On the LMO functor constructed by Cheptea, Habiro and Massuyeau

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Contents

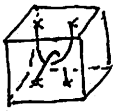
① Notation

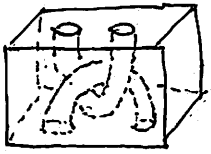
② q-tangles in $[-1,1]^3$ $\xrightarrow{\text{Kontsevich int. } \hat{Z}}$ Jacobi diagrams based on 1-mfd

≀ generalize

q-tangles in homology cubes $\xrightarrow{\text{LMO-Kontsevich integral } Z}$ //

③ bottom-top tangles in homology cubes $\overset{D}{\cong}$ Lagrangian cobordisms between cpt surfaces





≀ ②③

Lagrangian q-cobordisms $\xrightarrow[Z\circ D^{-1}]{\text{not homo.}}$ Jacobi diagrams based on 1-mfd

$$\boxed{\begin{array}{c} M \\ \hline N \end{array}} \longmapsto \oslash \neq \boxed{\begin{array}{c} Z(M) \\ \hline Z(N) \end{array}}$$

≀ modify

④ Lagrangian q-cobordisms $\xrightarrow{\text{monoid } \tilde{Z} \text{ homo}}$ top-substantial Jacobi diagrams

⑤
$$\begin{tikzcd}
(\underbrace{\mathcal{I}(F_g)}_{\text{Torelli grp}}\subset)\mathcal{L}\mathrm{Cob}_q(g,g) \arrow[rr, "\tilde{Z}"] & & {}^{ts}\mathcal{A}(g,g) \arrow[dl, "\text{tree part}"'] \arrow[dr, "\text{loop part}"] & \\
& \text{Johnson homo.} & & \text{finite type inv. e.g. Casson inv.}
\end{tikzcd}$$

Reference

[CHM]: D. Cheptea, H. Habiro, G.Massuyeau, ``A functorial LMO invariant for Lagrangian Cobordisms", Geom. Topol. 12 (2008), 1091-1170

[BGRT1]: D. Bar-Natan, S. Garoufalidis, L. Rozansky, D.P. Thurston, ``The Århus integral of rational homology 3-spheres I", Selecta Math. (N.S.) 8 (2002), 315-339

[BGRT2]: D. Bar-Natan, S. Garoufalidis, L. Rozansky, D.P. Thurston, ``The Århus integral of rational homology 3-spheres II", Selecta Math. (N.S.) (2002), 341-371

[CDM]: S.Chmutov, S. Duzhin, J. Mostovoy, ``Introduction to Vassiliev knot invariants", Cambridge University Press, 2012

[JMM]: D.M. Jackson, I. Moffatt, A. Morales, ``On the group-like behavior of the Le-Murakami-Ohtsuki invariant", J. Knot Theory Ramifications 16 (2007), 699-718

[LM]: Representation of the category of tangles by Kontsevich's iterated integral, Comm. Math.Phys. 168 (1995), 535-562

[O]: T. Ohtsuki, ``Quantum invariants", Series on Knots and Everything 29, World Scientific Publishing Co. (2002)

§ Jacobi diagrams and operations

X : cpt oriented 1-mfd,

C : finite set.

Jacobi diagrams based on (X, C)

D : Jacobi diagram based on (X, C).

$\underset{\text{def}}{\Leftrightarrow}$ D is a uni-trivalent graph

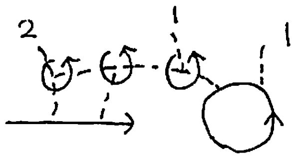
whose univalent vertices are either embedded into X

or colored with elements of C

and trivalent vertices are oriented.

Example

X =  , $C = \{1, 2\}$.

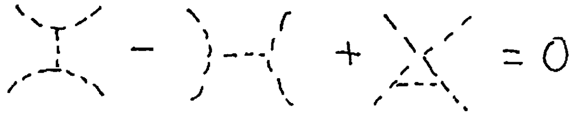


A(X, C)

$$A(X,C) = \frac{\mathbb{Q}\{\text{Jacobi diagrams based on } (X,C)\}}{AS,\ IHX,\ STU}$$: the space of Jacobi diagrams.

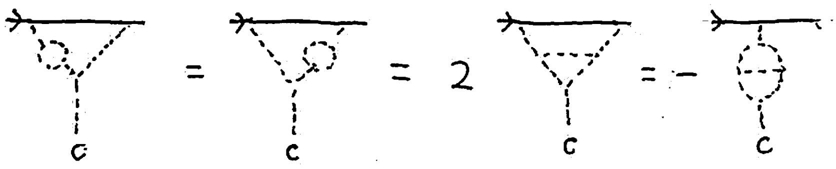
AS

IHX

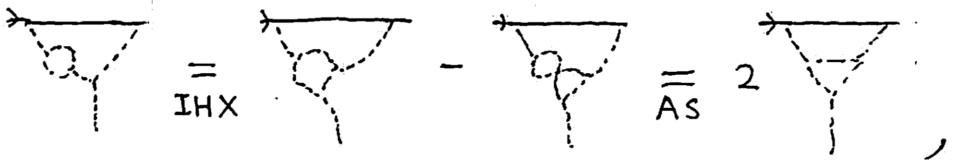


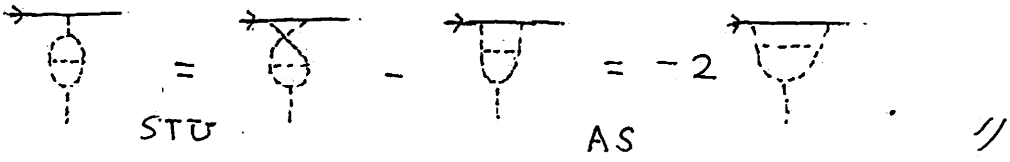
STU

- Example



(∵)

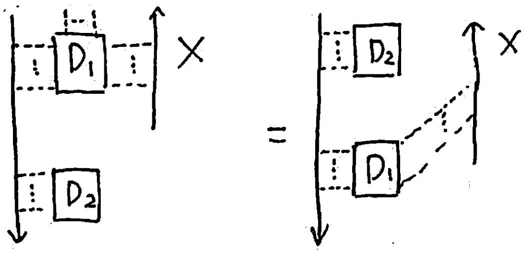




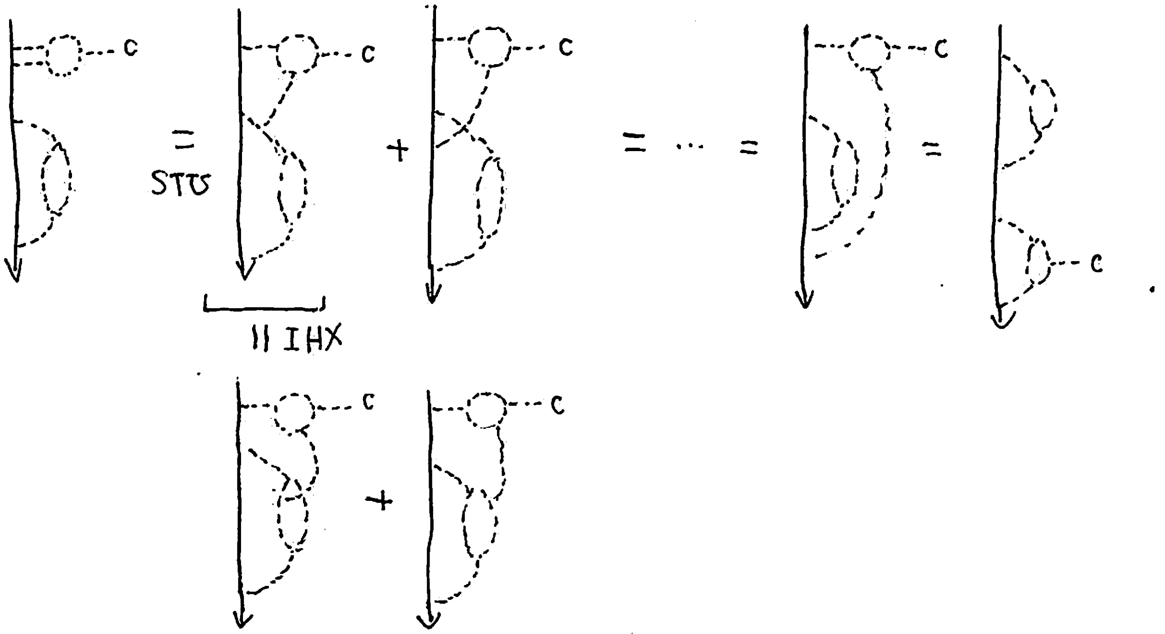
Lemma (see [0] Prop 6.1)

D_1 : a Jacobi diagram based on $(X\uparrow, C)$,
D_2 : based on $\uparrow$.

Then, in $\mathcal{A}(X \sqcup \uparrow, C)$,



Example



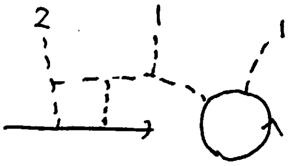
degree of Jacobi diagrams

i-deg = # trivalent vertices,

e-deg = # univalent vertices,

$$\deg = \frac{(\text{i-deg}) + (\text{e-deg})}{2}.$$

example



i-deg = 4 $\qquad \deg = \frac{4+8}{2} = 6.$

e-deg = 8

Rem

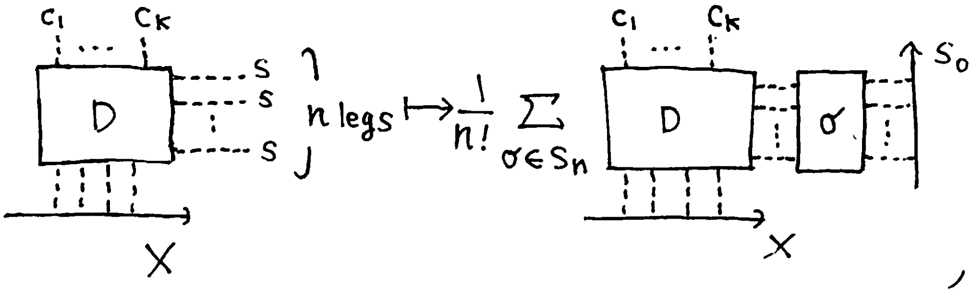
The AS and IHX relations preserve i-deg, e-deg and deg.

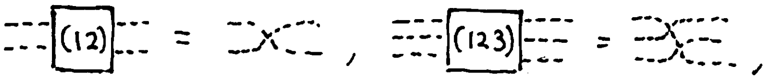
The STU relation preserves deg.

In the following, we consider the degree completion of $\mathcal{A}(X, C)$, and denote it also by $\mathcal{A}(X, C)$.

$\chi_s : \mathcal{A}(X, C \amalg \{s\}) \to \mathcal{A}(X \uparrow^s, C)$

$$\mathcal{A}(X, C \amalg \{s\}) \xrightarrow{\chi_s} \mathcal{A}(X \uparrow^s, C)$$

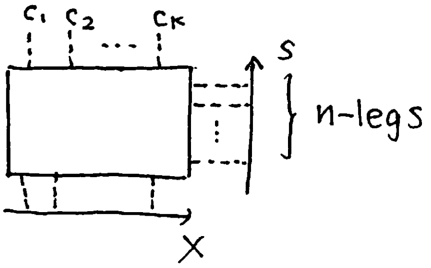


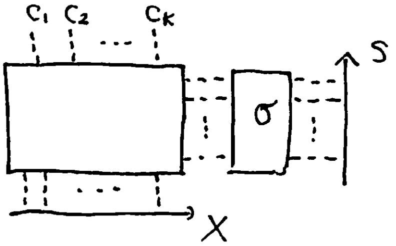
where  and so on.

For $S = \{s_1, s_2, \cdots, s_n\}$, we denote $\chi_S = \chi_{s_1} \circ \chi_{s_2} \circ \cdots \circ \chi_{s_n}$.

