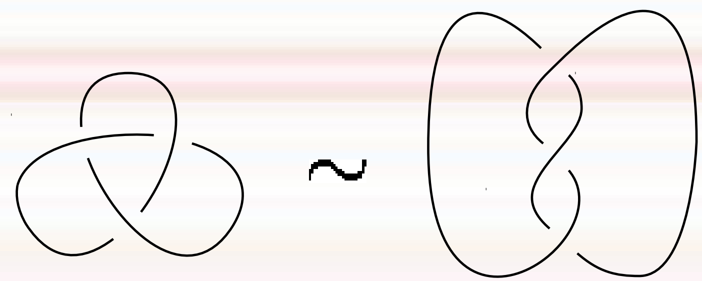


K_{31} # K_{41}

1. Knots

K∼K'　⇔　We can transform K to K' continuously

<example>

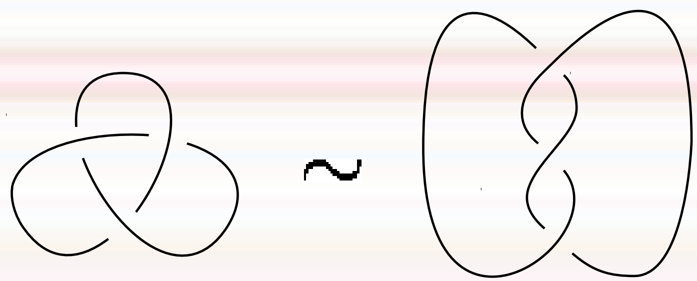


Can you see?

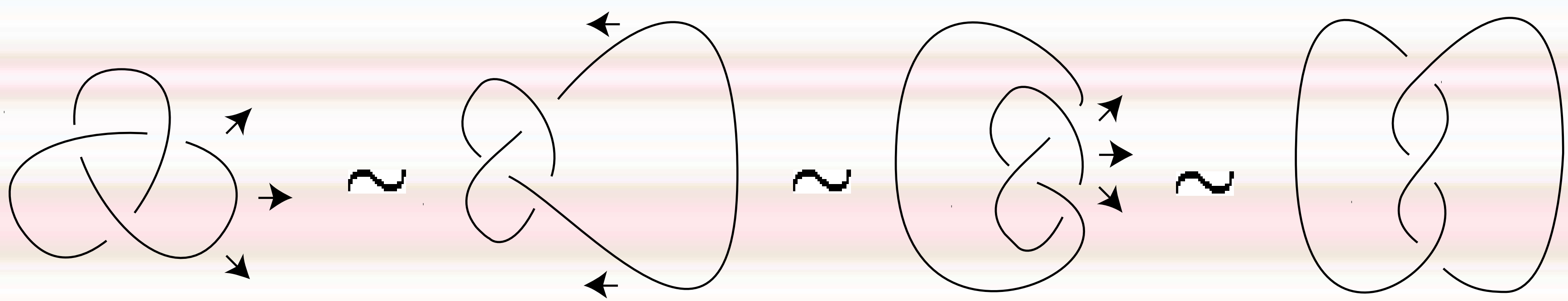
1. Knots

$K \sim K'$ $\Leftrightarrow$ We can transform K to K' continuously

<example>



Can you see?



Interests

1. Equivalence Problem

"Judge whether two given knots are the same or not!"

2. Classification Problem

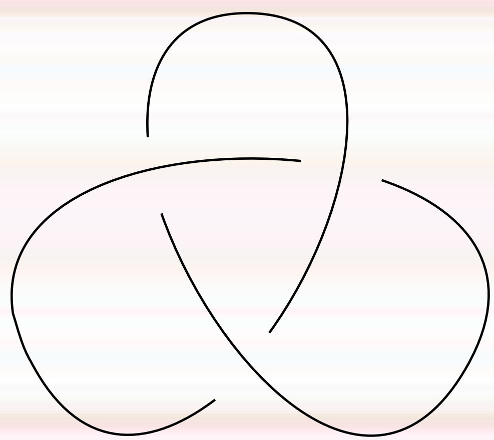
"Make a table of knots!"

3. Properties of knots

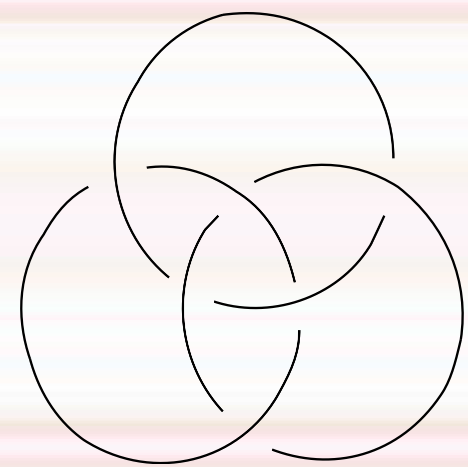
"How is a knot knotted?"

2． Invariants of Knots

Are these links are the same?

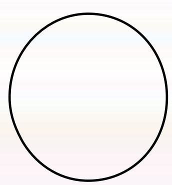
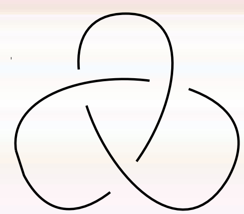
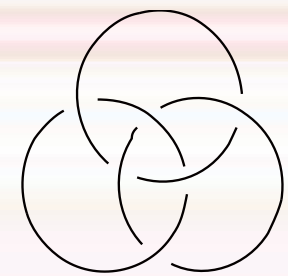


$\overset{?}{\simeq}$



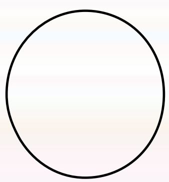
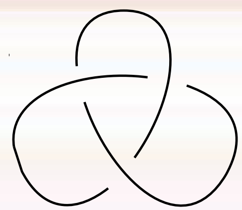
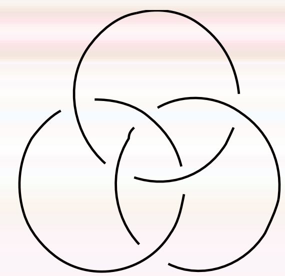
Number of components･･･Invariant under transformation！

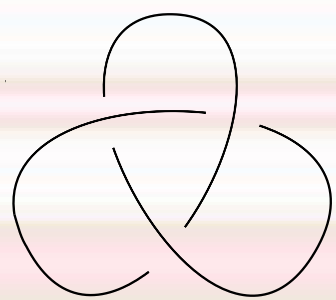
- Number of components　s : {links} → {1, 2, 3, 4, , , }

s()=1,　s()=1,　s()=3

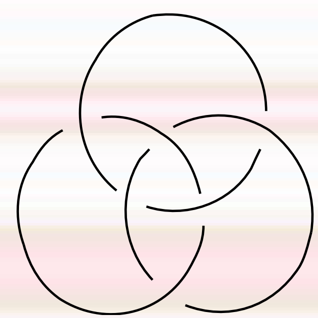
Number of components…Invariant under transformation !

