The universal sl_2 invariant and Milnor's invariants

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Low dimensional topology and number theory VI
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Introduction

Jacobi diagrams

Milnor's invariant

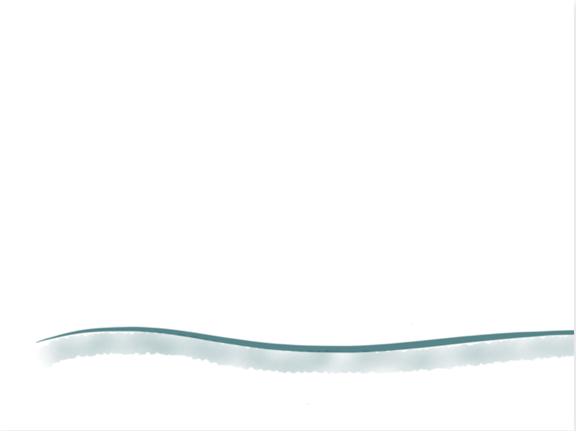
Universal sl_2 invariant

Results

Introduction

- Background
- Result

Background



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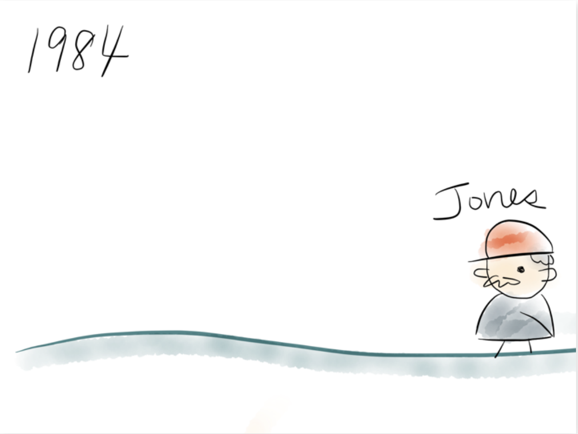

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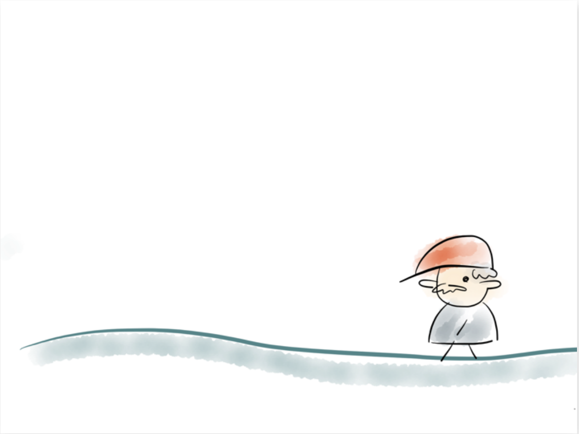

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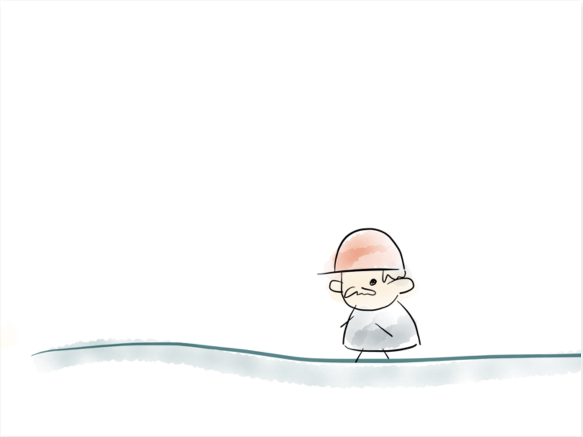