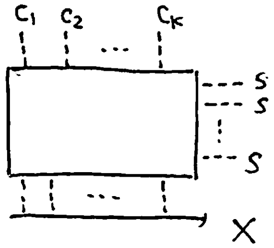
Lemma

Let D be a Jacobi diagram based on $(X\uparrow^s, C)$ with n-legs attached to $\uparrow^s$.

$D =$ 

, $\sigma D =$ 

$(\sigma \in S_n)$,

$\tilde{D} =$ 

.

(1) For $\sigma \in S_n$, $D - \sigma D \in \mathcal{A}(X\uparrow^s, C)$ is represented by a sum $\Gamma_\sigma(D)$ of Jacobi diagrams with at most $(n-1)$-legs attached to $\uparrow^s$.

(2) Define, for a Jacobi diagram D with at most n-legs are attached to $\uparrow^s$,

$$\tau_n(D) = \begin{cases} \tilde{D} + \frac{1}{n!}\sum_{\sigma \in S_n} \tau_{n-1}(\Gamma_\sigma(D)) \in \mathcal{A}(X, C \sqcup \{s\}) & \text{if } \#\text{attached legs} = n, \\ \tau_{n-1}(D) & \text{if } \qquad\qquad < n, \end{cases}$$

inductively. $(\tau_1(D) = \tilde{D})$.

Then, $\tau_n(D)$ does not depend on the choice of $\Gamma_\sigma(D)$, and $\tau : \mathcal{A}(X\uparrow^s, C) \to \mathcal{A}(X, C\sqcup\{s\})$ defined by $\tau|_{\#\text{attached legs} \leq n} = \tau_n$ is the inverse map of $\chi_s : \mathcal{A}(X, C\sqcup\{s\}) \to \mathcal{A}(X\uparrow^s, C)$.

Rem

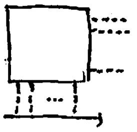
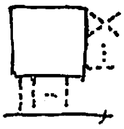
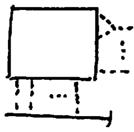
We can also define $\chi_s : \mathcal{A}(X, C\sqcup\{s\}) \to \mathcal{A}(X\circlearrowleft^s, C)$, but it is not an isomorphism. Instead, we have.

$$\begin{tikzcd}
\mathcal{A}(X, C\sqcup\{s\}) \arrow[r, "\chi_s", "\cong"'] \arrow[d] & \mathcal{A}(X\uparrow^s, C) \arrow[d] \\
\dfrac{\mathcal{A}(X, C\sqcup\{s\})}{\text{s-link relation}} \arrow[r, phantom, "\cong"] & \mathcal{A}(X\circlearrowleft^s, C)
\end{tikzcd}$$

.

Proof [CDM, § 5.7.1]

(1) When $\sigma = (12)$,

$D - \sigma D =$  $-$  $=$  $= \Gamma_{(12)}(D)$

For general $\sigma \in S_n$,

σ can be written as a product $\sigma_1 \cdots \sigma_n$ of $\{(i\ i{+}1) \mid i = 1, \cdots, n-1\}$.

Then, $D - \sigma D = (D - \sigma_1 D) + \sigma_1 (D - \sigma_2 D) + \cdots + \sigma_1 \cdots \sigma_{n-1} (D - \sigma_n D)$.

(2) For welldefinedness, see CDM. For a Jacobi diagram D with n-legs attached,

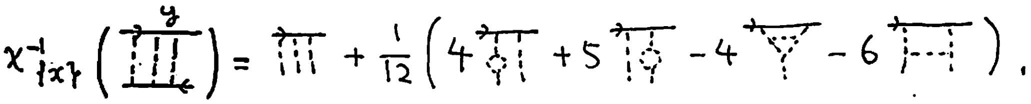
$$(\chi \circ \tau)(D) = \chi\Big(\tilde{D} + \frac{1}{n!} \sum_{\sigma \in S_n} \tau_{n-1}(\Gamma_\sigma(D))\Big)$$

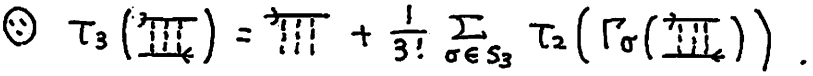
$$= \frac{1}{n!} \sum_{\sigma \in S_n} \sigma D + \frac{1}{n!} \sum_{\sigma \in S_n} \underbrace{(\chi \circ \tau)(\Gamma_\sigma(D))}_{\substack{\shortparallel \ \ \text{induction} \\ \Gamma_\sigma(D)}}$$

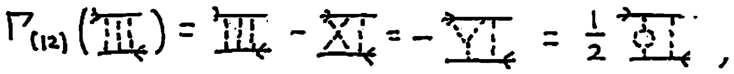
$$= \frac{1}{n!} \sum_{\sigma \in S_n} \sigma D + \frac{1}{n!} \sum_{\sigma \in S_n} (D - \sigma D)$$

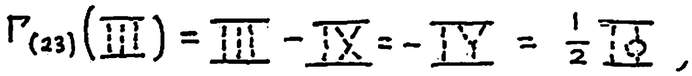
$$= D. \ //$$

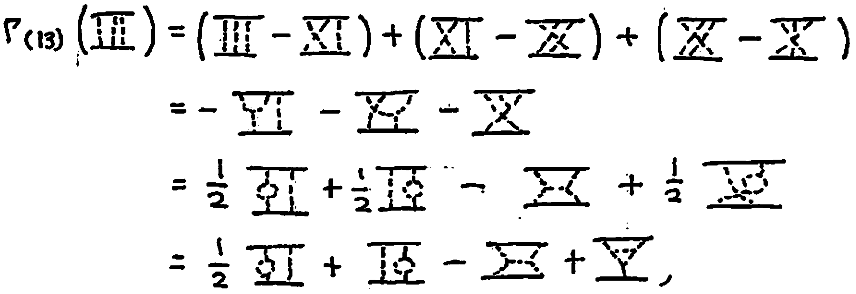
Example

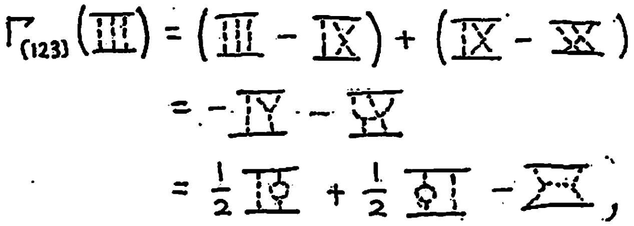


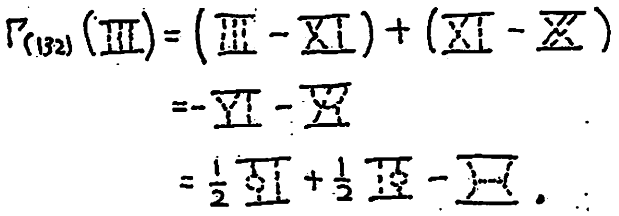


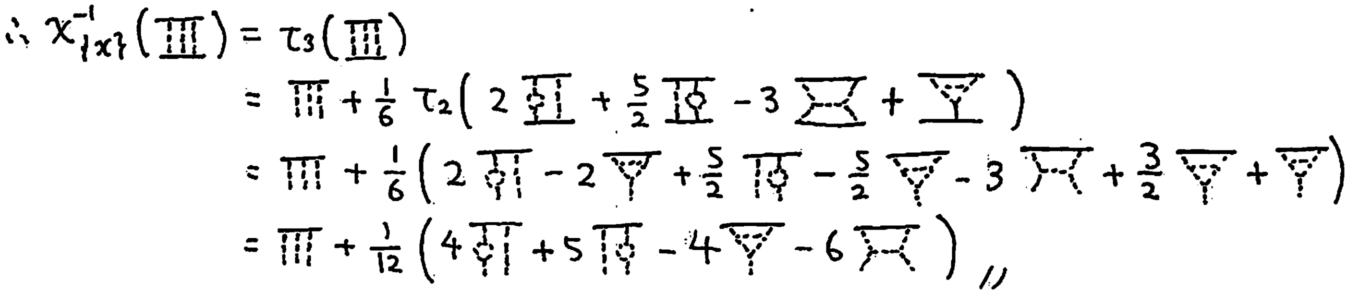












- comultiplication

Let $\hat{\Delta} : \mathcal{A}(\emptyset, C) \to \mathcal{A}(\emptyset, C) \otimes \mathcal{A}(\emptyset, C)$ be the map defined by

$$\hat{\Delta}(D) = \sum_{D_1 \amalg D_2 = D} D_1 \otimes D_2 .$$

Rem

$\alpha \in \mathcal{A}(\emptyset, C)$

$\hat{\Delta}(\alpha) = \alpha \otimes \emptyset + \emptyset \otimes \alpha$ (i.e. α is primitive)

$\Leftrightarrow$ α is a linear combination of connected Jacobi diagrams.

Lemma ([O] Lemma 6.10)

$\alpha \in \mathcal{A}(\emptyset, C)$.

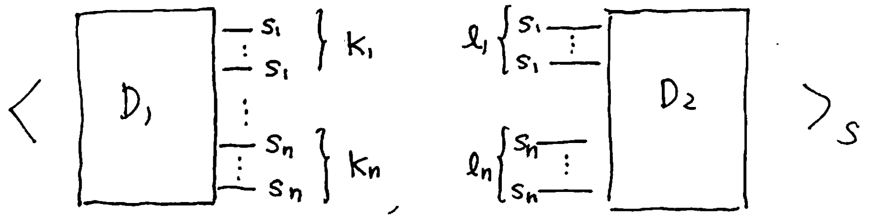
$\hat{\Delta}(\alpha) = \alpha \otimes \alpha$ (i.e. α is grouplike)

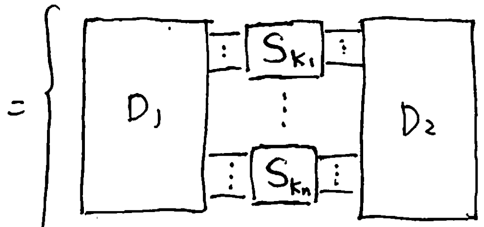
$\Leftrightarrow$ $^\exists \beta \in \mathcal{A}(\emptyset, C)$: linear combination of conn. Jacobi diagrams (primitive) s.t. $\alpha = \exp \beta$.

$\langle \, , \rangle_S$

D_1, D_2 : Jacobi diagrams based on $(X, C \cup S)$,

$S = \{s_1, \cdots, s_n\}$, Let us define



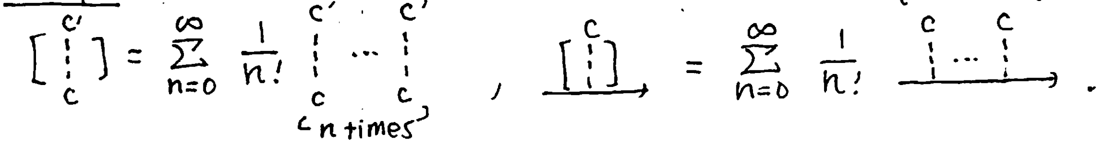


if $k_i = l_i$ for all i

0 otherwise.

For a Jacobi diagram D, we denote $\exp D = [D]$.

Example



Rem

For $\alpha \in \mathcal{A}(X, C\cup\{s\})$, and $\chi_s : \mathcal{A}(X, C\cup\{s\}) \to \mathcal{A}(X\uparrow^s, C)$,

$$\langle\, [\overset{s}{\vdots}]_{\rightarrow} ,\ \alpha \,\rangle_s = \chi_s(\alpha).$$

Let $C = \{c_1, c_2, \cdots, c_n\}$.

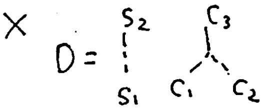
For a rational symmetric $n\times n$ matrix,

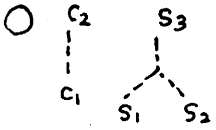
we denote $[L] = [\sum_{i,j=1}^{n} L_{ij} \overset{c_j}{\underset{c_i}{\vdots}}\,] \in \mathcal{A}(\emptyset, C)$.

S-substantial

A Jacobi diagram D based on $(X, C\cup S)$ is S-substantial

if it has no strut component both ends are labeled by S.