- Number of components s : {links} → {1, 2, 3, 4, , , }

s()=1, s()=1, s()=3



$\not\sim$



2. Invariants

An **invariant** is a map $f: \{\text{links}\} \to G$ (Any set) such that

$$L \sim L' \;\Rightarrow\; f(L)=f(L').$$

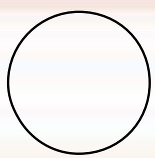
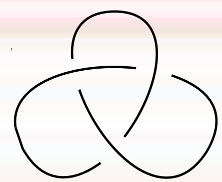
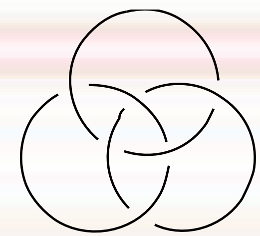
An **invariant** is a map f: {links} → G (Any set) such that

$$L \sim L' \Rightarrow f(L)=f(L').$$

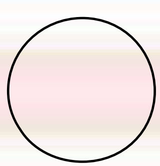
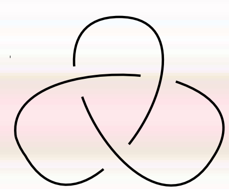
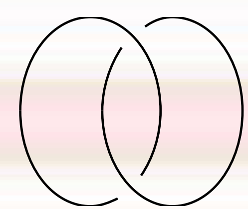
A language representing "How knots are knotted"

⟨examples⟩

- Number of components s: {links} → {1, 2, 3, 4, , , }

s()=1, s()=1, s()=3

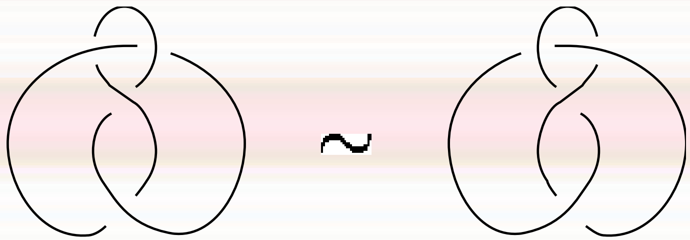
- Minimal crossing numbers m: {links} → {0, 1, 2, 3, , ,}

m()=0, m()=3, m()=2

3 . Background and motivation for Q. topology

3. Background and Motivation

- 1849, Notes by J.B.Listing (Lord Kelvin's Vortex Atom Hypothesis ?)

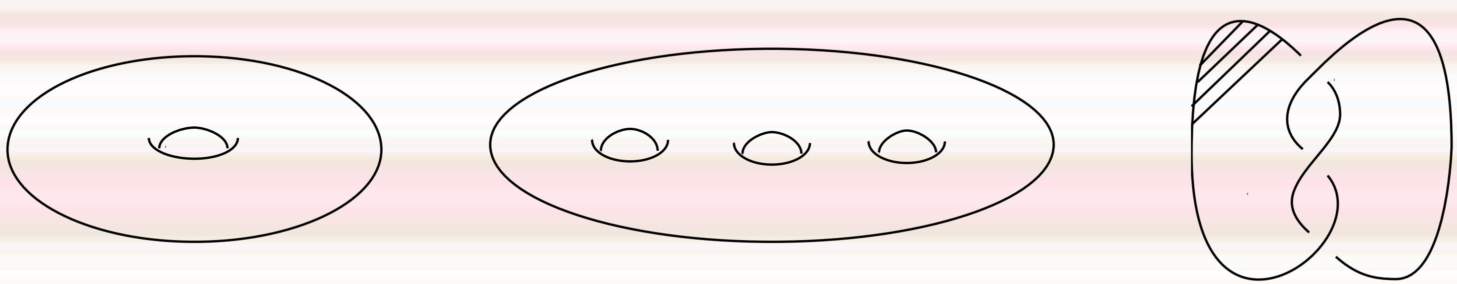


- 1930's, K. Reidemeister, H. Seifert, J. W. Alexander
- 1940's～1970's, R. H. Fox, T. Honma, S. Kinoshita, K. Sugimoto, A. Kawauchi... (foundations)
- 1980's, V. F. R. Jones, T. Ohtsuki, K. Habiro, J. Murakami,... (Developing Involving Quantum mechanics, Statistic, mathematical physics, etc...)

Background

<u>Classical invariants</u>

- Simple invariants
  〈ex〉 Number of components
- Invariants described in topological language
  〈ex〉 Fundamental group, Homology, Surfaces,...



Background

<u>Quantum invariants</u>

Turning point!

In 1984・・・ The discovery of the Jones polynomial

↓ A lot of invariants

" Quantum invariants "
(Quantum groups + representations)

which are described in algebraic language, such as representation theory, mathematical physics, ...

Motivation

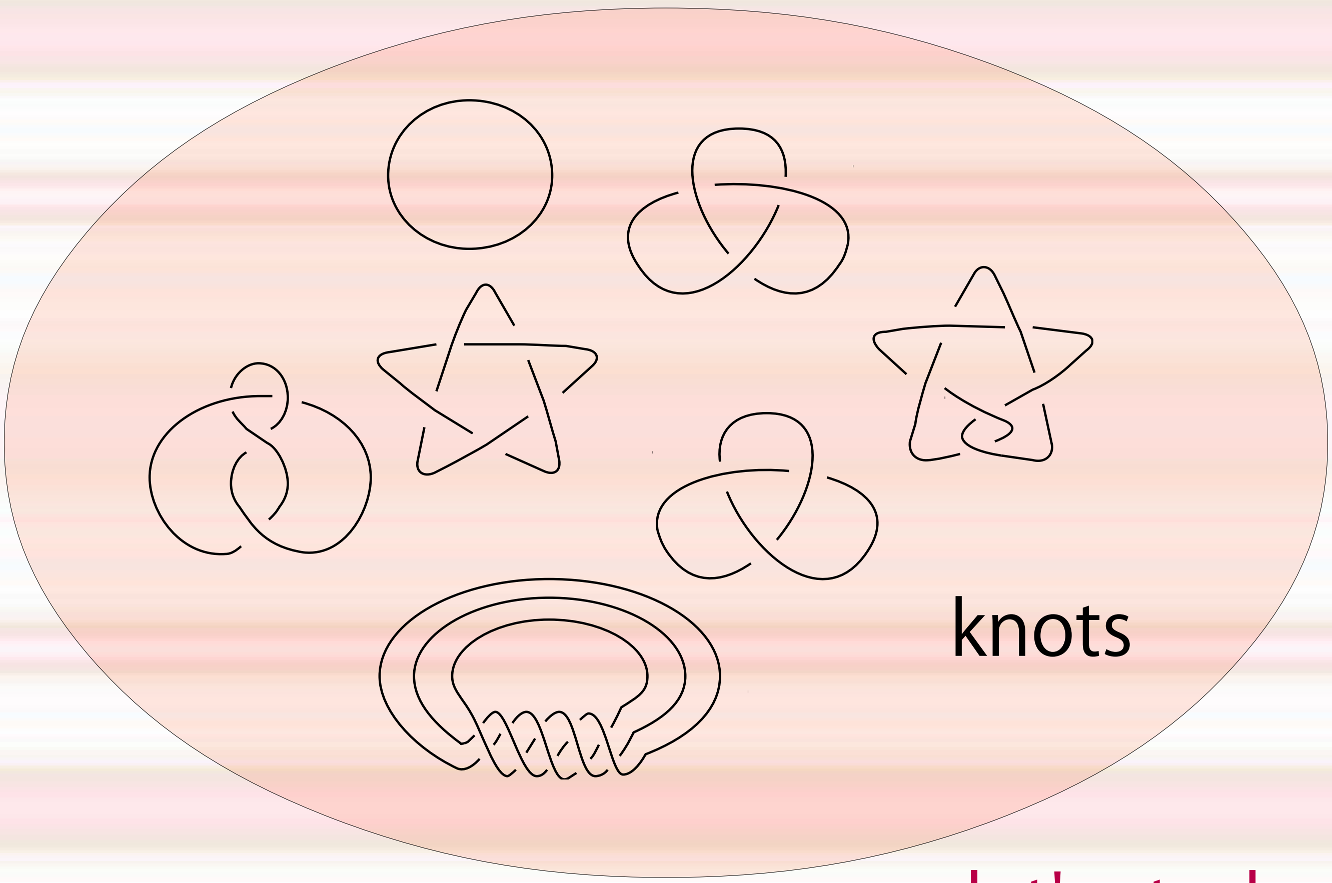
- ～1980's
  - — Equivalence Problem
  - — Classification Problem
  - — Properties of knots
- 1980's～ （After the discovery of Quantum invariants）
  - — Structure of the set of knots (sociology!)
  - — Application to low dimensional topology and others (Rep. theory, Category theory, Math. Physics,...)