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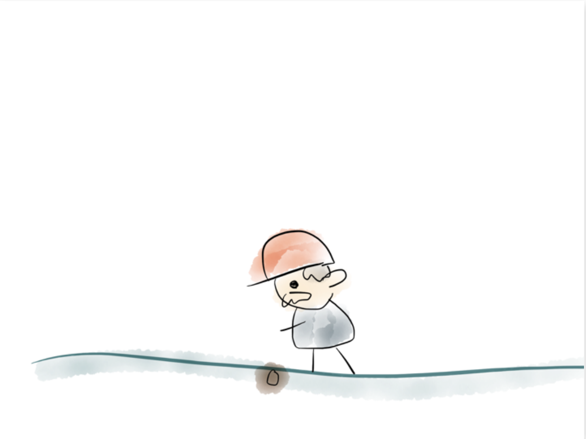

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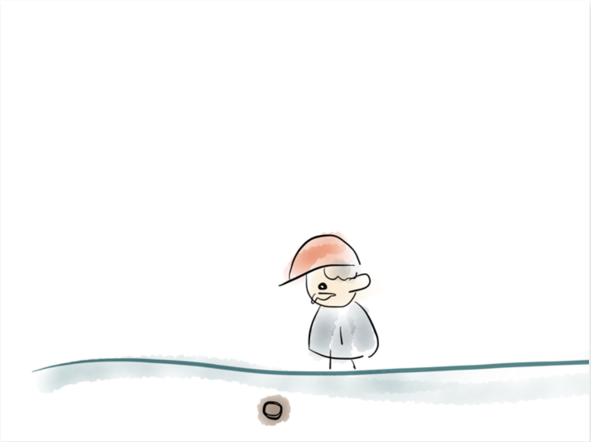

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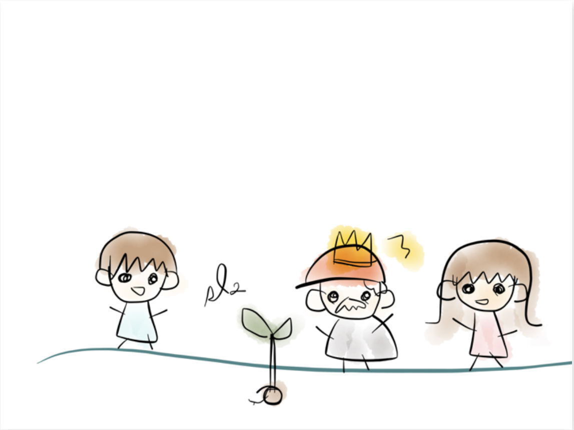

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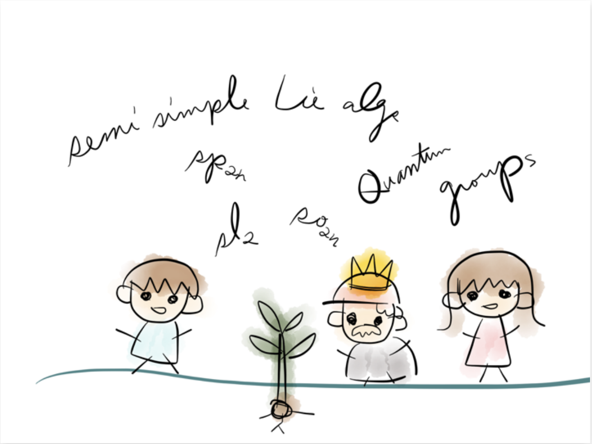

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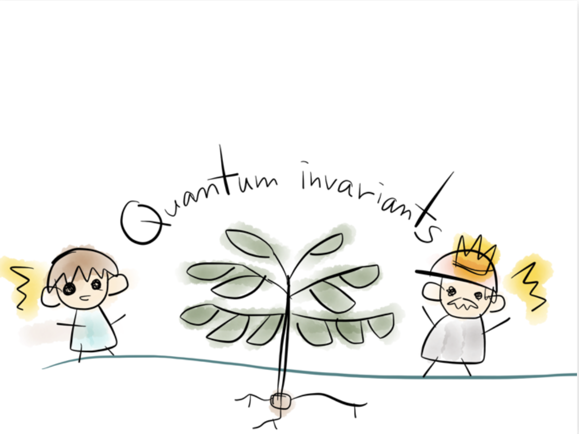

