Bing doubling and the colored Jones polynomial

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Introduction

Colored Jones polynomial

Unified WRT invariant

Results

Introduction

Quantum invariants for links

$L = L_1 \cup \cdots \cup L_n$: framed link

$$\begin{array}{lccc} \text{Kontsevich inv.} & Z_L & \in & \hat{\mathcal{A}}(\cup_n S^1) \\ & \downarrow W_{sl_2} & & \\ \text{Universal } sl_2 \text{ inv.} & J_L & \in & U_h(sl_2)^{\hat{\otimes} n}/I \\ & \downarrow \mathrm{tr}^{V_1} \otimes \cdots \otimes \mathrm{tr}^{V_n} & & \\ \text{Colored Jones poly.} & J_{L;V_1,\ldots,V_n} & \in & \mathbb{Z}[q^{1/4}, q^{-1/4}] \end{array}$$

Quantum invariants for 3-mfds

M: Integral homology sphere (=IHS)

$$
\begin{array}{lccc}
\text{LMO inv.} & Z_M & \in & \hat{\mathcal{A}}(\emptyset) \\
 & \downarrow W_{sl_2}^m & & \\
\text{Unified WRT inv.} & J_M & \in & \widehat{\mathbb{Z}[q]} \\
 & \downarrow \mathrm{ev}_\zeta & & \\
\text{WRT inv.} & \tau_M^\zeta & \in & \mathbb{Z}[\zeta]
\end{array}
$$

Quantum invariants

Links	IHS
Kontsevich inv.	LMO inv.
Universal sl_2 inv.	Unified WRT inv.
Colored Jones poly.	WRT inv.

Quantum invariants

Links

Kontsevich inv.

Universal sl_2 inv.

Colored Jones poly.

IHS

LMO inv.

Unified WRT inv.

WRT inv.

Problem

Q : Topological meaning of quantum invariants ?

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Answer the question mathematically.

What is "Topological" ? $\sim$ Classical

- Fundamental groups
  (Alexander poly., Milnor μ inv., ...)
- Coverings
  (Alexander poly., ...)
- Seifert surfaces
  (Alexander poly., boundary links, ...)
- Cobordisms
  (slice, ribbon, ...)
- Local moves
  (crossing change, mutant, satellite, ...)

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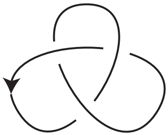
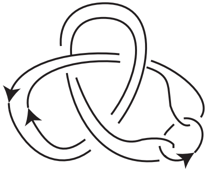
Bing doubling

$K =$ 

$B(K) =$ 

Bing doubling and Link concordance

Fact: K is slice $\Rightarrow B(K)$ is slice.

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$\Updownarrow$

"Surgery conjecture" in 4-dim. topology.

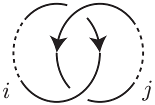
Bing doubling and Milnor $\bar{\mu}$ invariant

Roughly

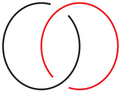
Milnor $\bar{\mu}$ invariants of length $l \geq 2$

$$||$$

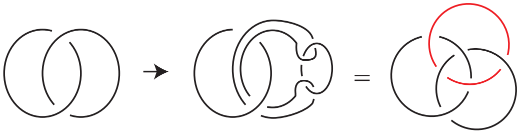
"linking numbers of degree l"

linking number of degree $2 =$ number of 

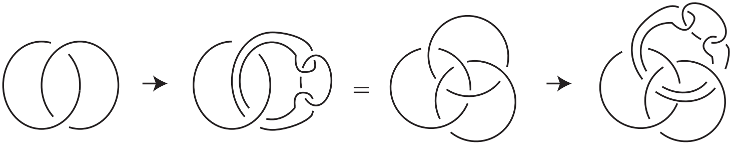
Bing doubling and Milnor $\bar{\mu}$ invariant



Bing doubling and Milnor $\bar{\mu}$ invariant

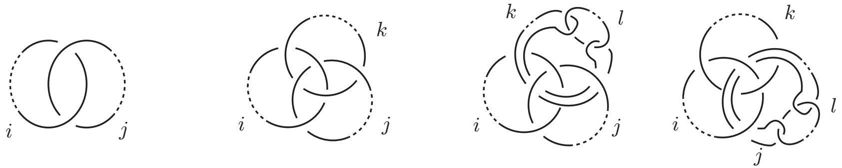


Bing doubling and Milnor $\bar{\mu}$ invariant



Bing doubling and Milnor $\bar{\mu}$ invariant

Milnor $\bar{\mu}$ invariants count the following parts:

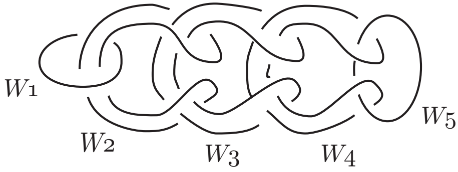


length 2 length 3 length 4

Colored Jones polynomial

Colored Jones polynomial

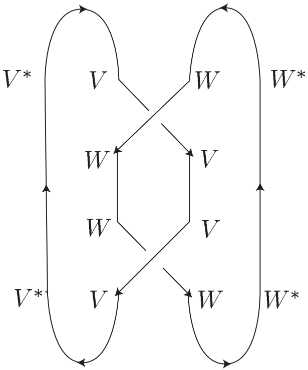
$$W_1, \ldots, W_5 \in \mathrm{Mod}_f(U_q(sl_2)).$$