Motivation

- ～1980's
  - — Equivalence Problem
  - — Classification Problem
  - — Properties of knots
- 1980's～ （After the discovery of Quantum invariants）
  - — Structure of the set of knots (sociology!)
  - — Application to low dimensional topology and others (Rep. theory, Category theory, Math. Physics,...)

4. Jones polynomial

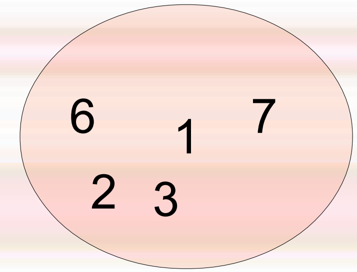
Q：What properties does the set of knots has?



let's study sociology!

4． Jones polynomial

<ex> Natural numbers N= { 1, 2, 3 , , , }



product：2×3 = 6, 4×7=28, , , ,

An operation : input two numbers (a,b)
⇒ output a number a×b

(2, 3)→ 6=2×3, (4,7)→28=4×7,

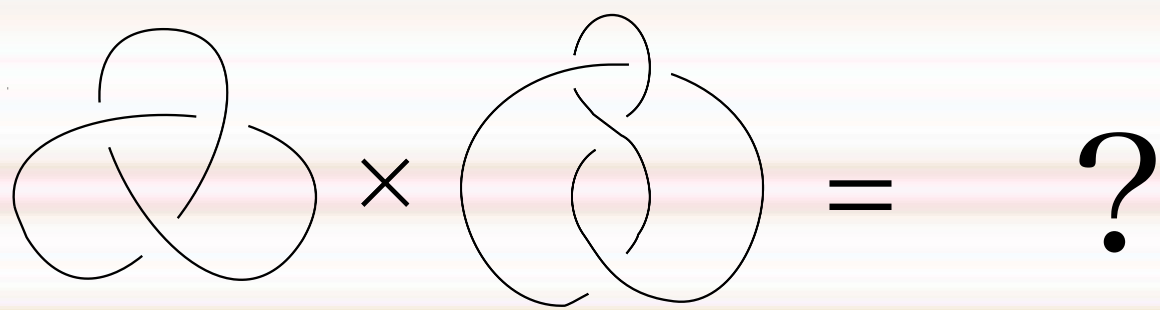
satisfying

1． Associativity：(a×b)×c=a×(b×c)

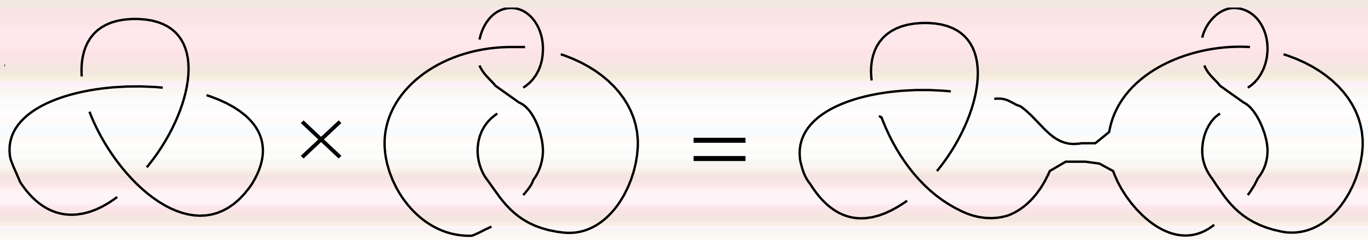
2． Unitarity：1×n=n=n×1

Define a product of knots!

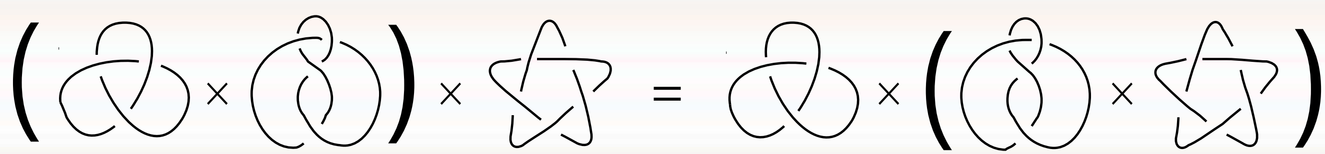
Product : input two numbers (a,b)
$\Rightarrow$ output a number $a \times b$
+ Associativity, Unitarity ★



<An example>

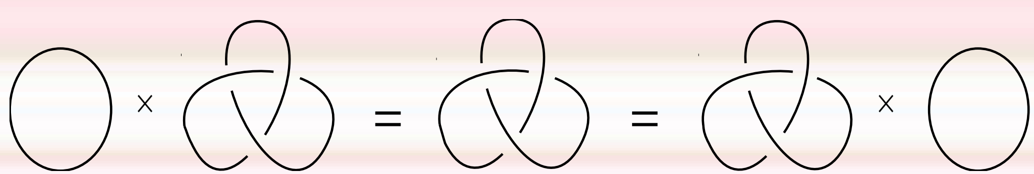


Associativity :



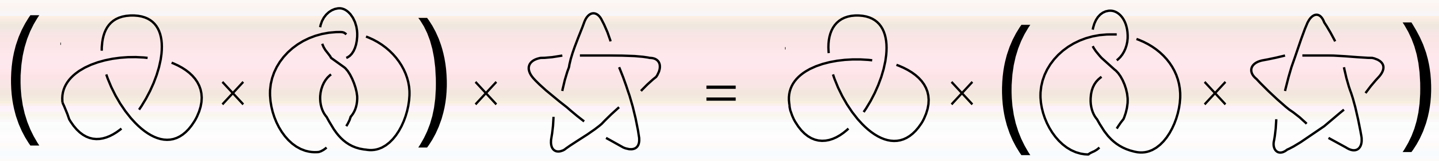
?