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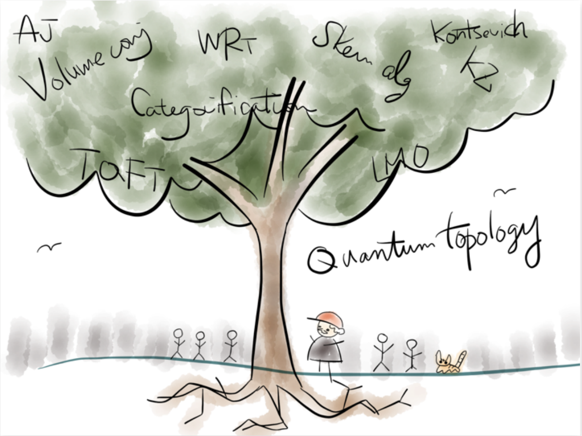
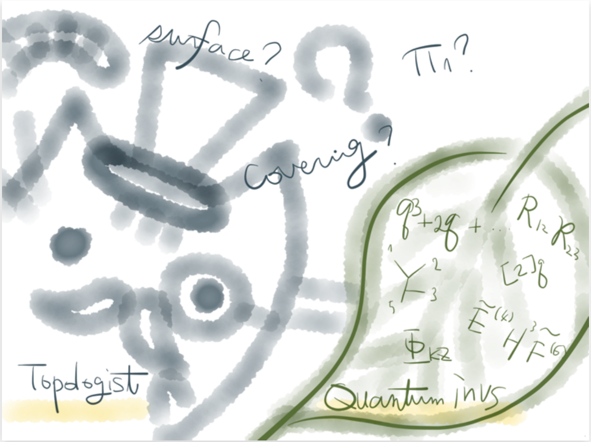
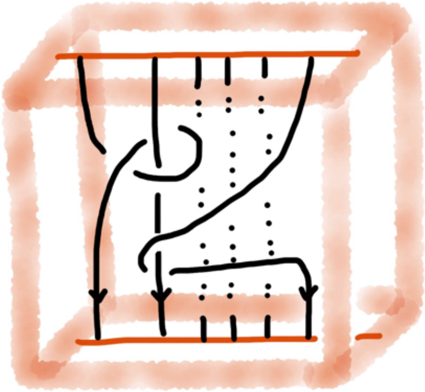

AJ
Volume conj
WRT
Skein alg
Kontsevich
KZ
Categorification
TQFT
LMO
Quantum topology

Background


surface?
π_1?
covering?
Topologist
Quantum invs
$q^3+2q+\dots$
$R_{12}R_{23}$
$[2]_q$
Φ_{KZ}

Background

String links

$\bigcup_{i=1}^{l}[0,1]_i \hookrightarrow$ 

$SL(l) := \{l\text{-component string links }\}/\sim$

Quantum invariants for $T \in SL(l)$

Kontsevich inv.	Z_T	$\in$	$\hat{\mathcal{A}}(l)$
Universal sl_2 inv.	J_T	$\in$	$U_\hbar(sl_2)^{\hat{\otimes} l}$
Colored Jones poly.	$J_{\mathrm{cl}(T)}^{(V_1,\ldots,V_l)}$	$\in$	$\mathbb{Z}[q^{1/4}, q^{-1/4}]$