- As Operator invariant
- By Skein relation
- From Kontsevich invariant
- From Universal sl_2 invariant

Colored Jones polynomial (Operator invariant)

$V, W \in \mathrm{Mod}_f(U_q(sl_2))$



$$
\begin{array}{ccc}
\mathbb{C}(q) & & 1 \\
\downarrow \text{coev}^* \otimes \text{coev} & & \\
V^* \otimes V \otimes W \otimes W^* & & \\
\downarrow 1 \otimes R_{V,W} \otimes 1 & & \\
V^* \otimes W \otimes V \otimes W^* & & \downarrow \\
\downarrow 1 \otimes R_{W,V} \otimes 1 & & \\
V^* \otimes V \otimes W \otimes W^* & & \\
\downarrow \text{ev} \otimes \text{ev}^* & & \\
\mathbb{C}(q) & & J_{L;V,W}
\end{array}
$$

Generalized colored Jones polynomial

Set

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(q^{\frac{1}{2}})}\{V_m : m\text{-dim. irr. rep.} \mid m \geq 1\}.$$

Definition

For a link $L = L_1 \cup \cdots \cup L_n$ and

$$X_i = \sum_{j_i} x_{j_i}^{(i)} V_{j_i} \in \mathcal{R}, \quad x^{(i)} \in \mathbb{Q}(q^{\frac{1}{2}}),$$

set

$$J_{L;X_1,\dots,X_n} = \sum_{j_1,\dots,j_n} x_{j_1}^{(1)} \cdots x_{j_n}^{(n)} J_{L;V_{j_1},\dots,V_{j_n}}.$$

For $l \geq 0$, set

$$\begin{aligned}
P'_l &= \frac{1}{\{l\}!}\prod_{i=0}^{l-1}(V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R}, \\
\tilde{P}'_l &= q^{-\frac{1}{4}l(l-1)} P'_l \in \mathcal{R}, \\
\mathcal{P}_k &= \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{\tilde{P}'_l \mid l \geq k\}, \\
\hat{\mathcal{P}} &= \varprojlim_{k \geq 0} \mathcal{P}_0/\mathcal{P}_k, \\
\omega^{\pm 1} &= \sum_{l=0}^{\infty} (\pm 1)^l q^{\pm\frac{1}{4}l(l+3)} P'_l \in \hat{\mathcal{P}}.
\end{aligned}$$

Theorem (K. Habiro)

$L = L_1 \cup \cdots \cup L_n$: *algebraically-split link*

$$J_{L;\omega^{\epsilon_1},\ldots,\omega^{\epsilon_n}} = \sum_{l_1,\ldots,l_n=0}^{\infty} \Big(\prod_{i=1,\ldots n} \epsilon_i^{l_i} q^{\epsilon_i \frac{1}{4} l_i(l_i+3)} \Big) J_{L;P'_{l_1},\ldots,P'_{l_n}} \in \widehat{\mathbb{Z}[q]}$$

Here $\epsilon_i, \ldots, \epsilon_n \in \{\pm 1\}$ and

$$\widehat{\mathbb{Z}[q]} = \varprojlim_{n \geq 0} \mathbb{Z}[q]/((1-q)(1-q^2)\cdots(1-q^n)).$$

Unified WRT invariant

Unified WRT invariant

$L = L_1 \cup \cdots \cup L_n$: link with framings $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$
$M = S^3_L$: IHS obtained by surgery along L in S^3

Definition (Unified WRT invariant)

Set

$$J_M = J_{L^0; \omega^{\epsilon_1}, \ldots, \omega^{\epsilon_n},} \in \widehat{\mathbb{Z}[q]}.$$

(L^0 : L with all framings 0.)

Results

Notation

We use the following q-integer notations:

$$\{i\} = q^{\frac{i}{2}} - q^{-\frac{i}{2}},$$
$$\{i\}_n = \{i\}\{i-1\}\cdots\{i-n+1\},$$
$$\{n\}! = \{n\}_n,$$
$$\begin{bmatrix} i \\ n \end{bmatrix} = \{i\}_n / \{n\}!,$$

for $i \in \mathbb{Z}, n \geq 0$.

Results

Theorem (S. Suzuki)

Let K be a knot with 0-framing. For $i, j \geq 0$, we have

$$J_{B(K);P_i',P_j'} = \sum_{l \geq 0} a_{i,j}^{(l)} J_{K;P_l'},$$

where

$$a_{i,j}^{(l)} = \delta_{i,j}(-1)^i \frac{\{2i+1\}!\{l\}!}{\{2l+1\}!} \lambda_{l,i},$$

$$\lambda_{l,i} = \sum_{k=0}^{l} (-1)^k \begin{bmatrix} 2l+1 \\ k \end{bmatrix} \begin{bmatrix} 2l+i-2k+1 \\ 2i+1 \end{bmatrix}.$$

$\Phi_m \in \mathbb{Z}[q]$: m-th cyclotomic polynomial.

$$\Phi_1 = q - 1, \quad \Phi_2 = q + 1, \quad \Phi_3 = q^2 + q + 1.$$

Theorem (S. Suzuki)

Let K be a knot with 0-framing and M the integral homology sphere obtained by surgery along $B(K)$ with ± 1 framing in S^3. We have

$$J_M - 1 \in \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_6 \widehat{\mathbb{Z}[q]}.$$