Unitarity :

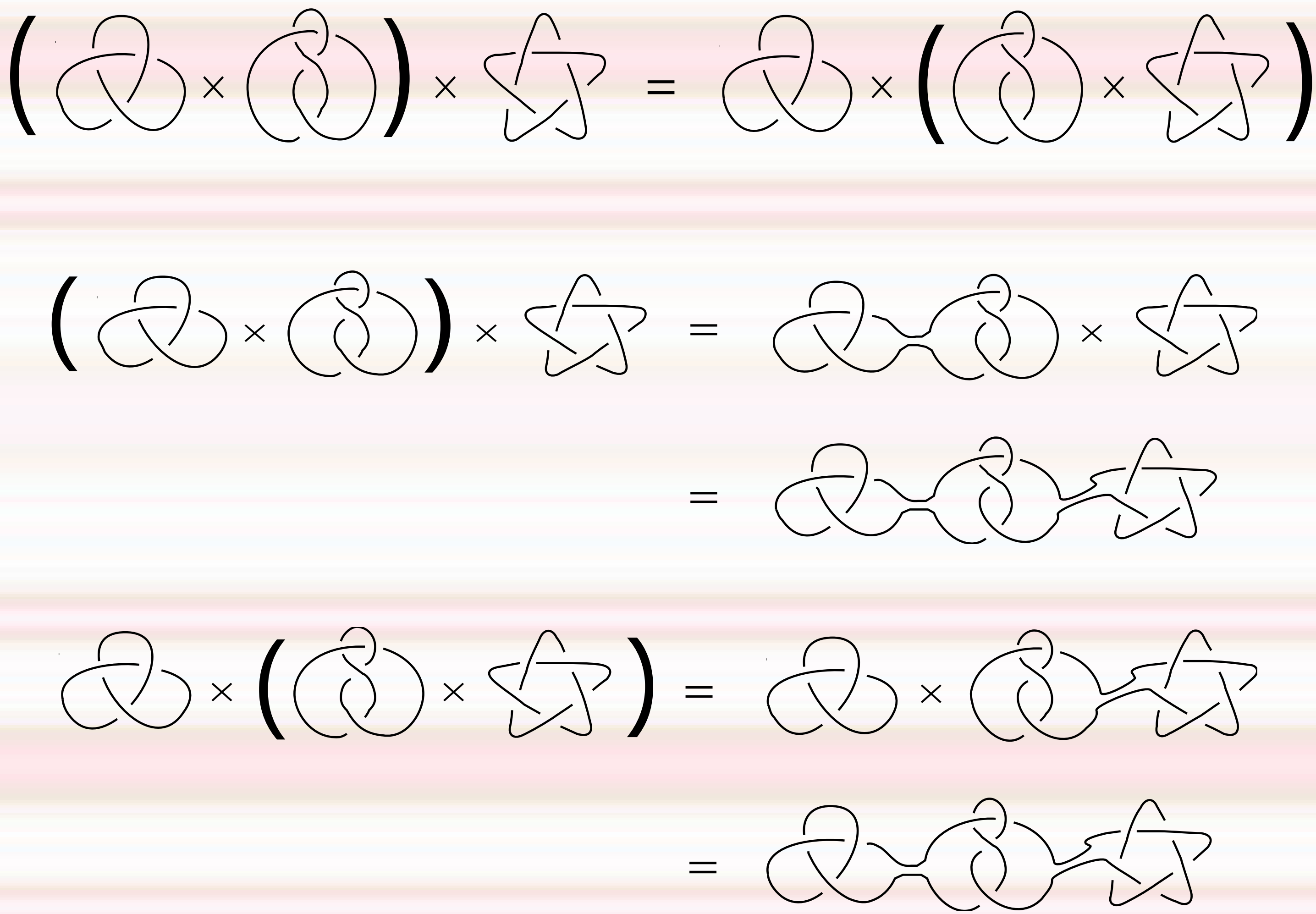


?

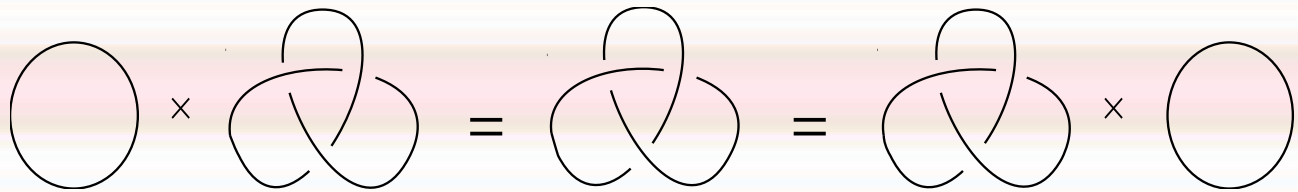
Associativity :



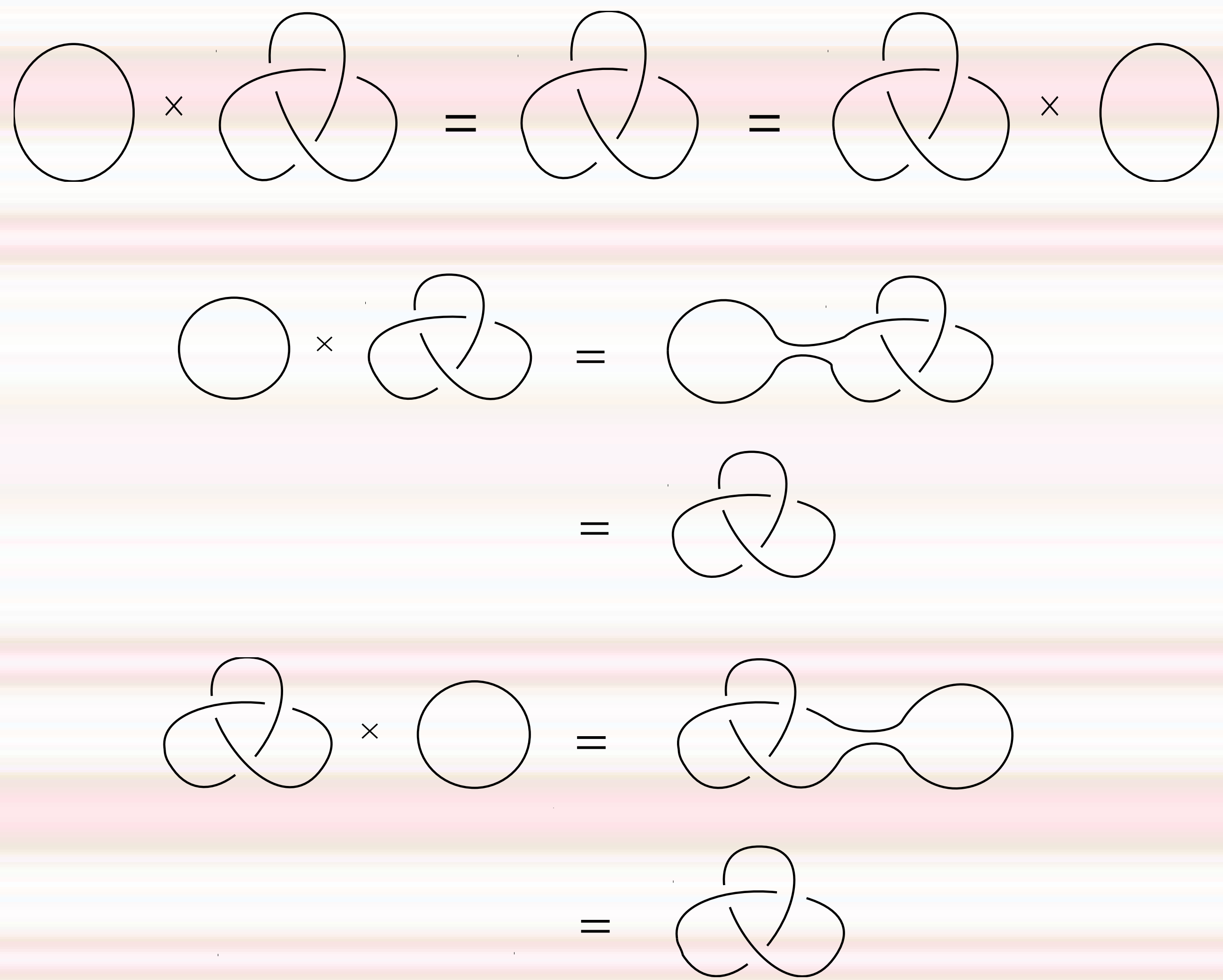
Associativity :