Quantum invariants for $T \in SL(l)$

$$Z_T \in \hat{\mathcal{A}}(l)$$

$$\hat{\mathcal{A}}(l) \xrightarrow{W} U(sl_2)^{\otimes l}[[\hbar]]$$

$$J_T \in U_\hbar(sl_2)^{\hat{\otimes} l} \simeq U(sl_2)^{\otimes l}[[\hbar]]$$

$$U_\hbar(sl_2)^{\hat{\otimes} l} \xrightarrow{\mathrm{tr}_q^{\otimes l}} \mathbb{Z}[q^{1/4}, q^{-1/4}] \qquad U(sl_2)^{\otimes l}[[\hbar]] \xrightarrow{\mathrm{tr}_\nu^{\otimes l}} \mathbb{Z}[q^{1/4}, q^{-1/4}]$$

$$J_{\mathrm{cl}(T)}^{(V_1,\ldots,V_l)} \in \mathbb{Z}[q^{1/4}, q^{-1/4}]$$

Result

[Habegger-Masbaum, 2000]

$$Z_T^t = 1 + \mu_m(T) + (\textit{higher})$$

Theorem (JB-S)

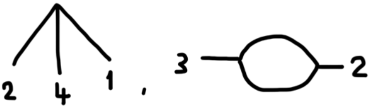
$$J_T^t = W(\mu_m(T)) + (\textit{higher})$$

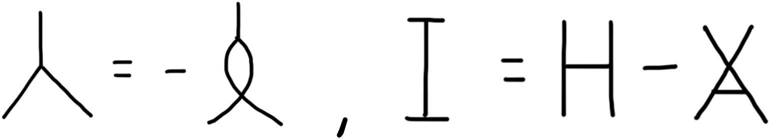
Note: (i) These results are essentially independent.
(ii) W is not injective for $m \geq 6$.

Jacobi diagrams

- The space $\mathcal{B}(l)$ of labeled Jacobi diagrams
- Subspaces of $\mathcal{B}(l)$

The space $\mathcal{B}(l)$ of labeled Jacobi diagrams

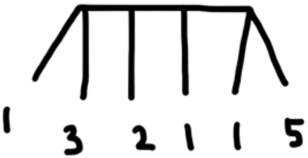
$\mathcal{B}(l) = \langle$  $\dots \rangle_{\mathbb{Q}}/$ AS, IHX

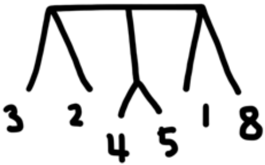


AS IHX

Subspaces of $\mathcal{B}(l)$

$$\mathcal{C}^t(l) = \langle \text{ simply connected, connected diagrams } \rangle_{\mathbb{Q}}$$
$$\cup$$
$$\mathcal{C}^h(l) = \langle \text{ non-repeated labeled diagrams } \rangle_{\mathbb{Q}}$$

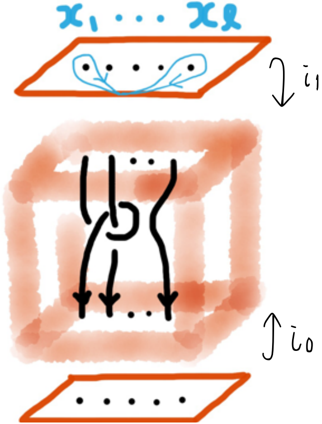




Milnor's invariant

- Artin representation
- Milnor numbers
- Milnor map

Artin representation



$$F_l/(F_l)_{n+1}$$

$$\downarrow \; i_1^*$$

$$\pi_1([0,1]^3 \setminus T)/(\pi_1)_{n+1}$$

$$\uparrow \; i_0^*$$

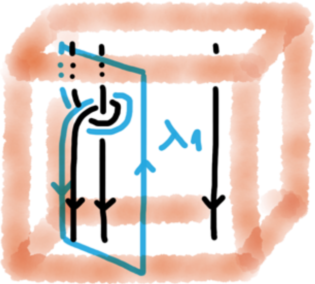
$$F_l/(F_l)_{n+1}$$

$$\Rightarrow \quad \mathcal{A}_n : \; SL(l) \to Aut\big(F_l/(F_l)_{n+1}\big)$$

Milnor numbers

λ_i: the longitude of the ith component.

