The universal sl_2 weight system on the space of tree Jacobi diagrams

Sakie Suzuki
(Joint work with J. B. Meilhan)

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Introduction

Jacobi diagrams

Universal sl_2 weight system

Results

Introduction

- Quantum invariants
- Motivation to study the universal sl_2 weight system

Quantum invariants for links

Jones polynomial (1984, Jones)

$\downarrow$ R matrix with respect to $(U_\hbar(\mathfrak{g}), V)$

Quantum $(\mathfrak{g}, V)$ invariant

$\downarrow$ omit V

Universal $\mathfrak{g}$ invariant (1990-, Lawrence, Ohtsuki)

$\downarrow$ KZ-eq. (Kohno, Drinfeld) omit $\mathfrak{g}$

Kontsevich integral (1993, Kontsevich)

Quantum invariants

Classical invariants

Milnor invariants
Alexander invariant, ...

- Equivalence Probrem
- Classification Problem
- Property of knots

Quantum invariants

Quantum $(\mathfrak{g}, V)$ invariant
Universal $\mathfrak{g}$ invariant
Kontsevich integral

- Structure of the set of knots
  - Algebraic structures
  - Filtrations
  - Classification by weaker equivalence relations

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Quantum invariants

Quantum $(\mathfrak{g}, V)$ invariant
Universal $\mathfrak{g}$ invariant $(\mathfrak{g} = sl_2)$
Kontsevich integral

- Structure of the set of knots
  - Algebraic structures
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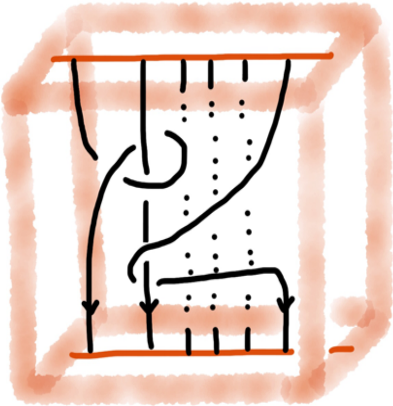
  - Classification by weaker equivalence relations

String links

$\bigcup_{i=1}^{l}[0,1]_i \hookrightarrow$ 

$$SL(l) := \{l\text{-component string links }\}/\sim$$

Quantum invariants for $T \in SL(l)$

Kontsevich inv.	Z_T	$\in$	$\hat{\mathcal{A}}(l)$
Universal sl_2 inv.	J_T	$\in$	$U_\hbar(sl_2)^{\hat{\otimes} l}$
Colored Jones poly.	$J_{\mathsf{cl}(T)}^{(V_1,\ldots,V_l)}$	$\in$	$\mathbb{Q}[[\hbar]]$

Quantum invariants for $T \in SL(l)$

$$Z_T \in \hat{\mathcal{A}}(l)$$

$$\hat{\mathcal{A}}(l) \xrightarrow{W^U} U_\hbar(sl_2)^{\hat{\otimes} l}, \qquad \hat{\mathcal{A}}(l) \xrightarrow{W} U(sl_2)^{\otimes l}[[\hbar]]$$

$$J_T \in U_\hbar(sl_2)^{\hat{\otimes} l} \simeq U(sl_2)^{\otimes l}[[\hbar]]$$

$$U_\hbar(sl_2)^{\hat{\otimes} l} \xrightarrow{\mathrm{tr}_q^{\otimes l}} \mathbb{Q}[[\hbar]], \qquad U(sl_2)^{\otimes l}[[\hbar]] \xrightarrow{\mathrm{tr}_\nu^{\otimes l}} \mathbb{Q}[[\hbar]]$$

$$J_{\mathsf{cl}(T)}^{(V_1,\ldots,V_l)} \in \mathbb{Q}[[\hbar]]$$

Quantum invariants and Milnor invariants

[Habegger-Masbaum, 2000]

$$Z^t_T = 1 + \mu_m(T) + (\textit{higher})$$

Theorem (Meilhan-S, 2014)

$$J^t_T = W(\mu_m(T)) + (\textit{higher})$$

Note: These results are essentially independent.