Unitarity：



Unitarity：

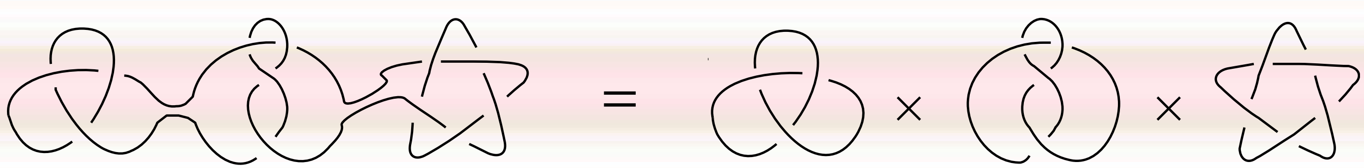


We obtained a product of knots!

unique-prime-factorization theorem

prime : factors ⇒ 1 and itself
prime-factorization : 6=2×3, 8=2×2×2, unique

prime knot : factors ⇒ ○ and itself
prime-factorization : unique



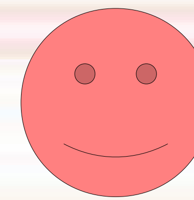
Break

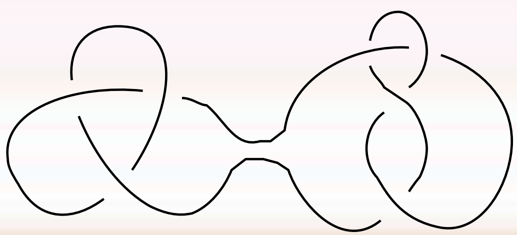
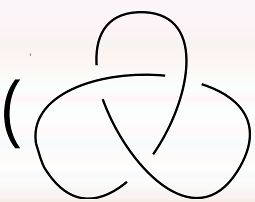
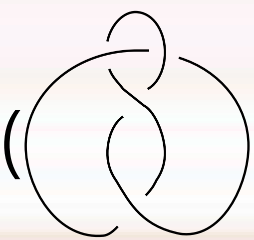
Let's compute the Jones polynomial!

$$V: \{\text{knots}\} \to Z[A, A^{-1}]$$

4. Jones polynomial

The Jones polynomial keeps the products!



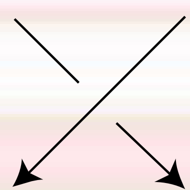
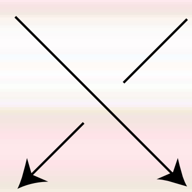
V () = V () V ()

- detect a prime knot
- study the structure of products

Jones polynomial $V(K) \in Z[A, A^{-1}]$

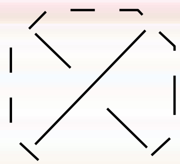
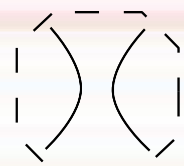
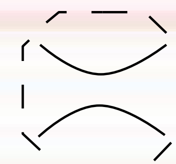
Step 1． Kauffman bracket $\langle D \rangle \in Z[A, A^{-1}]$

Step 2． $V(K) = (-A^3)^{-w(D)} \langle D \rangle$

w(D)= #  — # 

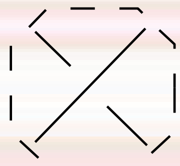
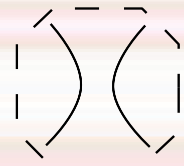
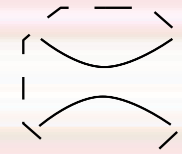
4． Jones polynomial

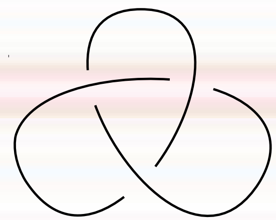
1． Kauffman bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$

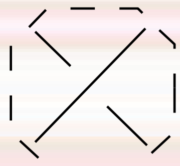
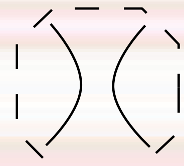
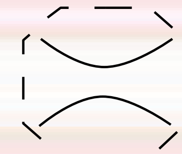
Rule 1 ： $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

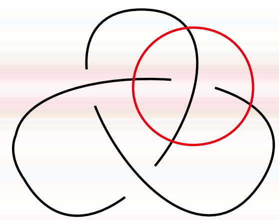
Rule 2 ： $\langle D \bigcirc \rangle = -(A^2 + A^{-2}) \langle D \rangle$