Gaussian

$G \in \mathcal{A}(X, C\cup S)$ is Gaussian w.r.t. S

$\Leftrightarrow$ $G = [\frac{L}{2}] \amalg P$, where

$P \in \mathcal{A}(X, C\cup S)$: S-substantial,

L : rational symmetric $S\times S$ matrix

formal Gaussian integral

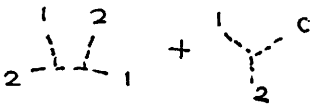
$G = [\frac{L}{2}] \amalg P$: Gaussian w.r.t. S, and $\det L \neq 0$.

Then, we define

$$\int_S G := \langle\, [-\tfrac{L^{-1}}{2}] ,\ P \,\rangle_S \in \mathcal{A}(X, C).$$

Example

$L = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$, $P =$

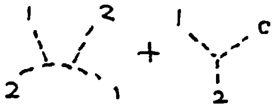


$\in \mathcal{A}(\emptyset, C\cup\{1,2\})$

$(L^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix})$

$G = [\frac{L}{2}] \amalg P$.

$\int_{\{1,2\}} G = \langle\, [\, -\overset{2}{\underset{1}{\vdots}} - \overset{2}{\underset{2}{\vdots}}\,],$



$\rangle_{\{1,2\}} =$

Category of q-tangles in $[-1,1]^3$

$\mathcal{T}_q$: the (nonstrict monoidal) category of q-tangles in $[-1,1]^3$.

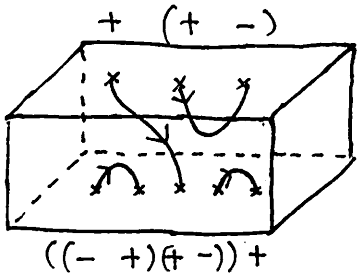
object : the free non-associative magma gen. by $\{+,-\}$,

morphism : framed oriented tangles $\gamma \in \mathcal{T}_q(u,v)$

(1) $\# \partial \gamma = |u| + |v|$,

(2) $\partial \gamma$ are uniformly distributed along $[-1,1] \times \{0\} \times \{\pm 1\}$,

(3) In the top square,
γ starts from "+" and ends at "−".
In the bottom square,
γ starts from "−" and ends at "−".



composition & tensor product

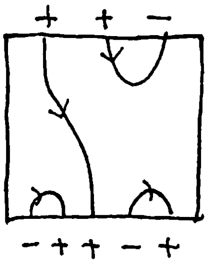
$$\gamma_1 \circ \gamma_2 = \begin{array}{|c|}\hline \gamma_2 \\ \hline \gamma_1 \\ \hline \end{array}, \qquad \gamma_1 \otimes \gamma_2 = \begin{array}{|c|c|}\hline \gamma_1 & \gamma_2 \\ \hline \end{array}.$$

Category of Jacobi diagrams on 1-mfds

$\mathcal{A}$: the (strict monoidal) category of Jacobi diagrams on (oriented) 1-mfds.

object : associative words in the letter $\{+,-\}$.

morphism : $\mathcal{A}(u,v) = \bigcup_{\substack{\partial X \\ \text{corr. to }(u,v)}} \mathcal{A}(X)$



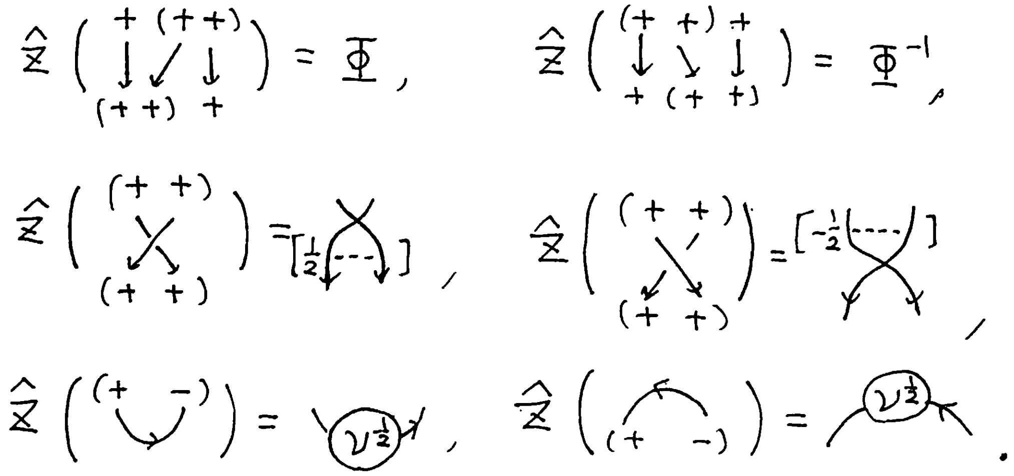
Thm [LM]

There exists a tensor-preserving functor

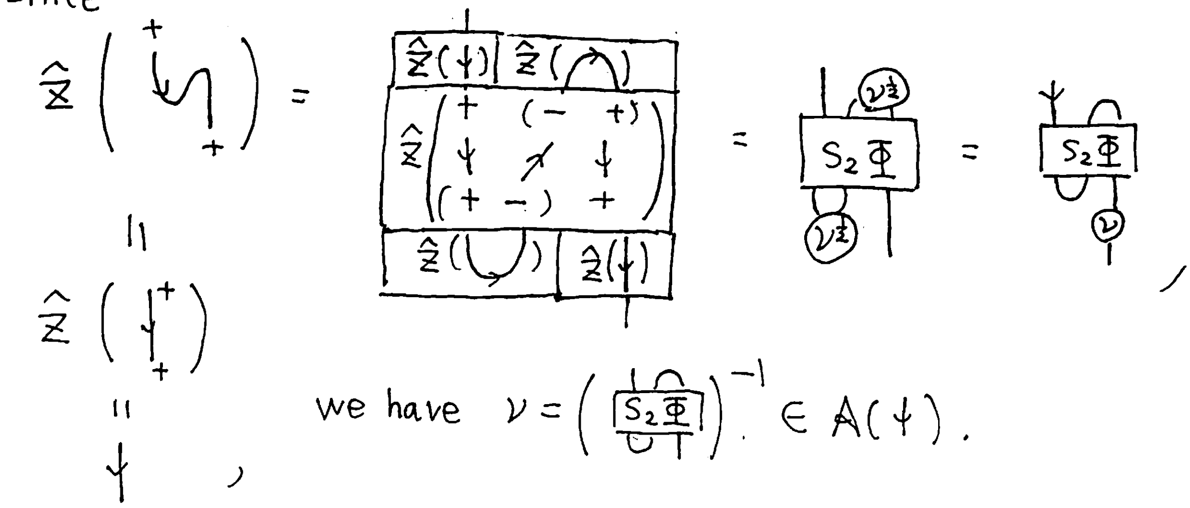
$$\hat{Z} : \mathcal{T}_q \longrightarrow \mathcal{A}$$

$$\gamma \longmapsto \hat{Z}_f(\gamma) \in \mathcal{A}(\gamma, \emptyset).$$

Values of $\hat{Z}$

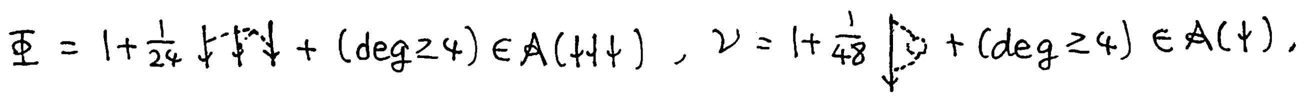


Since



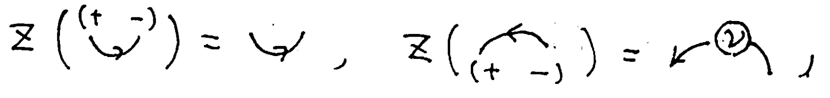
we have $\nu = \left(\quad \right)^{-1} \in \mathcal{A}(\downarrow)$.

If we choose a rational and even Drinfeld associator,



$\Phi = 1 + \frac{1}{24}$ + (deg ≥ 4) $\in \mathcal{A}(\downarrow\downarrow\downarrow)$, $\nu = 1 + \frac{1}{48}$ + (deg ≥ 4) $\in \mathcal{A}(\downarrow)$.

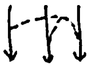
We modify $\hat{Z}$, and define $Z : \mathcal{T}_q \to \mathcal{A}$ by

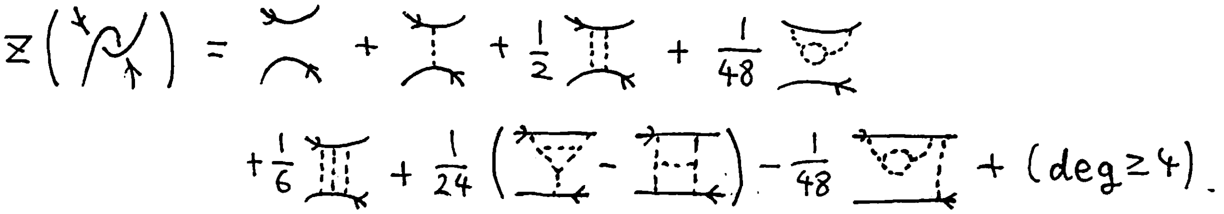


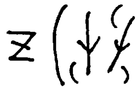
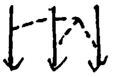
and for other elementary q-tangles, $Z = \hat{Z}$.

Example

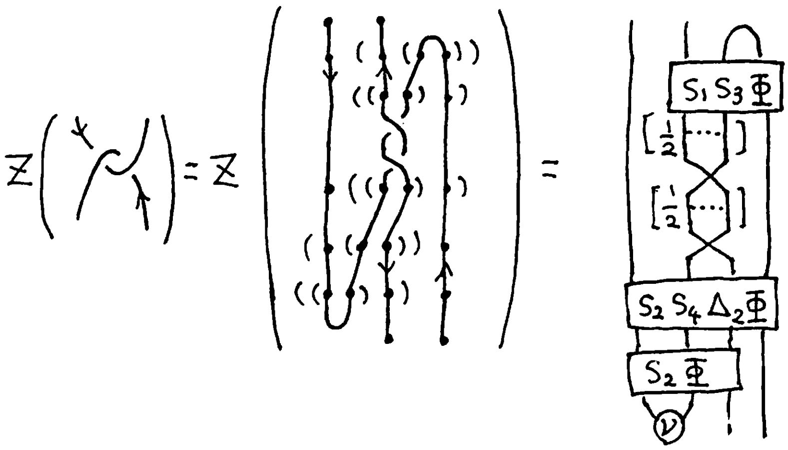
Assume that chosen Drinfeld associator Φ is rational and even.