$$\Rightarrow \mathcal{A}_n(T)(x_i) = \lambda_i x_i \lambda_i^{-1}$$



Magnus Expansion:

$$\mu : F_l \to \mathbb{Z}[[X_1, \ldots, X_l]]$$

$$x_i \mapsto 1 + X_i$$

$$\lambda_i \mapsto \sum \mu_{i_1,\ldots,i_p;i}(T) X_{i_1} \cdots X_{i_p}$$

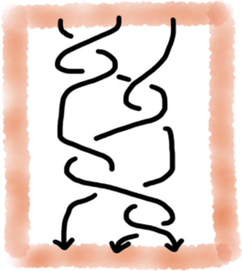
$$T \in SL_m(l) \quad \Leftrightarrow \quad \forall \mu_{i_1,\ldots,i_p;i}(T) = 0, \ \forall p < m$$

Milnor map for $T \in SL_m(l)$

$$\mu_m(T) \in \mathrm{Ker}\{[-,-]\colon \mathrm{Lie}_1(l) \otimes \mathrm{Lie}_m(l) \to \mathrm{Lie}_{m+1}\}$$

Let $\bar{\lambda}_i \in (F_l)_m/(F_l)_{m+1}$

Set $\mu_m(T) = \sum_{i=1}^{l} X_i \otimes \bar{\lambda}_i$

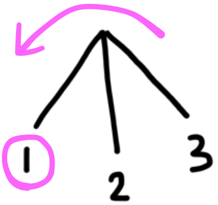


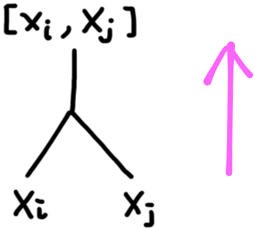
$$\mu_2(T) = X_1 \otimes [X_2, X_3] + X_2 \otimes [X_3, X_1] + X_3 \otimes [X_1, X_2]$$

Milnor map and Jacobi diagrams

$$\mu_m(T) \in \mathrm{Ker}\{[-,-]: \ \mathrm{Lie}_1(l) \otimes \mathrm{Lie}_m(l) \to \mathrm{Lie}_{m+1}\}$$

$$\sim \mathcal{C}_m^t(l)$$





$$X_1 \otimes [X_2, X_3] + X_2 \otimes [X_3, X_1] + X_3 \otimes [X_1, X_2]$$

Universal sl_2 invariant

- The quantized enveloping algebra $U_\hbar(sl_2)$
- Universal sl_2 invariant
- Universal sl_2 weight system

The quantized enveloping algebra $U_\hbar(sl_2)$

: the $\hbar$-adically complete $\mathbb{Q}[[\hbar]]$-algebra

- generaters: $H, E,\ F$
- relations:

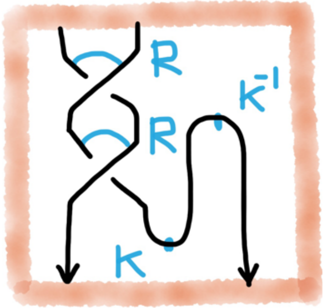
$$HE - EH = 2E, \quad HF - FH = -2F,$$

$$EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where $\quad q = \exp \hbar, \quad K = q^{H/2} = \exp \frac{\hbar H}{2}.$

Universal sl_2 invariant

$$R = q^{\frac{H \otimes H}{4}\hbar}\left(\sum_{n\geq 0} q^{\frac{1}{2}n(n-1)}\frac{(q-1)^n}{[n]_q!}F^n \otimes E^n\right)$$

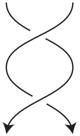


(1) Choose a nice diagram

(2) Put labels

(3) Read the labels

Universal sl_2 invariant

J ()

$$= q^{\frac{H\otimes H}{2}\hbar} \sum_{m,n\geq 0} q^{\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)+m^2} \frac{(q-1)^{m+n}}{[m]_q![n]_q!} F^m K^{-m} E^n \otimes E^m K^m F^n$$

$$= 1 + (\frac{1}{2}H\otimes H + F\otimes E + E\otimes F)\hbar + (\hbar^2)$$

Universal sl_2 invariant

Set $c = \frac{1}{2}H \otimes H + F \otimes E + E \otimes F$.