Quantum invariants for $T \in SL(l)$

$$Z_T \in \hat{\mathcal{A}}(l)$$

$$J_T \in U_\hbar(sl_2)^{\hat{\otimes} l} \simeq_{\mathbb{Q}[[\hbar]]} U(sl_2)^{\otimes l}[[\hbar]]$$

$$J_{\mathsf{cl}(T)}^{(V_1,\dots,V_l)} \in \mathbb{Q}[[\hbar]]$$

$$W^U : \hat{\mathcal{A}}(l) \dashrightarrow U_\hbar(sl_2)^{\hat{\otimes} l}, \qquad W : \hat{\mathcal{A}}(l) \to U(sl_2)^{\otimes l}[[\hbar]]$$

$$\mathrm{tr}_q^{\otimes l} : U_\hbar(sl_2)^{\hat{\otimes} l} \to \mathbb{Q}[[\hbar]], \qquad \mathrm{tr}_\nu^{\otimes l} : U(sl_2)^{\otimes l}[[\hbar]] \to \mathbb{Q}[[\hbar]]$$

Today

Today, we study

$$W\colon \hat{\mathcal{A}}(l) \to U(sl_2)^{\otimes l}[[\hbar]]$$

restricting on the space of tree Jacobi diagrams.

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② Universal sl_2 weight system

③ Results

Jacobi diagrams

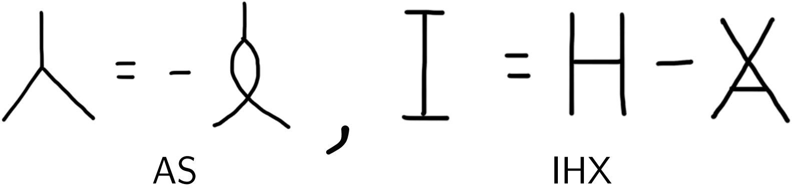
- The space $\hat{\mathcal{B}}(l)$ of labeled Jacobi diagrams
- Subspaces $\mathcal{C}^t(l), \mathcal{C}^h(l)$, and $\mathcal{C}_l$ of $\hat{\mathcal{B}}(l)$

The space $\hat{\mathcal{B}}(l)$ of labeled Jacobi diagrams

$\mathcal{B}(l) = \langle$ 

, 

$\dots \rangle_{\mathbb{Q}} /$ AS, IHX



$\deg(D) = \frac{1}{2}\#\{\text{vertices in } D\}$

Subspaces $\mathcal{C}^t(l), \mathcal{C}^h(l)$, and $\mathcal{C}_l$ of $\hat{\mathcal{B}}(l)$

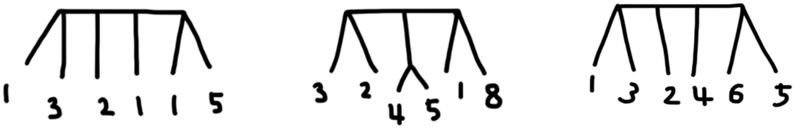
$\mathcal{C}^t(l) = \langle$ simply connected, connected diagrams $\rangle_{\mathbb{Q}}$

$\cup$

$\mathcal{C}^h(l) = \langle$ non-repeated labeled diagrams $\rangle_{\mathbb{Q}}$

$\cup$

$\mathcal{C}_l = \langle$ each label appears exactly once $\rangle_{\mathbb{Q}}$



Remark

Recall that we have $\mu_m(T) \in C^t_m(l)$ for $T \in SL_m(l)$.

For l even;

$$C^t_k(2g) \simeq \mathfrak{h}^{\mathbb{Q}}_{g,1}(k)$$

where

$$\mathfrak{h}_{g,1}(k) = \mathrm{Ker}\{[-,-]\colon\ \mathrm{Lie}_g(1) \otimes \mathrm{Lie}_g(k+1) \to \mathrm{Lie}_g(k+2)\}$$

is a target space of the Johnson homomorphism of Mapping class groups (Morita, Kontsevich).

$$\mathcal{C}^t_m(l) \simeq (\mathcal{C}_{m+1} \otimes (\mathbb{Q}^l)^{\otimes m+1})_{\mathfrak{S}_{m+1}}$$

($\mathcal{C}_l \simeq \mathcal{O}S[l]$: $\mathcal{O}$-spider for Lie operad)

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($\mathcal{C}_l \simeq \mathcal{O}S[l]$: $\mathcal{O}$-spider for Lie operad)

$\mathcal{C}_l$ is a brick!