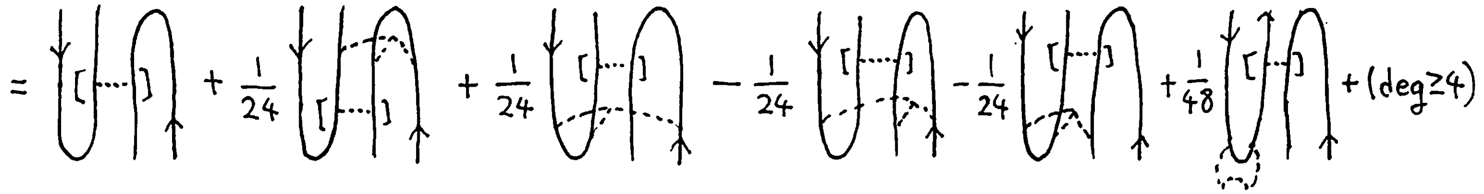
Then, $\Phi = 1 + \frac{1}{24}$  $+ (\deg \geq 4)$, and

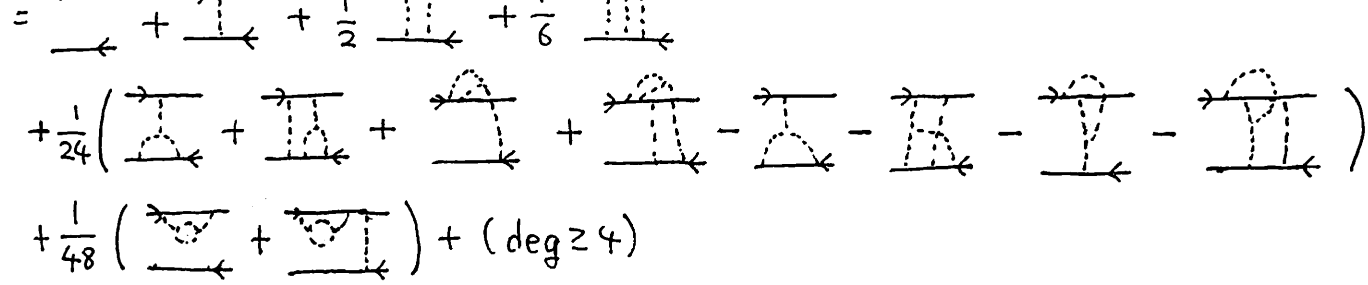


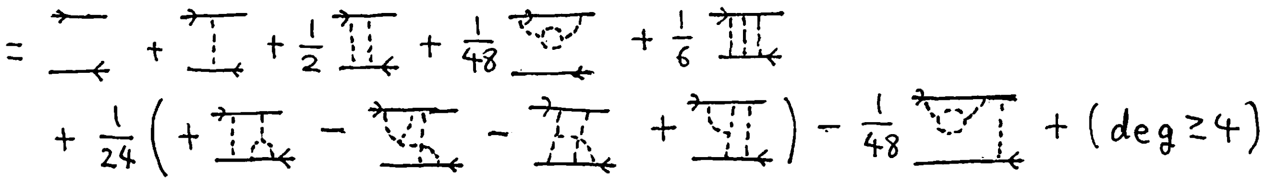
☺ Recall that $\Phi = Z($  $) = 1 + \frac{1}{24}$  $+ (\deg \geq 4) \in \mathcal{A}(\downarrow\downarrow\downarrow, \emptyset)$,

and $\nu = Z($  $) = 1 + \frac{1}{48}$  $+ (\deg \geq 4) \in \mathcal{A}($  $)$.





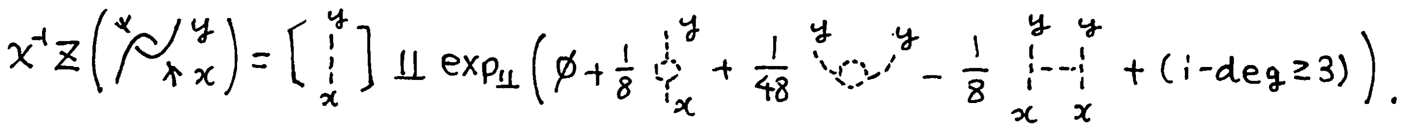




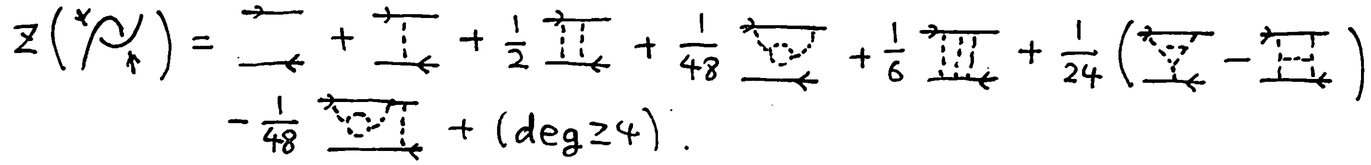
= (RHS). //

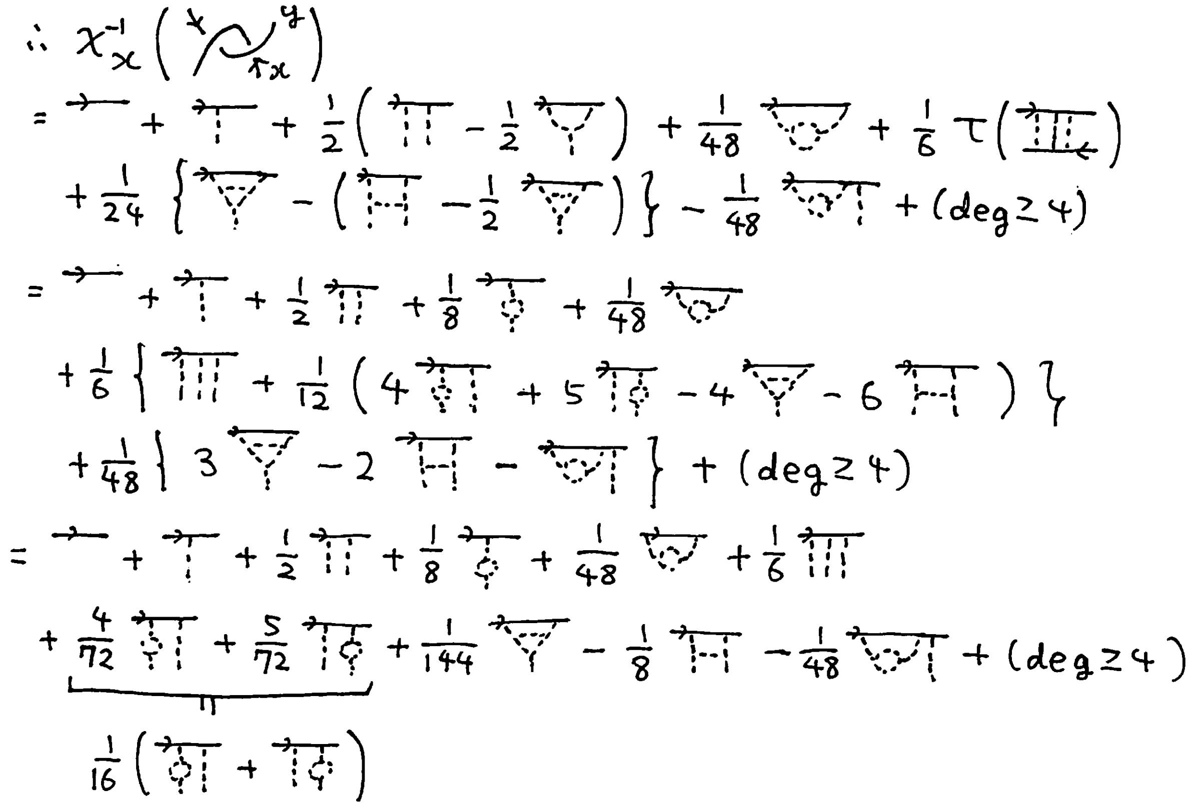
Example

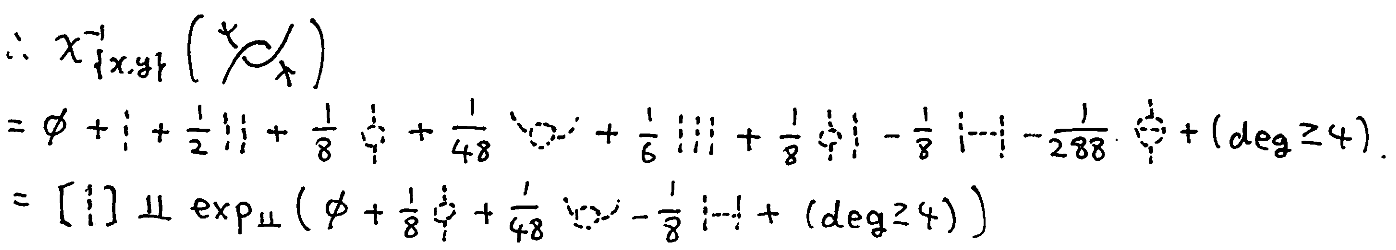
Assume that chosen Drinfeld associator is rational and even.



☺ Recall that







= RHS.

↑

For connected diagrams, i-deg ≥ deg−1. //

Homology cubes

A homology cube is a pair (M, m), where

M is a cpt conn. oriented 3-mfd s.t. $H_*(M) \cong H_*([-1,1]^3)$.

$m : \partial[-1,1]^3 \xrightarrow{\cong} \partial M$: ori-pres. homeo.

Category of q-tangles in homology cubes

$\mathcal{T}_q \mathcal{C}ub$: the (nonstrict monoidal) category of q-tangles in homology cubes

object : the free non-ass. magma gen. by $\{+,-\}$,

morphism : framed oriented tangles γ in homology cubes B

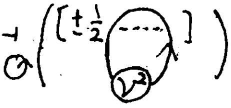
$(B,\gamma) \in \mathcal{T}_q \mathcal{C}ub(u,v)$

(1) $\# \partial\gamma = |u| + |v|$

(2) $\partial\gamma$ are uniformly distributed along $m([-1,1]\times\{0\}\times\{\pm1\})$.

(3) The orientation of $\partial\gamma$ corr. to u and v.

$U_\pm \in \mathcal{A}(\emptyset)$

$$U_\pm = \int \chi^{-1}_{\bigcirc}\Big(\underbrace{\underbrace{\nu \# Z(\bigcirc_{\pm1})}_{\overset{\cap}{\mathcal{A}(\bigcirc;\emptyset)}}\Big)}_{\mathcal{A}(\{*\})} = \int \chi^{-1}_{\bigcirc}\Big(\Big[$$

$$\Big]\Big)$$

$$= \int \exp\Big(\pm\tfrac{1}{2}$$

$$\Big) \amalg \Big(\emptyset + \tfrac{1}{16}$$

$$+ \deg \geq 3\Big)$$

$$= \emptyset \mp \tfrac{1}{16}$$

$$+ \deg \geq 3 \in \mathcal{A}(\emptyset).$$

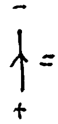
Normalization

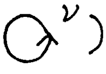
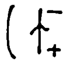
$L \cup \gamma$: a q-tangle in $[-1,1]^3$.

We set $Z(L^\nu \cup \gamma) := \nu^{\otimes \pi_0(L)} \underset{\pi_0(L)}{\#} Z(L\cup\gamma) \in \mathcal{A}(L\cup\gamma)$.

Example

$L =$


$= \gamma \subset [-1,1]^3$

$Z(L^\nu \cup \gamma) = Z\big($

$^{\nu}\big) \amalg Z\big($

$\big)$

$=$

$\in \mathcal{A}\big($

$\big)$.

Surgery presentation

$(B,\gamma) \in \mathcal{T}_q\mathcal{C}ub(u,v)$.

(L,γ) : a surgery presentation of (B,γ).

$\overset{\Leftrightarrow}{\text{def}}$ (1) L is a framed oriented link in $[-1,1]^3$ s.t. $[-1,1]^3_L = B$,

(2) γ is a 〃 tangle in $[-1,1]^3$ s.t.

surgery along L maps $([-1,1]^3, \gamma)$ to (B,γ).

Kontsevich-LMO invariant

$(B,\gamma) \in \mathcal{T}_q\mathcal{C}ub(u,v)$.

$$Z(B,\gamma) := U_+^{-\sigma_+(L)} \sqcup U_-^{-\sigma_-(L)} \sqcup \int_{\pi_0(L)} \chi^{-1}_{\pi_0(L)} Z(L^\nu \cup \gamma) \in \mathcal{A}(\gamma),$$

where (L,γ) is a surgery presentation of (B,γ),

$(\sigma_+(L), \sigma_-(L))$ denotes the signature of the linking matrix of L.

Rem

When $B = [-1,1]^3$ $(L=\emptyset)$,

$Z(B,\gamma) = Z(\gamma) \in \mathcal{A}(\gamma)$: (modified) Kontsevich integral.

When $\gamma = \emptyset$,

$$Z(B,\emptyset) = U_+^{-\sigma_+(L)} \sqcup U_-^{-\sigma_-(L)} \sqcup \int_{\pi_0(L)} \chi^{-1}_{\pi_0(L)} Z(L^\nu) \in \mathcal{A}(\emptyset)$$

: LMO invariant.

Thm

$Z(B,\gamma)$ does not depend on the choice of a surgery presentation.

Lemma 3.17

(B,γ) : a bottom-top q-tangle in a homology cube B.