Proposition

For $T \in SL(l)$ with $Lk(T) = (m_{ij})_{1 \leq i,j \leq l}$, we have

$$J_T = 1 + \Big(\sum_{1 \leq i < j \leq l} m_{ij} c_{ij}^{(l)} + \frac{1}{2} \sum_{1 \leq i \leq l} m_{ii} c_{ii}^{(l)} \Big) \hbar + (\hbar^2).$$

We generalize this result, using Milnor's invariants.

Universal sl_2 weight system $W\colon \mathcal{B}(l) \to S^{\otimes l}[[\hbar]]$

$U(sl_2)$: The universal enveloping algebra of sl_2
$S(sl_2)$: The symmetric algebra of sl_2

$$\begin{array}{ccccc} \mathcal{A}(l) & \xrightarrow{W} & U(sl_2)^{\otimes l}[[\hbar]] & & \\ \uparrow \chi & & \uparrow \beta & & \\ \mathcal{B}(l) & \xrightarrow{W} & S(sl_2)^{\otimes l}[[\hbar]] & \sim_{\mathbb{Q}[[\hbar]]} & U_\hbar(sl_2)^{\hat{\otimes} l} \end{array}$$

$$\sum f^i h^j e^k \otimes \cdots \quad \mapsto \quad \sum F^i H^j E^k \otimes \cdots$$

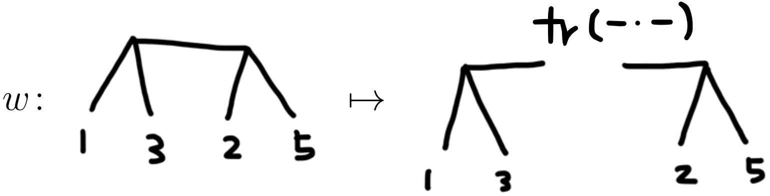
Universal sl_2 weight system $W\colon \mathcal{B}(l) \to S^{\otimes l}[[\hbar]]$

$$c = \tfrac{1}{2} H \otimes H + F \otimes E + E \otimes F \in sl_2^{\otimes 2}$$

$$b = \textstyle\sum_{\sigma \in \mathfrak{S}_3} (-1)^{|\sigma|} \sigma(H \otimes E \otimes F) \in sl_2^{\otimes 3}$$

$c_{i,j}$ $\qquad$ $b_{i,j,k} \quad \in S(sl_2)^{\otimes l}$

Universal sl_2 weight system $W\colon\ \mathcal{B}(l) \to S^{\otimes l}[[\hbar]]$



$$= \sum \mathrm{tr}(b_3 b_1') b_1 \otimes b_2' \otimes b_2 \otimes 1 \otimes b_3'$$

$$D \in \mathcal{B}_m(l) \quad \Rightarrow \quad W(D) = w(D)\hbar^m$$

Universal sl_2 weight system $W\colon \mathcal{B}(l) \to S^{\otimes l}[[\hbar]]$

Proposition

$W(\mathcal{C}^t_m(l)) \subset (S^{\otimes l})_{m+1}\hbar^m$ *and*
$W(\mathcal{C}^h_m(l)) \subset \langle sl_2\rangle^{(l)}_{m+1}\hbar^m$.