① We have

$$\mathcal{C}_m^h(l) = \bigoplus_{1\leq i_1<\cdots<i_{m+1}\leq l} \mathcal{C}_{m+1}^{(i_1,\ldots,i_{m+1})}$$

② For $p \geq m$,

$$\bar{D}^{(p)}\colon \mathcal{C}_m^t(l) \to \mathcal{C}_m^h(pl)$$

is injective.

Universal sl_2 weight system

- Lie algebra sl_2
- Universal sl_2 weight system

Lie algebra sl_2

Let sl_2 be the Lie algebra $/\mathbb{Q}$ with basis $\{H, E, F\}$ and Lie bracket defined by

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

Here

$$sl_2 \simeq \{A \in \mathrm{Mat}(2,\mathbb{Q}) \mid \mathrm{Tr}(A) = 0\}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Universal sl_2 weight system

$U = U(sl_2)$: The universal enveloping algebra of sl_2
$S = S(sl_2)$: The symmetric algebra of sl_2

$$\begin{array}{ccc} \hat{\mathcal{A}}(l) & \xrightarrow{W} & U^{\otimes l}[[\hbar]] \\ \wr\downarrow \text{formal PBW} & & \wr\downarrow \text{PBW} \\ \hat{\mathcal{B}}(l) & \xrightarrow{W} & S^{\otimes l}[[\hbar]] \end{array}$$

$*$ (formal) PBW is not an algebra homomorphism.

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Universal sl_2 weight system

$$c = \frac{1}{2} H \otimes H + F \otimes E + E \otimes F \in sl_2^{\otimes 2}$$

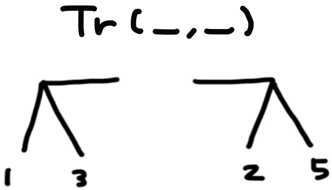
$$b = \sum_{\sigma \in \mathfrak{S}_3} (-1)^{|\sigma|} \sigma(H \otimes E \otimes F) \in sl_2^{\otimes 3}$$

$c_{i,j}$ $\qquad$ $b_{i,j,k} \quad \in S^{\otimes l}$

Universal sl_2 weight system

$w:$

$\mapsto$



$$= \sum \mathrm{Tr}(b_3 b_1') b_1 \otimes b_2' \otimes b_2 \otimes 1 \otimes b_3'$$

$$D \in \mathcal{B}_m(l) \quad \Rightarrow \quad W(D) = w(D)\hbar^m$$

Universal sl_2 weight system on $\mathcal{C}_l$

Recall $\mathcal{C}_l \subset \mathcal{C}^t(l)$

$$\mapsto \quad \sum \mathrm{Tr}(b_3 b_1') \, b_1 \otimes b_2' \otimes b_2 \otimes b_3'$$

Proposition

For $l \geq 2$, we have $w(\mathcal{C}_l) \subset \mathrm{Inv}(sl_2^{\otimes l})$.

Universal sl_2 weight system on $\mathcal{C}_l$

Set

$$w_{C_l} = w|_{\mathcal{C}_l} \colon \mathcal{C}_l \to \mathrm{Inv}(sl_2^{\otimes l}).$$

l	2	3	4	5	6	7	8	9	...	n
$\dim \mathcal{C}_l$	1	1	2	6	24	120	720	5040	...	$(n-2)!$
$\dim \mathrm{Inv}(sl_2^{\otimes l})$	1	1	3	6	15	36	91	232	...	R_n

R_n: Riordan number

Results

Result

Theorem (Meilhan-S, 2014)

(i) *For $l = 2$ or $l > 2$ odd, $w_{\mathcal{C}_l}$ is surjective.*

(ii) *For $l \geq 4$ even, $\operatorname{coker}(w_{\mathcal{C}_l})$ is spanned by $\overline{c^{\otimes \frac{l}{2}}}$.*

l	2	3	4	5	6	7	8	9
$\dim \mathcal{C}_l$	1	1	2	6	24	120	720	5040
$\dim \operatorname{Inv}(sl_2^{\otimes l})$	1	1	3	6	15	36	91	232
$\dim \operatorname{coker}(w_{\mathcal{C}_l})$	0	0	1	0	1	0	1	0
$\dim \ker(w_{\mathcal{C}_l})$	0	0	0	0	10	84	630	4808

$\mathfrak{S}_l$-module structure

Proposition (Kontsevich)

As a $\mathfrak{S}_l$-module, the character of C_l is

$$\chi(1^l) = (l-2)!, \quad \chi(1^1 a^b) = (b-1)! a^{b-1} \mu(a), \quad \chi(a^b) = -(b-1)! a^{b-1} \mu(a),$$

and $\chi_{\mathcal{C}_l}(*) = 0$ for other conjugacy classes.