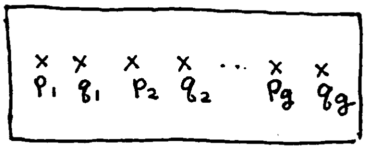
$\chi^{-1} Z(B,\gamma) \in \mathcal{A}(\emptyset, \pi_0(\gamma))$ is grouplike,

and its strut-part is $\left[\frac{Lk_B(\gamma)}{2}\right]$.

§ bottom-top tangles & Lagrangian cobordism

· bottom-top tangle

For $g \geq 1$, take $(2g)$-points in $[0,1]^2$ as below.

$[0,1]^2 =$ 

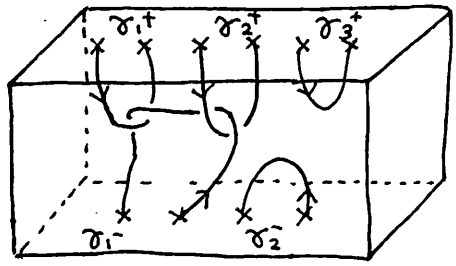
(B, γ) is a bottom-top tangle of type (g_+, g_-)

$\Leftrightarrow$ $B = (B, b)$ is a cobordism from $[0,1]^2$ to $[0,1]^2$, and

$\gamma = (\gamma^+, \gamma^-)$ is a framed oriented tangle s.t.

γ_j^+ runs from $p_j \times \{1\}$ to $q_j \times \{1\}$.

γ_j^- $\quad q_j \times \{-1\}$ $\quad p_j \times \{-1\}$.

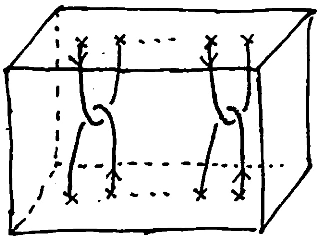


type $(3,2)$

composition & tensor product

(B, γ) : bottom-top tangle of type (g_+, g_-),

(C, ν) : $\quad (h_+, h_-)$, $g_+ = h_-$.

$T_g =$  $\subset [0,1]^3$.

$$(B,\gamma)\circ(C,\nu) = \begin{array}{|c|}\hline \nu \subset C \\ \hline T_{g_+} \subset [-1,1]^3 \\ \hline \gamma \subset B \\ \hline \end{array}$$ & surgery along $\gamma^+ \cup T_{g_+} \cup \nu^-$.

Category of bottom-top tangles ${}^t_b\mathcal{T}$

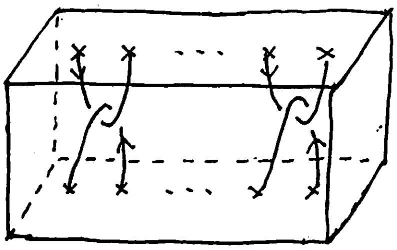
${}^t_b\mathcal{T}$: the (strict monoidal) category of bottom-top tangles

object : $\mathbb{Z}_{\geq 0}$,

morphism : $(B, b) \in {}^t_b\mathcal{T}(g_+, g_-)$: bottom-top tangle of type (g_+, g_-).

· identity

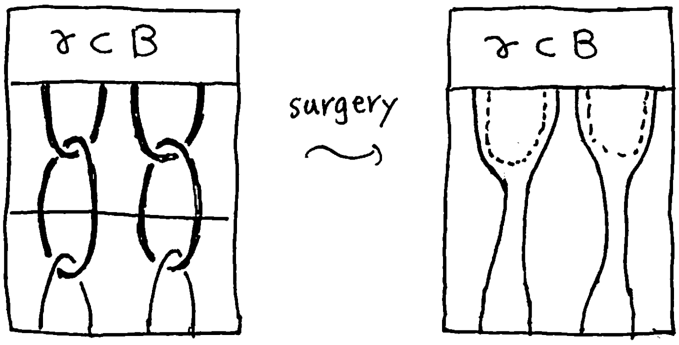
$\mathrm{id}_g =$



$\in {}_b^t\mathcal{T}(g,g)$

☺

$(B,\gamma)\circ \mathrm{id}_g =$



$= (B,\gamma).$

//

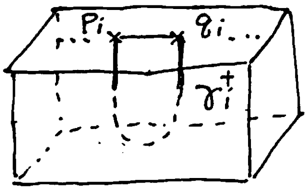
linking matrix

(B,γ) : a bottom-top tangle in a homology cube,

$\hat{B} := B \cup_b (S^3 \setminus [-1,1]^3)$.

$\hat{\gamma}_i^+ = \gamma_i^+ \cup \overline{p_i q_i}$

$\hat{\gamma}_i^- = \gamma_i^- \cup \overline{p_i q_i}$

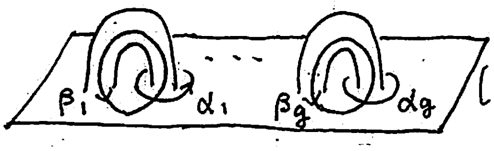


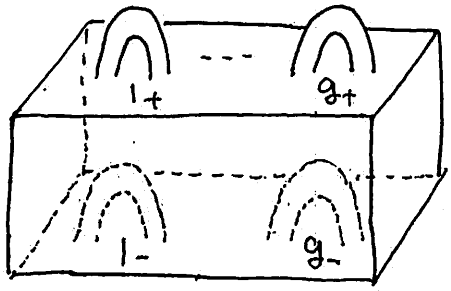
.

Let us define the linking matrix of γ by

$$\mathrm{Lk}_B(\gamma) := \mathrm{Lk}_{\hat{B}}(\hat{\gamma}).$$

- cobordism

$F_g =$  $(g \geq 0)$,

$C_{g_-}^{g_+} =$  $(g_+, g_- \geq 0)$.

(M, m) : a cobordism from F_{g_+} to F_{g_-}

$\Leftrightarrow$ M : cpt conn. ori. 3-mfd ,

$m : \partial C_{g_-}^{g_+} \xrightarrow{\cong} \partial M$ ori.-pres. homeo .

Category of cobordisms Cob

Cob : the (strict monoidal) category of cobordisms

object : $\mathbb{Z}_{\geq 0}$

morphism : $(M, m) \in \mathrm{Cob}(g_+, g_-)$: a cobordism from F_{g_+} to F_{g_-}

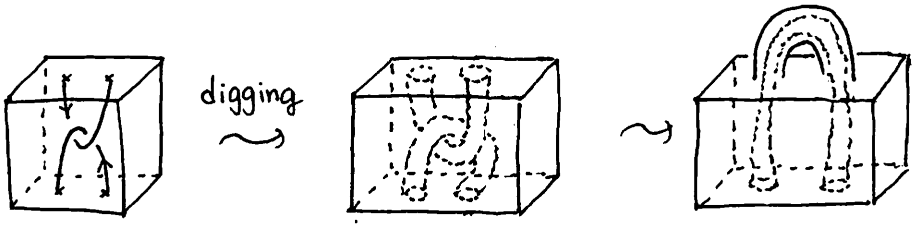
$M \circ N = \begin{array}{|c|}\hline M \\ \hline N \\ \hline\end{array}$, $M \otimes N = \begin{array}{|c|c|}\hline M & N \\ \hline\end{array}$.

Thm 2.10

$^{\exists}D : {}^{t}_{b}\mathcal{T} \longrightarrow \mathrm{Cob}$: isomorphism.

Proof

$(B, \gamma) \in {}^{t}_{b}\mathcal{T}(g_+, g_-)$

$D:$


$\in \mathrm{Cob}(g_+, g_-)$,

where the markings of $F_{g_\pm}$ are given by the meridians and longitudes of tangles.

$(M, m) \in \mathrm{Cob}$.

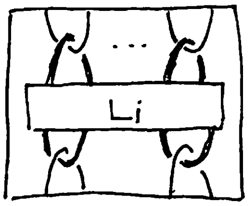
$= F_g$

By attaching 2-hdles along $m_-(\alpha_i)$ and $m_+(\beta_j)$ $(1 \leq i \leq g_-, 1 \leq j \leq g_+)$, we obtain a cobordism from $[0,1]^2$ to $[0,1]^2$, and a tangle comes from the cocores of these handles.

This construction gives the inverse of D.

- functoriality

$(B, \gamma) \in {}^{t}_{b}\mathcal{T}(g_+, g_-)$ and $(C, v) \in {}^{t}_{b}\mathcal{T}(h_+, \underset{\substack{\shortparallel \\ g_+}}{h_-})$ can be written as the form

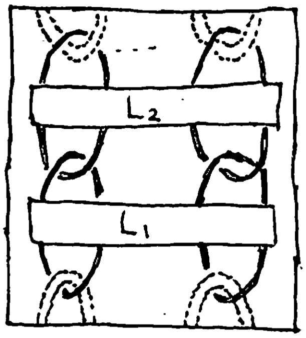


$(i = 1, 2)$,

where thick lines and Li are links along which surgeries applied.

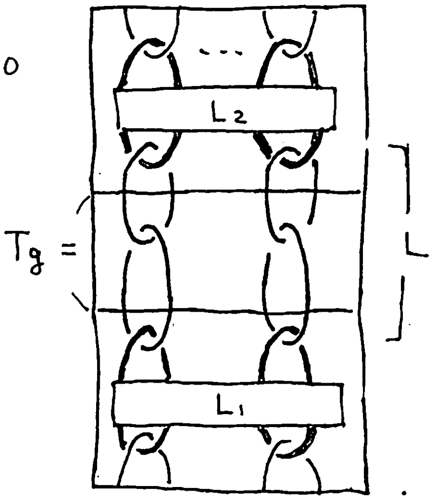
Then, the composition of corr. 3-mfds M, N is

$M \circ N =$



,

and it corresponds to



& surgery along L. //

<u>Lagrangian cobordism</u>

$A_g = \langle \alpha_1, \alpha_2, \cdots, \alpha_g \rangle \subset H_1(F_g)$,

$B_g = \langle \beta_1, \beta_2, \cdots, \beta_g \rangle$.

$(M, m) \in \mathrm{Cob}(g_+, g_-)$ is Lagrangian

$\Leftrightarrow$ (1) $m_- \oplus m_+ : A_{g_-} \oplus B_{g_+} \to H_1(M)$ is an isomorphism,

(2) $m_+(A_{g_+}) \subset m_-(A_{g_-})$ in $H_1(M)$.

<u>Rem</u>

(1) $\iota : MCG(F_g) \to \mathrm{Cob}(g, g)$

$h \mapsto (F_g \times [-1,1], (\mathrm{Id} \times \{-1\}) \cup (h \times \{1\}))$

is an injective homo.

(2) $\iota(h)$ is Lagrangian $\Leftrightarrow$ $h(A_g) = A_g$.

Lemma 2.12

$(B, \gamma) \in {}^{+}_{b}\mathcal{T}(g_+, g_-)$.

$D(B, \gamma)$ is Lagrangian $\Leftrightarrow$ $\mathrm{Lk}(\gamma^+)$ is trivial, and B is a homology cube.

Proof

$(M, m) := D(B, \gamma)$.

Since $B = M \cup \left(\coprod_{i=1}^{g} N(\gamma_i^+) \cup \coprod_{i=1}^{g} N(\gamma_i^-) \right)$, we have

$$H_2(B) \to H_1\left(\coprod_{i=1}^{g_+} \partial N(\gamma_i^+) \cup \coprod_{i=1}^{g_-} \partial N(\gamma_i^-)\right)$$

$$\to H_1(M) \oplus \underbrace{H_1\left(\coprod_{i=1}^{g_+} N(\gamma_i^+) \cup \coprod_{i=1}^{g_-} N(\gamma_i^-)\right)}_{\substack{\| \\ 0}} \to H_1(B).$$

$\therefore$ $H_1(B) = H_2(B) = 0$.