$S_n \subset S$: the $\mathbb{Q}$-subsp spanned by $a_1 \cdots a_n$, $a_i \in sl_2$

$$(S^{\otimes l})_n = \bigoplus_{n_1+\cdots+n_l=n} S_{n_1} \otimes \cdots \otimes S_{n_l}$$

$$\langle sl_2\rangle^{(l)}_n = \bigoplus_{\substack{n_1+\cdots+n_l=n \\ 0\le n_1,\ldots,n_l\le 1}} S_{n_1} \otimes \cdots \otimes S_{n_l}$$

Result

Set

$$J^t := p^t \circ J\colon\ SL(l) \to \prod_{m\geq 1} (S^{\otimes l})_{m+1} h^m,$$

where

$$p^t\colon\ U_h^{\hat{\otimes} l} \to \prod_{m\geq 1} (S^{\otimes l})_{m+1} h^m,$$

denotes the projection as $\mathbb{Q}$-modules.

Result

Theorem (JB-S)

For $T \in SL_m(l)$, we have

$$J_T^t = W(\mu_m(T)) + (\textit{higher}).$$

Result (homotopy version)

Set

$$J^h = p^h \circ J\colon\ SL(l) \to \bigoplus_{m=1}^{l-1} \langle sl_2 \rangle_{m+1}^{(l)} \hbar^m,$$

where

$$p^h\colon\ U_h^{\hat{\otimes} l} \to \bigoplus_{m=1}^{l-1} \langle sl_2 \rangle_{m+1}^{(l)} h^m$$

denotes the projection as $\mathbb{Q}$-modules.

Result (homotopy version)

Theorem (JB-S)

For $T \in SL^h_m(l)$, *we have*

$$J^h_T = W(\mu^h_m(T)) + (\textit{higher}).$$

$SL^h_m(l)$: the set of string links whose Milnor homotopy invariants of length $\leq m$ vanish.
$\mu^h_m(T)$: the non-repeated part of $\mu_m(T)$.

Proof

1. Prove the result for homotopy version.
2. Deduce the general case.

$$\begin{array}{ccc} SL_m(l) & \xrightarrow{W^s\circ\mu_{m+1}} & (S^{\otimes l})_{m+1}\hbar^m \\ \downarrow D^{(p)} & & \downarrow \pi\circ\Delta^{(p)} \\ SL_m(pl) & \xrightarrow{W^s\circ\mu^h_{m+1}} & \langle sl_2\rangle^{(pl)}_{m+1}\hbar^m, \end{array} \qquad \begin{array}{ccc} SL_m(l) & \xrightarrow{\pi\circ J^t} & (S^{\otimes l})_{m+1}\hbar^m \\ \downarrow D^{(p)} & & \downarrow \pi\circ\Delta^{(p)} \\ SL_m(pl) & \xrightarrow{\pi\circ J^h} & \langle sl_2\rangle^{(pl)}_{m+1}\hbar^m. \end{array}$$

$$(p \geq m)$$

Apprication

Theorem (JB-S)

For $T \in SL(l)$ with vanishing all Milnor invariants, we have

$$J_T \in \prod_{0 \le i \le j} (S^{\otimes l})_i \hbar^j.$$

Future

- Find an integrality property for Milnor's invariant at the level of the universal sl_2 invariant.
- Find the weight system for the universal sl_2 invariant.
- Make an algebraic isomorphism between $U_\hbar(sl_2)$ and $U(sl_2)[[\hbar]]$ which is compatible with the weight systems.

Quantum invariants for $T \in SL(l)$

$$
\begin{array}{ccccc}
Z_T & \in & \hat{\mathcal{A}}(l) & & \\
 & & & \searrow^{W} & \\
J_T & \in & U_\hbar(sl_2)^{\hat{\otimes} l} & \simeq & U(sl_2)^{\otimes l}[[\hbar]] \\
 & & \downarrow \mathrm{tr}_q^{\otimes l} & \swarrow_{\mathrm{tr}_\nu^{\otimes l}} & \\
J_{\mathrm{cl}(T)}^{(V_1,\ldots,V_l)} & \in & \mathbb{Z}[q^{1/4}, q^{-1/4}] & &
\end{array}
$$