Lemma

$$\mathrm{Inv}(sl_2^{\otimes l}) \simeq \bigoplus V_\lambda$$

with the summation over partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of l s.t. each λ_i is odd or each λ_i is even, and $n \leq 3$.

$\mathfrak{S}_l$-module structure

Corollary (Meilhan-S, 2014)

(i) For $l = 2$ or $l > 2$ odd, we have

$$\chi_{\ker(w_{\mathcal{C}_l})} = \chi_{\mathcal{C}_l} - \chi_{\mathrm{Inv}(sl_2^{\otimes l})},$$

$$\chi_{\mathrm{Im}(w_{\mathcal{C}_l})} = \chi_{\mathrm{Inv}(sl_2^{\otimes l})}.$$

(ii) For $l \geq 4$ even, we have

$$\chi_{\ker(w_{\mathcal{C}_l})} = \chi_{\mathcal{C}_l} - \chi_{\mathrm{Inv}(sl_2^{\otimes l})} + \chi_{(l)},$$

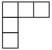
$$\chi_{\mathrm{Im}(w_{\mathcal{C}_l})} = \chi_{\mathrm{Inv}(sl_2^{\otimes l})} - \chi_{(l)}.$$

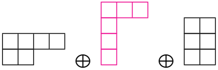
$\mathfrak{S}_l$-module type of C_l with $l \leq 8$

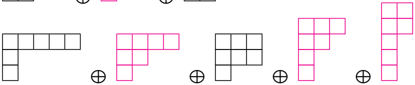
$l = 2, 3 :$ 

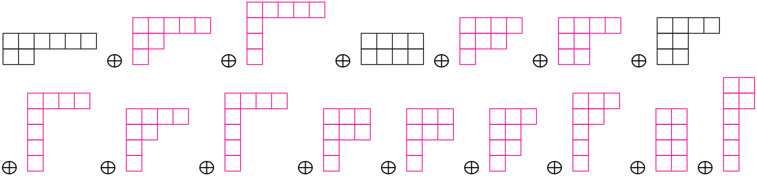
$l = 4 :$ 

Red components are in kernel

$l = 5 :$ 

$l = 6 :$ 

$l = 7 :$ 

$l = 8 :$ 

Key lemmas for Proof

Lemma

If a simply connected Jacobi diagram T has a trivalent vertex, then we have $w(T) \in w\left(\mathcal{C}^t(l)\right)$.

Lemma

If T consists of l cords for $n \geq 2$, then we have

(i) $w(T) \equiv w(\cap \cdots \cap)$ *modulo* $w\left(\mathcal{C}^t(l)\right)$, *and*

(ii) $w(\cap \cdots \cap) \not\equiv 0$ *modulo* $w\left(\mathcal{C}^t(l)\right)$.

Future research

- To describe the irreducible decomposition of $\ker(w_{\mathcal{C}_l})$ explicitly.
- To study w on $C^t(l)$.

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① We have

$$\mathcal{C}_m^h(l) = \bigoplus_{1\le i_1<\cdots<i_{m+1}\le l} \mathcal{C}_{m+1}^{(i_1,\dots,i_{m+1})}$$

② The following diagram commutes;

$$\begin{array}{ccc} \mathcal{C}_m^t(l) & \xrightarrow{\quad w \quad} & (S^{\otimes l})_{m+1} \\ {\scriptstyle \bar{D}^{(p)}}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle \bar{\Delta}^{(p)}} \\ \mathcal{C}_m^h(pl) & \xrightarrow{w} & \bigoplus \iota^{i_1,\dots,i_{m+1}}(sl_2^{\otimes m+1}) \subset S_{m+1}^{\otimes pl} \end{array}$$

Thank you !