$\Leftrightarrow$ $H_1\left(\coprod_{i=1}^{g_+} \partial N(\gamma_i^+) \cup \coprod_{i=1}^{g_-} \partial N(\gamma_i^-)\right) \cong H_1(M)$

(i.e. $H_1(M)$ is freely generated by the meridians of the tangle $\gamma_i^{\pm}$)

$\Leftrightarrow$ $H_1(M) = \langle \alpha_1^-, \cdots, \alpha_g^-, \beta_1^+, \cdots, \beta_g^+ \rangle$

$\Leftrightarrow$ $H_1(M) = m_-(A_{g_-}) \oplus m_+(B_{g_+})$. $\cdots (*)$

On the other hand,

$\mathrm{Lk}(\gamma^+)$ is trivial

$\Leftrightarrow$ $\mathrm{lk}(\alpha_i^+, \alpha_j^+) = 0$ $(1 \leq i \leq j \leq g_+)$.

When $(*)$ is satisfied, $\mathrm{lk}(x, \alpha_1^+) = \cdots = \mathrm{lk}(x, \alpha_{g_+}^+) = 0 \Leftrightarrow x \in m_-(A_{g_-})$.

Thus, $\mathrm{lk}(\alpha_i^+, \alpha_j^+) = 0 \Leftrightarrow m_+(A_{g_+}) \subset m_-(A_{g_-})$. //

Lagrangian q-cobordism

$(M, m, w_t(M), w_b(M))$ is a Lagrangian q-cobordism

$\Leftrightarrow$ (M, m) is a Lagrangian cobordism from F_g to F_f, and $w_t(M)$ and $w_b(M)$ are non-ass. words of length g and f in the single letter $\bullet$.

Category of Lagrangian q-cobordisms $\mathcal{L}Cob_q$

$\mathcal{L}Cob_q$: the (non-strict monoidal) category of Lagrangian q-cobordisms

object : the free non-ass. magma gen. by $\bullet$,

morphism : $(M, m, \underset{u}{\underset{\shortparallel}{w_t(M)}}, \underset{v}{\underset{\shortparallel}{w_b(M)}}) \in \mathcal{L}Cob(u, v)$,

where (M, m) is a Lagrangian cobordism from $|u|$ to $|v|$.

Rem

$D^{-1} : Cob \xrightarrow{\cong} {}_b^t\mathcal{T}$ induces a functor

$$\begin{array}{ccc} \mathcal{L}Cob_q & \longrightarrow & \mathcal{T}_qCub \\ \bullet & \longmapsto & (+-) \\ (M, m) & \longmapsto & D^{-1}(M, m) \end{array} .$$

$Z(M)$

For a Lagrangian q-cobordism (M, m) from F_g to F_f,

we denote $Z(M) = Z(B, \gamma) \in \mathcal{A}(\gamma) = \mathcal{A}(\curvearrowright^{\lfloor g \rceil} \curvearrowleft^{\lfloor f \rceil}, \emptyset)$

where $\lfloor g \rceil = \{1, 2, \cdots, g\}$.

Rem

By $\underbrace{\text{Lemma 2.12}}_{\substack{Lk(\gamma^+) \\ \text{is trivial}}}$ and $\underbrace{\text{Lemma 3.17}}_{\substack{\text{Strut-part} \\ \text{is } \left[\frac{Lk_B(\gamma)}{2}\right]}}$, $\chi^{-1} Z(M) \in \mathcal{A}(\emptyset, \lfloor g \rceil^+ \cup \lfloor f \rceil^-)$

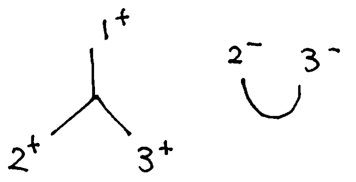
is $\lfloor g \rceil^+$-substantial.

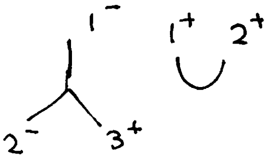
§ LMO functor $\tilde{Z} : \mathcal{L}\mathrm{Cob}_q \to {}^{ts}\mathcal{A}$

top-substantial

$f, g \geq 0$.

A Jacobi diagram based on $\mathcal{A}(\emptyset, [g]^+ \cup [f]^-)$ is top-substantial if it is $[g]^+$-substantial.

○ 

× 

${}^{ts}\mathcal{A}$

The (strict monoidal) category of top-substantial diagrams.

object : $\mathbb{Z}_{\geq 0}$

morphism : ${}^{ts}\mathcal{A}(g, f)$: the subspace of $\mathcal{A}(\emptyset, [g]^+ \cup [f]^-)$ spanned by top-substantial diagrams.

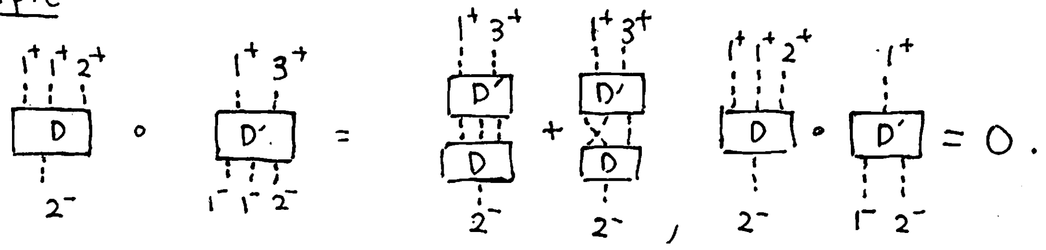
composition & tensor product

$x \in {}^{ts}\mathcal{A}(g, f)$, $y \in {}^{ts}\mathcal{A}(h, g)$

$$x \circ y = \langle (x/i^+ \mapsto i^*), (y/i^- \mapsto i^*) \rangle_{[g]^*}$$

$$= \left(\begin{array}{l} \text{sum of all ways of gluing the } i^+\text{-colored vertices of } x \\ \text{to the } i^-\text{-colored vertices of } y \text{ for all } i = 1, \cdots, g \end{array} \right).$$

Example



identity $\mathrm{id}_g = [\, \overset{1^+}{\underset{1^-}{\vdots}} + \cdots + \overset{g^+}{\underset{g^-}{\vdots}} \,]$.

Rem

For a Lagrangian q-cobordism (M, m) from F_g to F_f, $\chi^{-1} Z(M) \in \mathcal{A}(\emptyset, [g]^+ \cup [f]^-)$ is in ${}^{ts}\mathcal{A}(g, f)$.

Lemma 4.5

$a \in {}^{ts}\mathcal{A}(g, f)$, $b \in {}^{ts}\mathcal{A}(h, g)$ s.t.

$a = [\frac{A}{2}] \amalg a^{\Upsilon}$ and $b = [\frac{B}{2}] \amalg b^{\Upsilon}$,

where A is a symmetric $(\lfloor g \rceil^{+} \cup \lfloor f \rceil^{-}) \times (\lfloor g \rceil^{+} \cup \lfloor f \rceil^{-})$ matrix

and B $(\lfloor h \rceil^{+} \cup \lfloor g \rceil^{-}) \times (\lfloor h \rceil^{+} \cup \lfloor g \rceil^{-})$ matrix

of the form

$$A = \begin{pmatrix} 0 & A^{+-} \\ A^{-+} & A^{--} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B^{+-} \\ B^{-+} & B^{--} \end{pmatrix},$$

Then, $a \circ b = \left[\frac{1}{2} \begin{pmatrix} 0 & B^{+-} A^{+-} \\ A^{-+} B^{-+} & A^{--} + A^{-+} B^{--} A^{+-} \end{pmatrix} \right] \amalg (a^{\Upsilon} \overset{A,B}{\star} b^{\Upsilon})$,

where

$$x \overset{A,B}{\star} y = \langle (x /_{i^{+} \mapsto i^{*} + B^{+-} i^{-} + A^{-+} B^{--} i^{-}),$$
$$([B^{--}/2] /_{i^{-} \mapsto i^{*}}) \amalg (y /_{i^{-} \mapsto i^{*} + A^{-+} i^{+}) \rangle_{\lfloor g \rceil^{*}}$$

Rem

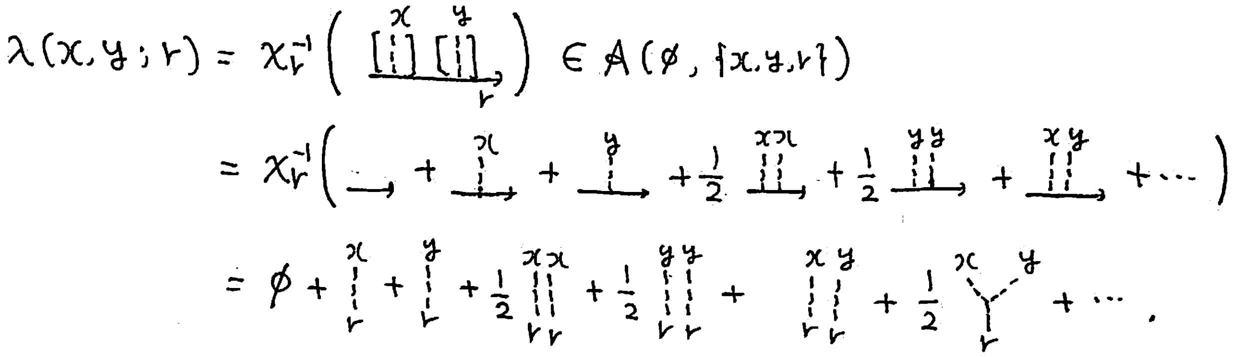
When $f = g = h$ and $A = B = \begin{pmatrix} 0 & I_g^{+-} \\ I_g^{-+} & 0 \end{pmatrix}$,

$$x \star y = \langle (x /_{i^{+} \mapsto i^{*} + i^{+}}), (y /_{i^{-} \mapsto i^{*} + i^{-}}) \rangle_{\lfloor g \rceil^{*}}.$$

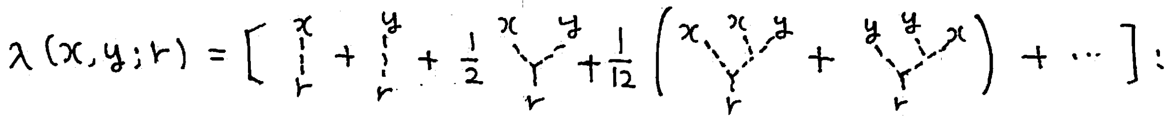
Rem

If a and b is grouplike, $a \circ b$ is grouplike as well.

$\lambda(x,y;r)$



Lemma

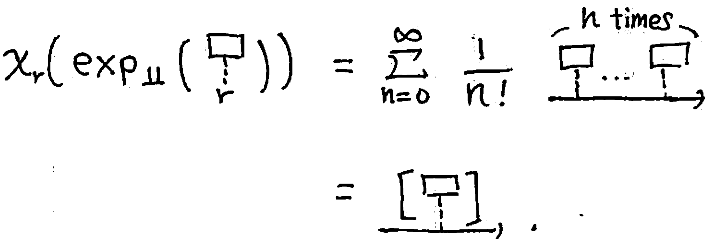


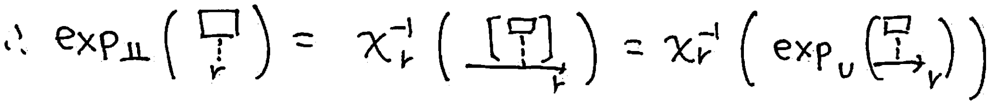
Baker-Campbell-Hausdorff series.

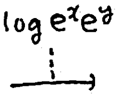
Thus, $\lambda(x,y;r)$ is grouplike.