Rule 3 ： $\langle \quad \rangle = 1$

Rule 1 ： ⟨  ⟩ ＝ A ⟨  ⟩ $+A^{-1}$ ⟨  ⟩

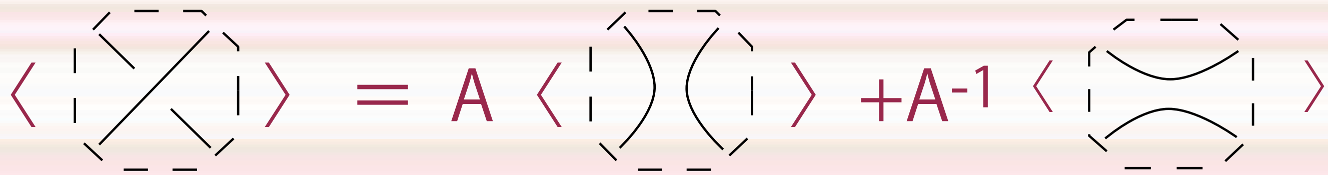


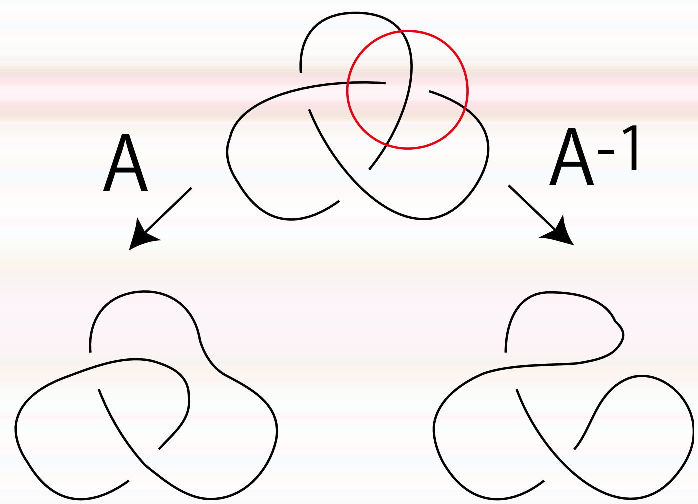
Rule 1 : ⟨  ⟩ = A ⟨  ⟩ + A^{-1} ⟨  ⟩



4． Jones polynomial

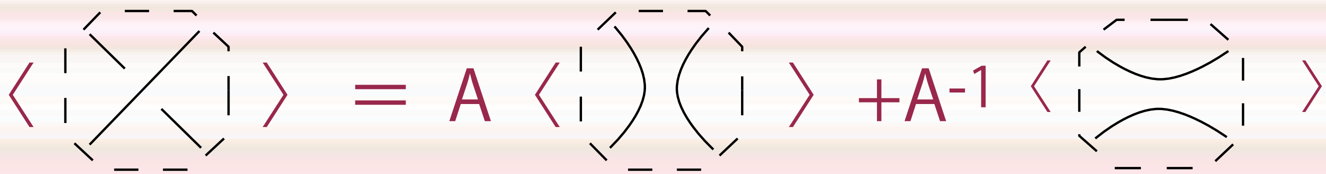
ルール１：

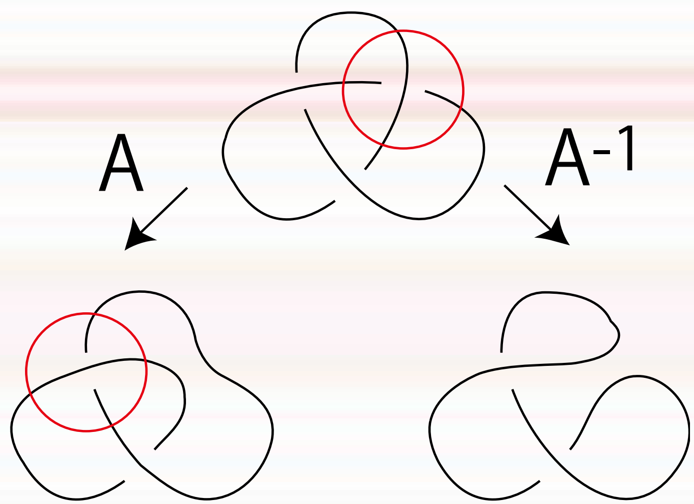




4． Jones polynomial

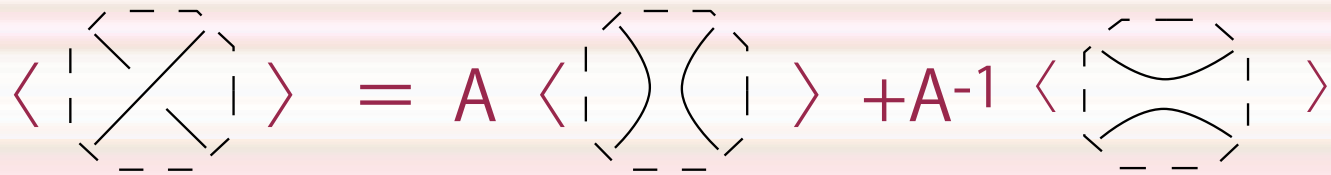
Rule 1 :

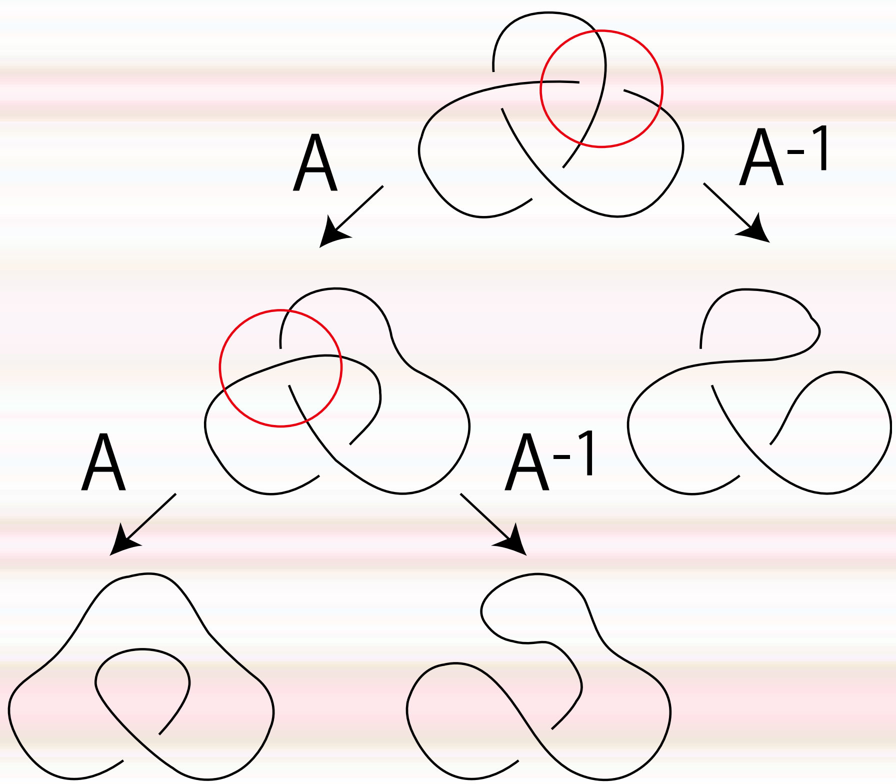




4． Jones polynomial

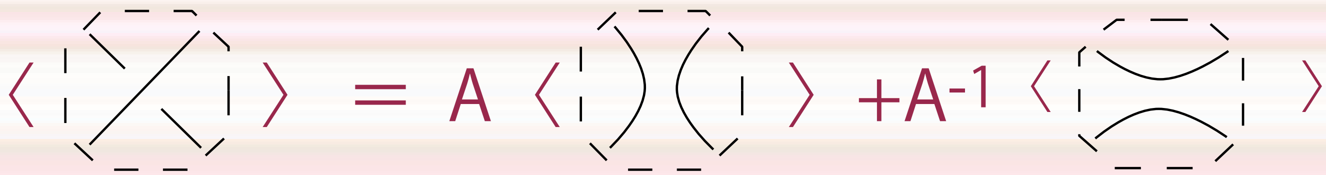
Rule 1 :

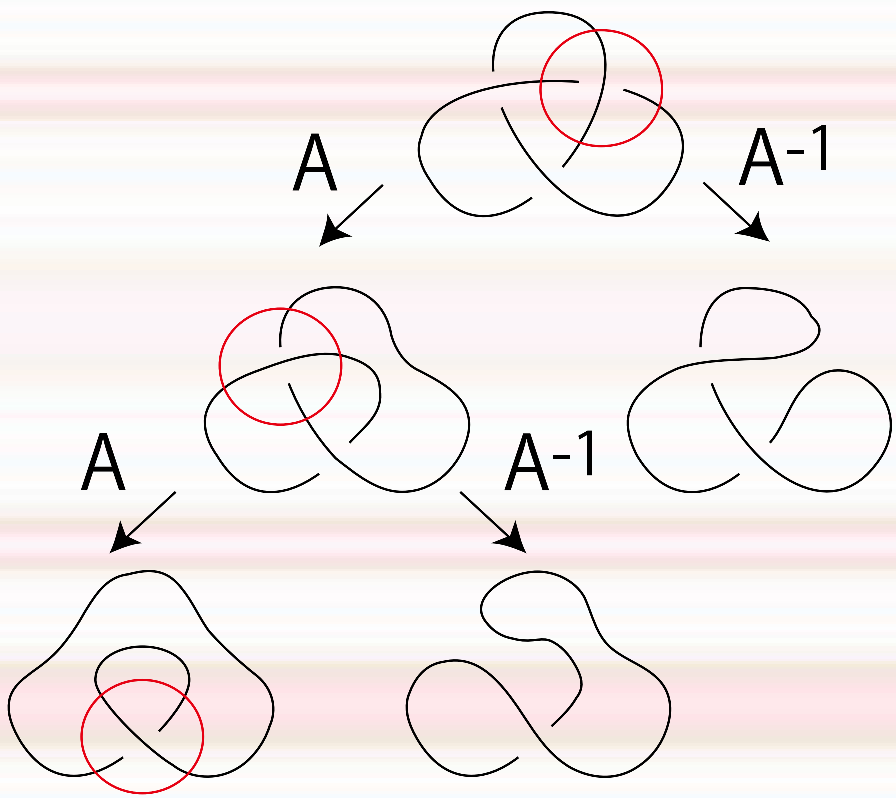




4． Jones polynomial

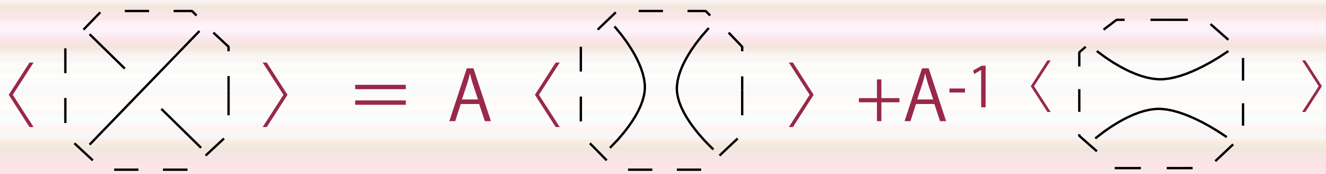
Rule 1 :