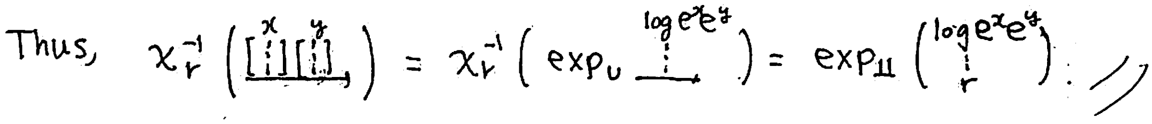
Proof

For a connected tree with one r-leg  $\in \mathcal{A}(\emptyset,\{x,y,r\})$,



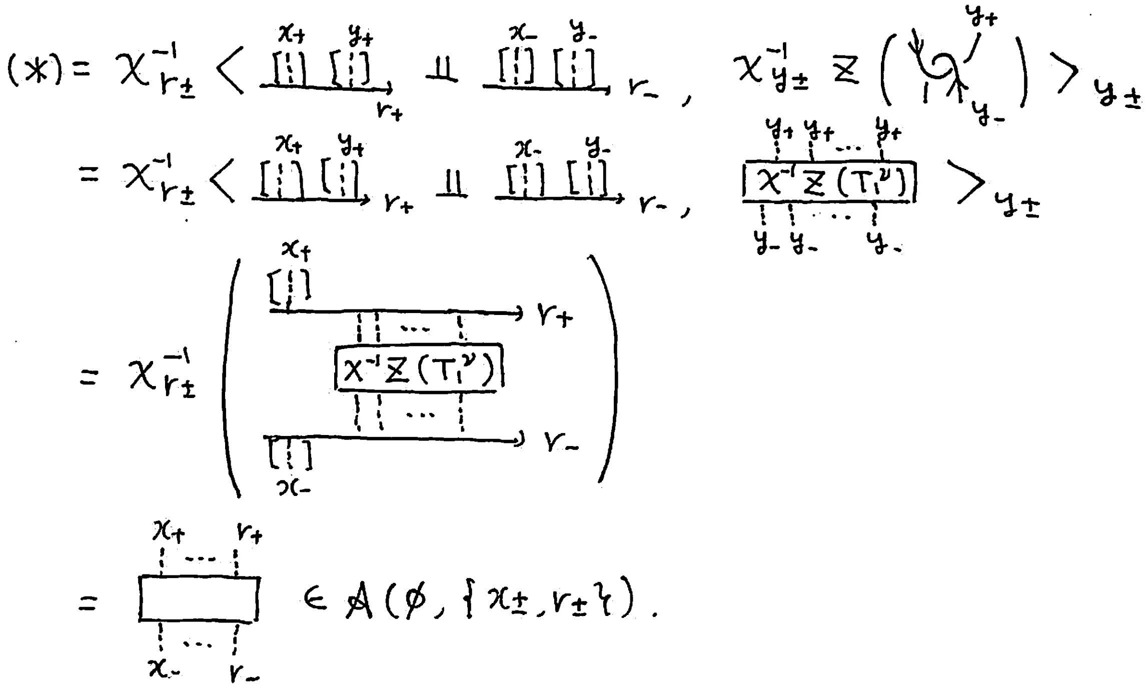


The element  is a linear combination of conn. diagrams.



- $\underline{\Pi(x_+,x_-)}$

$$\Pi(x_+,x_-)=\mho_+^{-1}\sqcup\mho_-^{-1}\sqcup\int_{r_\pm}\underbrace{\langle\lambda(x_-,y_-;r_-)\sqcup\lambda(x_+,y_+;r_+),\chi^{-1}Z(T_1^\nu)\rangle_{y_\pm}}_{(*)}$$

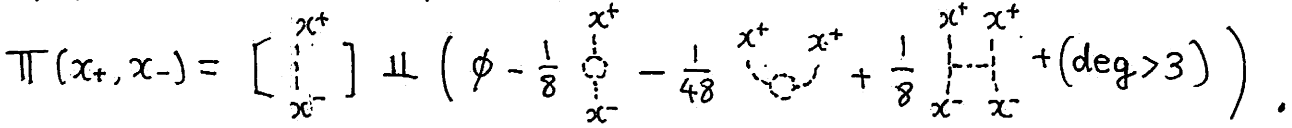


Lemma 4.9

$\Pi(x_+,x_-)$ is grouplike in $\mathcal{A}(\emptyset,\{x_+,x_-\})$, and its strut part is $[\,\overset{x_+}{\underset{x_-}{\vdots}}\,]$.

Rem

By straightforward computations, we have



Proof of Lemma 4.9

- strut part

The strut part of $\chi^{-1}Z(T_1^{\nu})$ coincides with the linking matrix $\left[-\overset{y_+}{\underset{y_-}{\vdots}}\right]$,

$\lambda(x,y;r)$ is, by definition, $\left[\overset{x}{\underset{r}{\vdots}}+\overset{y}{\underset{r}{\vdots}}\right]$.

($\chi^{-1}Z(T_1^{\nu})$ and $\lambda(x,y;r)$ are grouplike)

$$(*) = \left\langle \left[\overset{x_-}{\underset{r_-}{\vdots}}+\overset{y_-}{\underset{r_-}{\vdots}}+\overset{x_+}{\underset{r_+}{\vdots}}+\overset{y_+}{\underset{r_+}{\vdots}}\right] \amalg (\text{Y-part}),\ \left[-\overset{y_+}{\underset{y_-}{\vdots}}\right] \amalg (\text{Y-part}) \right\rangle_{y_\pm}$$

$$= \left[\overset{x_-}{\underset{r_-}{\vdots}}+\overset{x_+}{\underset{r_+}{\vdots}}\right] \amalg \left[-\overset{\frown}{r_- \quad r_+}\right] \amalg (\text{Y-part})$$

$$= \left[\overset{x_-}{\underset{r_-}{\vdots}}+\overset{x_+}{\underset{r_+}{\vdots}}-\overset{r_+}{\underset{r_-}{\vdots}}\right] \amalg (\text{Y-part}).$$

$$\therefore \int_{r_\pm}(*) = \left\langle \left[\overset{r_+}{\underset{r_-}{\vdots}}\right],\ \left[\overset{x_-}{\underset{r_-}{\vdots}}+\overset{x_+}{\underset{r_+}{\vdots}}\right] \amalg (\text{Y-part}) \right\rangle_{r_\pm}$$

$$= \left[\overset{x_+}{\underset{x_-}{\vdots}}\right] \amalg (\text{Y-part}). \quad //$$

- grouplike

Since χ preserves coproducts, $\chi^{-1}Z(T_1^{\nu})$ is grouplike, and $\lambda(x,y;r)$ is also grouplike by definition.

(Thm 3.6 (JMM)

$D, E \in \mathcal{A}(\emptyset, C \cup S)$: grouplike.

If D or E is S-substantial, $\langle E, D\rangle_S$ is grouplike.)

Since $\lambda(x_+, y_+; r_+) \amalg \lambda(x_-, y_-; r_-)$ is $\{y_\pm\}$-substantial, $(*)$ is also grouplike by Thm 3.6.

$\int_{r_\pm}(*) = \left\langle \left[\overset{r_+}{\underset{r_-}{\vdots}}\right],\ \left[\overset{x_-}{\underset{r_-}{\vdots}}+\overset{x_+}{\underset{r_+}{\vdots}}\right] \amalg (\text{Y-part}) \right\rangle_{r_\pm}$ is also grouplike by Thm 3.6. //

Π_g

For $g \geq 0$, we denote

$$\Pi_g = \pi(1^+, 1^-) \sqcup \cdots \sqcup \pi(g^+, g^-)$$
$$= \nu_+^{-g} \sqcup \nu_-^{-g} \sqcup \int_r \left\langle \coprod_{i=1}^{g} \lambda(x_i^-, y_i^-; r_i^-) \sqcup \coprod_{i=1}^{g} \lambda(x_i^+, y_i^+; r_i^+), \chi^{-1} Z(T_g^\nu) \right\rangle_y$$
$$\in \mathcal{A}(\emptyset, \lfloor g \rceil^+ \cup \lfloor g \rceil^-)$$

Rem

Π_g is a grouplike element, and the strut part is $\mathrm{Id}_g = \left[\sum_{i=1}^{g} \begin{smallmatrix} i^+ \\ \vdots \\ i^- \end{smallmatrix} \right]$.

Lemma 4.10

$M \in \mathcal{L}Cob_q(w_1, w_2)$,

$N \in \mathcal{L}Cob_q(w_2, w_3)$.

Then, $\chi^{-1} Z(M \circ N) = \chi^{-1} Z(M) \circ \Pi_g \circ \chi^{-1} Z(N)$.

Proof

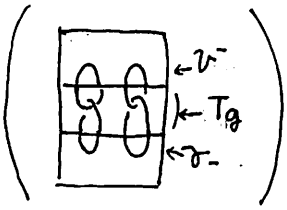
(B, γ) : the bottom-top q-tangle corr. to M,

(C, ν) : 〃 N.

(K, γ) : a surgery presentation of (B, γ),

(L, ν) 〃 (C, ν).

$M \circ N = \begin{array}{|c|} \hline \nu \cup L \\ \hline T_g \\ \hline \gamma \cup K \\ \hline \end{array}$ & surgery along $K \cup \underbrace{\gamma^+ \cup T_g \cup \nu^-}_{\overset{\shortparallel}{T}} \cup L$.



Step 1 $\sigma_\pm(K \cup T \cup L) = \sigma_\pm(K) + \sigma_\pm(L) + g$.

∵ $\mathrm{Lk}(K \cup T \cup L)$

$$= \begin{array}{cccc} \left(\begin{array}{c} \mathrm{Lk}(L) \\ \mathrm{Lk}(\nu^-, L) \\ 0 \\ 0 \end{array} \right. & \begin{array}{c} \mathrm{Lk}(L, \nu^-) \\ \mathrm{Lk}(\nu^-) \\ -I_g \\ 0 \end{array} & \begin{array}{c} 0 \\ -I_g \\ \mathrm{Lk}(\gamma^+) \\ \mathrm{Lk}(K, \gamma^+) \end{array} & \left. \begin{array}{c} 0 \\ 0 \\ \mathrm{Lk}(\gamma^+, K) \\ \mathrm{Lk}(K) \end{array} \right) \\ \underbrace{\hphantom{\mathrm{Lk}(\nu^-, L)}}_{L} & \multicolumn{2}{c}{\underbrace{\hphantom{\mathrm{Lk}(L, \nu^-) \quad \mathrm{Lk}(K, \gamma^+)}}_{T}} & \underbrace{\hphantom{\mathrm{Lk}(\gamma^+, K)}}_{K} \end{array}.$$

Denote

$$P = \begin{pmatrix} I_\ell & O & O & O \\ -Lk(v^-, L)Lk(L)^{-1} & I_g & O & O \\ O & O & I_g & -Lk(\gamma^+, K)Lk(K)^{-1} \\ O & O & O & I_K \end{pmatrix}.$$

Then, we have

$$P \cdot Lk(K \cup T \cup L) \cdot P^{-1} = \begin{pmatrix} Lk(L) & O & O & O \\ O & X_- & -I_g & O \\ O & -I_g & X_+ & O \\ O & O & O & Lk(K) \end{pmatrix},$$

where $X_- = Lk(v^-) - Lk(v^-, L) \cdot Lk(L)^{-1} \cdot Lk(L, v^-)$,

$X_+ = Lk(\gamma^+) - Lk(\gamma^+, K) \cdot Lk(K)^{-1} Lk(K, \gamma^+)$.

A homological argument shows $X_+ = Lk_{[-1,1]_K}(\gamma^+)$,

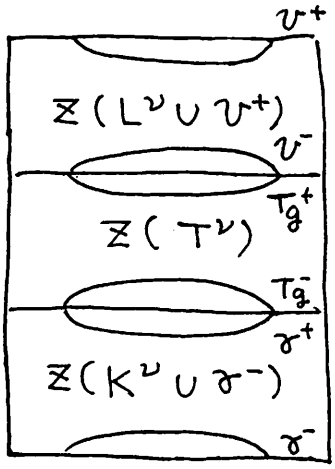
and by Lemma 2.12, $Lk_{[-1,1]_K}(\gamma^+) = 0$.