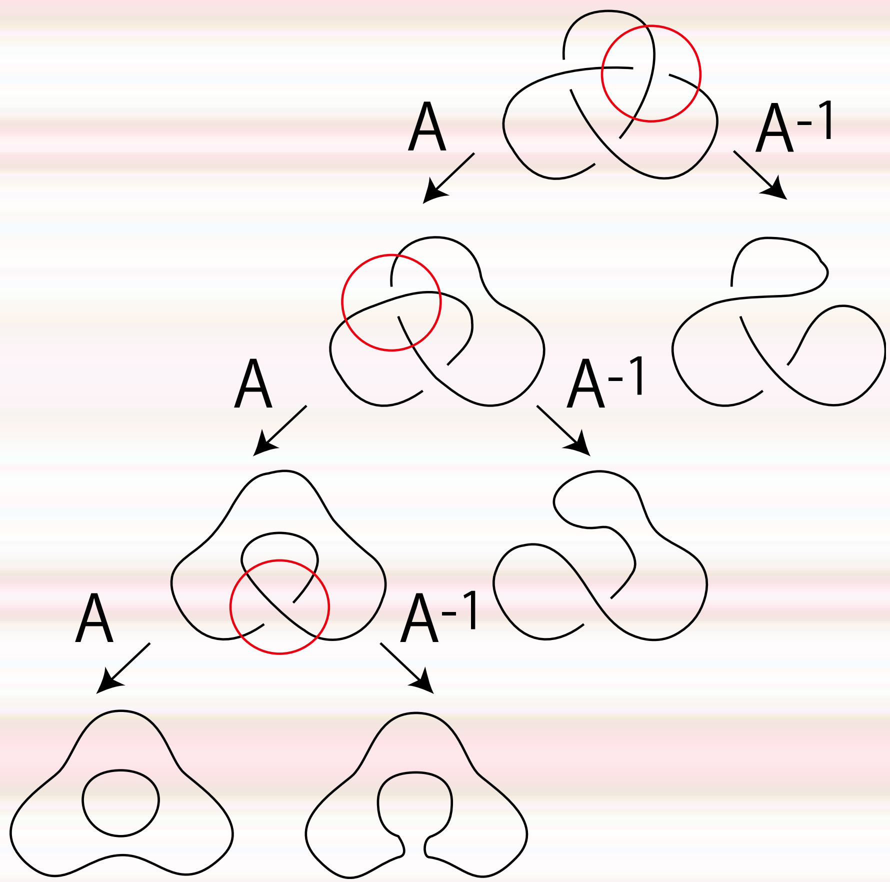




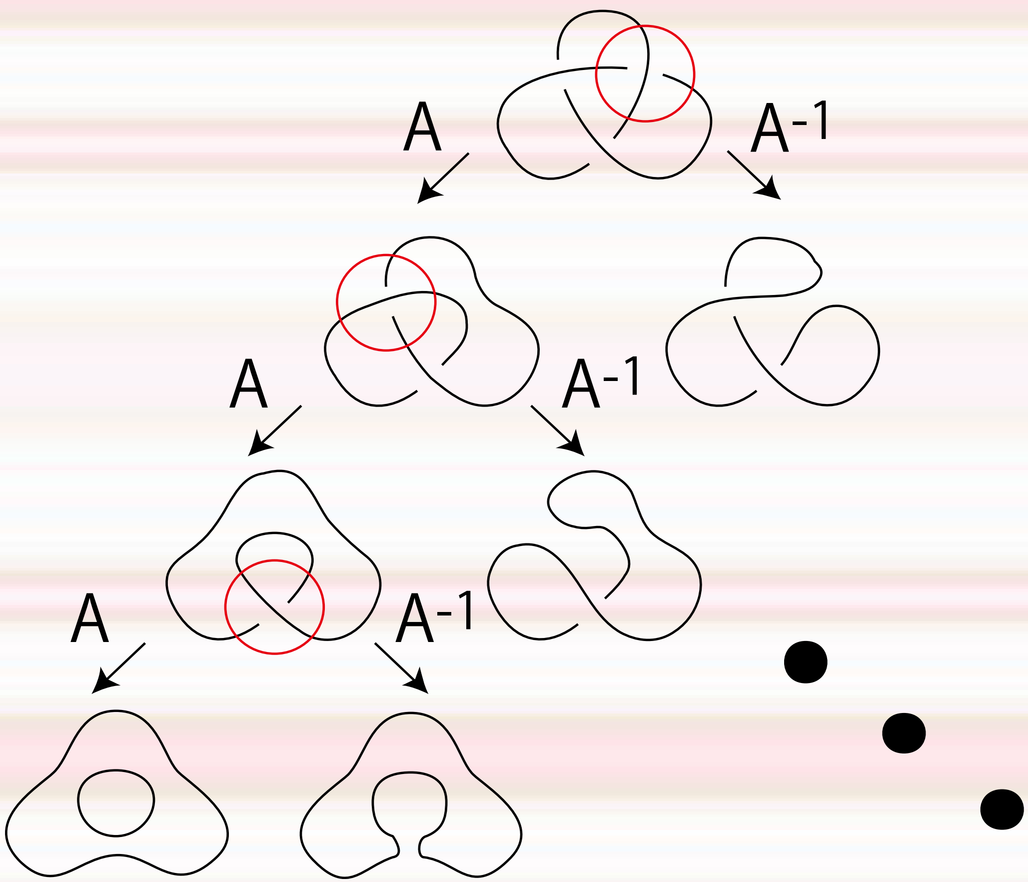
4． Jones polynomial

Rule 1 :





4． Jones polynomial



4. Jones polynomial

Rule 2 : $\langle D \bigcirc \rangle = -(A^{2}+A^{-2})\langle D \rangle$

$$\langle \bigcirc \rangle = 1$$

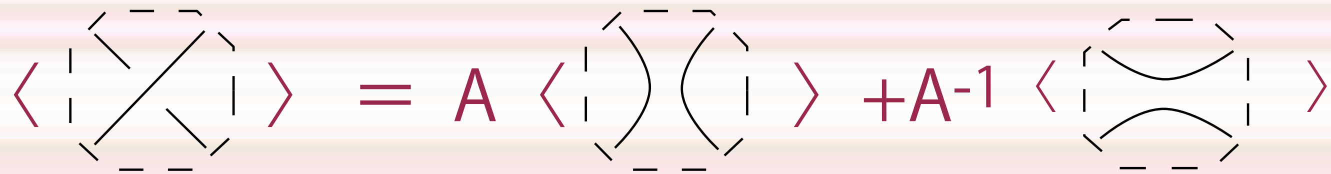
$$\langle \bigcirc \bigcirc \rangle = -(A^{2}+A^{-2})\langle \bigcirc \rangle = -(A^{2}+A^{-2})$$

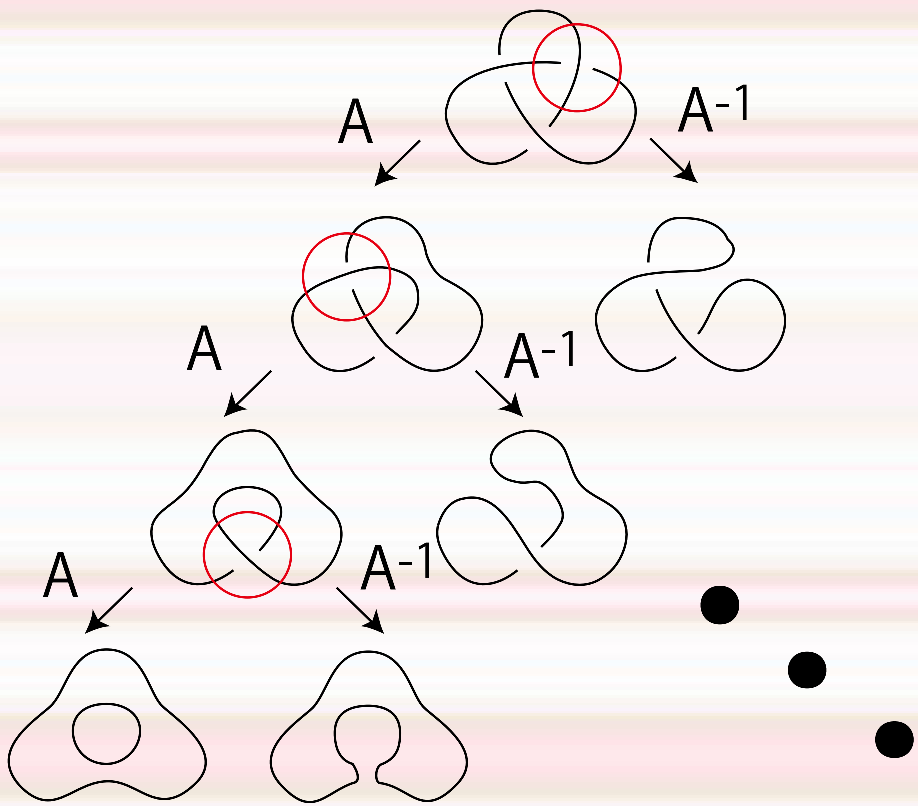
$$\langle \bigcirc \bigcirc \bigcirc \rangle = -(A^{2}+A^{-2})\langle \bigcirc \bigcirc \rangle = (A^{2}+A^{-2})^{2}$$

$$\langle \underbrace{\bigcirc \bigcirc \cdots \bigcirc}_{n} \rangle = (-1)^{n-1}(A^{2}+A^{-2})^{n-1}$$

4． Jones polynomial

Rule 1 ：



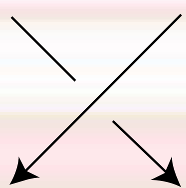
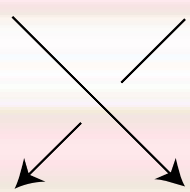


Rule 2 ： $-A^3(A^2+A^{-2}) + A + \cdots$

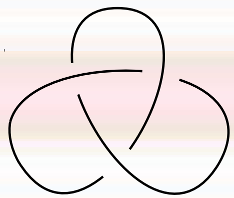
Jones polynomial $V(K) \in Z[A, A^{-1}]$

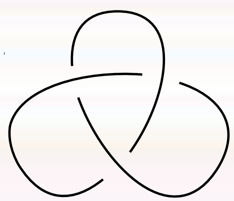
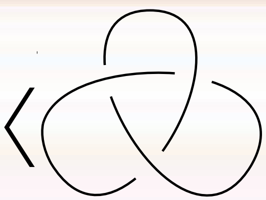
Step 1． Kauffman bracket $\langle D \rangle \in Z[A, A^{-1}]$

Step 2． $V(K) = (-A^{3})^{-w(D)} \langle D \rangle$

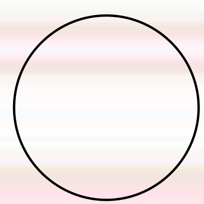
w(D)= #  — # 

4. Jones polynomial

$w($  $) = 3$

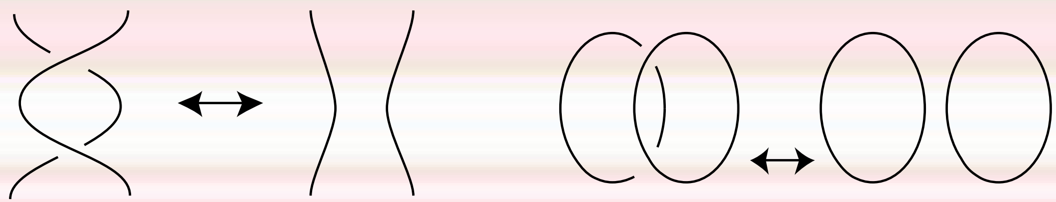
$V($  $) = (-A)^{-3} \langle$  $\rangle$

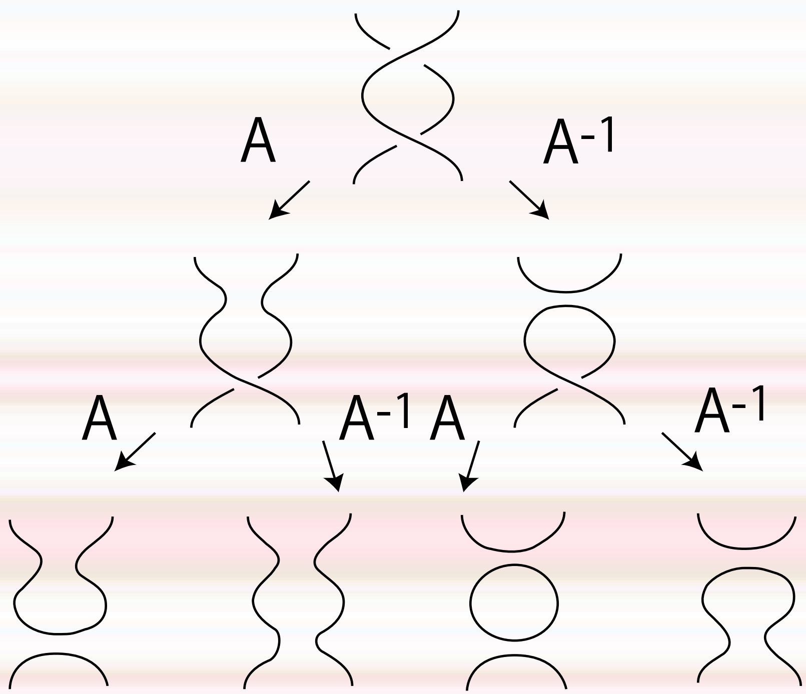
$= (-A)^{-3}(-A^{5} - A^{-3} + A^{-7})$

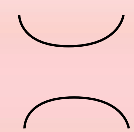
$\neq V($  $)$

Jones polynomial is invariant ?

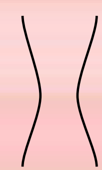
For instance…





Coefficient  =

$A^2 - (A^2+A^{-2}) - A^{-2}=0$

Coefficient  = 1

5． My research

Quantum invariants

Kontsevich invariant $Z(L) \in A(l)$

Hopf algebra structure

Complete invariant ?

Universal sl_2 invariant $J(L) \in U_h(sl_2)^{\otimes l}$

Hopf algebra structure

Studies of quantum groups

Colored Jones polynomial $V_n(L) \in Z[q^{1/4}, q^{-1/4}]$

(including Jones polynomial $V=V_2$)

Thank you