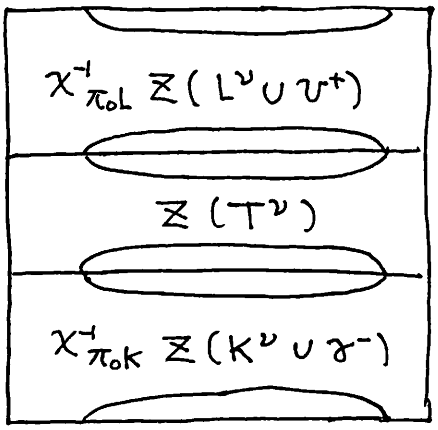
Thus, we obtain $\sigma_\pm(K \cup T \cup L) = \sigma_\pm(K) + \sigma_\pm(L) + g$. //

Step 2

$$Z(M \circ N) = \overbrace{v_+^{-\sigma_+(K \cup T \cup L)}}^{\substack{v_+^{-\sigma_+(K)-\sigma_+(L)-g} \\ \shortparallel}} \sqcup v_-^{-\sigma_-(K \cup T \cup L)}$$

$$\sqcup \int_{\pi_0(K \cup T \cup L)} \underbrace{\chi^{-1}_{\pi_0(K \cup T \cup L)} Z\left((K \cup T \cup L)^\nu \cup (\gamma^- \cup v^+)\right)}_{(*)}.$$

$(*) = \chi^{-1}_{\pi_0 T}\, \chi^{-1}_{\pi_0 K}\, \chi^{-1}_{\pi_0 L}$ 

$= \chi^{-1}_{\pi_0 T}$ 

,

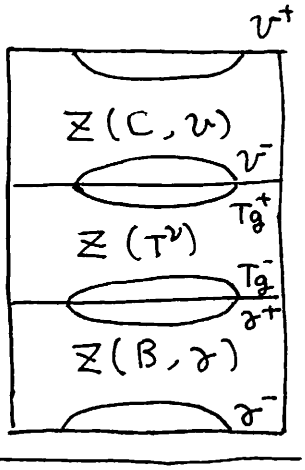
It is known that $\int_{\pi_0(K\cup T\cup L)} = \int_{\pi_0 T} \int_{\pi_0 K} \int_{\pi_0 L}$ ([BGRT2] Prop 2.11)

when $Lk_{[-1,1]^3}(K)$ and $Lk_{[-1,1]^3}(L)$ are invertible.

Recall that $Z(B,\gamma) = U_+^{-\sigma_+(L)} \amalg U_-^{-\sigma_-(L)} \amalg \int_{\pi_0 L} \chi^{-1}_{\pi_0 L} Z(L^\nu \cup \gamma)$,

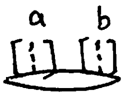
$Z(C,v) = U_+^{-\sigma_+(K)} \amalg U_-^{-\sigma_-(K)} \amalg \int_{\pi_0 K} \chi^{-1}_{\pi_0 K} Z(K^\nu \cup \gamma)$.

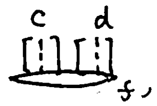
Thus, we have

$Z(M \circ N) = \quad U_+^{-g} \amalg U_-^{-g} \amalg \int_{\pi_0 T} \chi^{-1}_{\pi_0 T}$ 

.

(∗′)

$(*') = \int_{\{e,f\}} \chi^{-1}_{\{e,f\}} \langle$ 

$\amalg$ 

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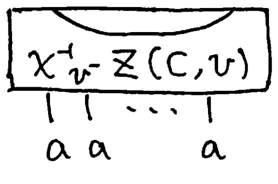
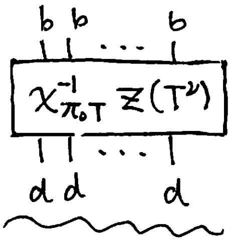
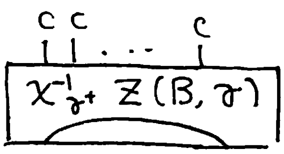


$\amalg$ 

$\amalg$ 

$\rangle$

$$= \left\langle \int_{\{e,f\}} \langle \lambda(a,b;e) \amalg \lambda(c,d;f),\ \chi^{-1}_{\pi_0 T} Z(T^\nu) \rangle_{b,d},\right.$$

$$\left. \chi^{-1}_{v^-} Z(C,v) \amalg \chi^{-1}_{\gamma^+} Z(B,\gamma) \right\rangle_{a,c}$$

Set $T_g = U_+^{-g} \amalg U_-^{-g} \amalg \int_{\{e,f\}} \langle \lambda(a,b;e) \amalg \lambda(c,d;f),\ \chi^{-1}_{\pi_0 T} Z(T^\nu) \rangle_{b,d}$.

Then, we have $\chi^{-1} Z(M \circ N) = \chi^{-1} Z(M) \circ T_g \circ \chi^{-1} Z(N)$.

Lemma 4.12

M : a Lagrangian $\mathbb{Q}$-cobordism from F_g to F_f.

$\tilde{Z}(M) = \chi^{-1}_{\pi_0\gamma}\, Z(M) \circ \mathbb{T}_g \in \mathcal{A}(\emptyset, [g]^+ \cup [f]^-)$

is grouplike and its strut part is $\left[\frac{Lk(M)}{2}\right]$.

Proof

(B, γ) : a bottom-top $\mathbb{Q}$-tangle in a homology cube corr. to M.

$\tilde{Z}(M) = \chi^{-1} Z(B, \gamma) \circ \mathbb{T}_g$.

$\chi^{-1} Z(B, \gamma)$ is grouplike by Lemma 3.17, and

$\mathbb{T}_g$ is 〃 by Lemma 4.9.

By Lemma 4.5, the composition $\chi^{-1} Z(B, \gamma) \circ \mathbb{T}_g$ is grouplike. //

Thm 4.13

$\tilde{Z} : \mathcal{L}Cob_{\mathbb{Q}} \longrightarrow {}^{ts}\mathcal{A}$ is a tensor-preserving functor.

Proof

w : non-ass. word of length g.

By Lemma 4.12, $\tilde{Z}$ preserves the composition law.

For $M \in \mathcal{L}Cob_{\mathbb{Q}}(u, u')$ and

$$\begin{aligned}
\tilde{Z}(M \otimes N) &= \chi^{-1} Z(\boxed{M|N}) \circ \mathbb{T}_{g+h} \\
&= \chi^{-1}\left(Z(M) \otimes Z(N)\right) \circ \mathbb{T}_{g+h} \\
&= \left(\chi^{-1} Z(M) \otimes \chi^{-1} Z(N)\right) \circ \left(\mathbb{T}_g \otimes \mathbb{T}_h\right) \\
&= \tilde{Z}(M) \otimes \tilde{Z}(N).
\end{aligned}$$

Since $\tilde{Z}(Id_w)$ is grouplike, we denote $\tilde{Z}(Id_w) = \left[\sum_{i=1}^{g} \overset{i^+}{\underset{i^-}{\vdots}}\right] \sqcup \underbrace{\tilde{Z}^{Y}(Id_w)}_{Y\text{-part}}$.

We denote $\tilde{Z}(Id_w) = \emptyset + T + (i\text{-deg} > k)$.

Then, by the equation $\tilde{Z}(Id_w) = \tilde{Z}(Id_w \circ Id_w) = \tilde{Z}(Id_w) \circ \tilde{Z}(Id_w)$,

we have $T = 2T$, i.e. $T = \emptyset$.

Thus, we obtain $\tilde{Z}^{Y}(Id_w) = \emptyset$.

§ Mapping class group & LMO functor

The monoid $\mathrm{Cyl}(F_g)$ of homology cylinders

(M, m) is a homology cylinder

$\Leftrightarrow$ (M, m) is a cobordism from F_g to F_g

s.t. $m_\pm : H_1(F_g) \to H_1(M)$ is an isom

and $m_+ = m_-$.

Rem

$\mathcal{I}(F_g) := \mathrm{Ker}(\mathrm{MCG}(F_g) \to \mathrm{Aut}\, H_1(F_g))$: Torelli group.

$$\begin{array}{ccl} \mathcal{I}(F_g) & \hookrightarrow & \mathrm{Cyl}(F_g) \qquad\qquad\qquad\qquad \subset \mathcal{L}\mathrm{Cob}(g, g). \\ h & \longmapsto & (F_g \times [-1,1],\ (\mathrm{Id} \times \{-1\}) \cup (h \times \{1\})) \end{array}$$

A choice of non-ass word, e.g. $w = \underbrace{((\cdots((\bullet\bullet)\bullet)\cdots)\bullet)\bullet}_{g}$,

gives a monoid homo. $\mathcal{I}(F_g) \xrightarrow{\iota} \mathcal{L}\mathrm{Cob}_q(w, w) \xrightarrow{\tilde{Z}} {}^{ts}\!\mathcal{A}(g, g)$.

Lemma

$$M \in \mathrm{Cyl}(F_g) \Leftrightarrow \mathrm{Lk}(M) = \begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}.$$

Denote

$$\tilde{Z}^Y : \mathcal{I}(F_g) \longrightarrow {}^{ts}\!\mathcal{A}(g, g) \xrightarrow{Y\text{-part}} \mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-).$$

Rem

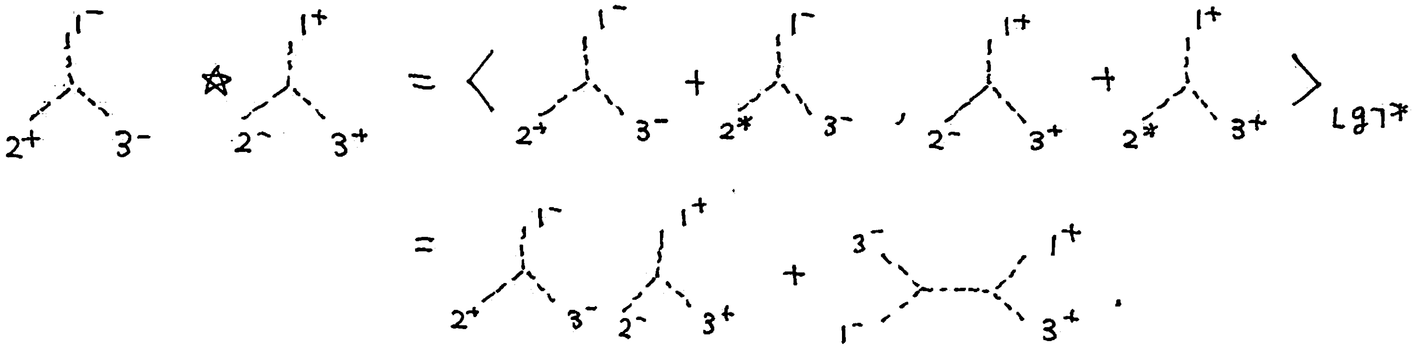
For $D, E \in \mathrm{Cyl}(F_g)$, $\tilde{Z}(D) = \left[\overset{1^+}{\underset{1^-}{\vdots}} + \cdots + \overset{g^+}{\underset{g^-}{\vdots}} \right] \sqcup \tilde{Z}^Y(D)$,
$\tilde{Z}(E) =$ 〃 $\sqcup \tilde{Z}^Y(E)$.

Then,

$$\begin{aligned} \tilde{Z}(D \circ E) &= \tilde{Z}(D) \circ \tilde{Z}(E) \in {}^{ts}\!\mathcal{A}(g, g) \\ &= \left[\overset{1^+}{\underset{1^-}{\vdots}} + \cdots + \overset{g^+}{\underset{g^-}{\vdots}} \right] \sqcup \tilde{Z}^Y(D) \star \tilde{Z}^Y(E), \end{aligned}$$

where $x \star y = \left\langle \left(x/_{i^+ \mapsto i^* + i^+} \right), \left(y/_{i^- \mapsto i^* + i^-} \right) \right\rangle_{\lfloor g \rceil^*}$.

Example



Thm 8.18

$M \in \mathcal{C}yl(F_g)$.

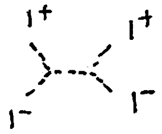
If $\tau_n(M)$ is its first nonvanishing Johnson homo.,

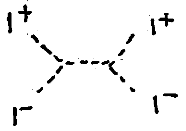
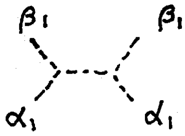
then the tree part of $\tilde{Z}^Y(M)$ is equal to

$\emptyset + \tau_n(M) + (\text{i-deg} > n)$.

Example

$F_1 =$  .

$\tilde{Z}^Y(t_c) = \emptyset - \frac{1}{2}$  $+ \frac{1}{2}$  $+ (\text{i-deg} \geq 3)$.

$\frac{1}{2}$  $\longleftrightarrow -\frac{1}{2}$  $\longleftrightarrow \tau_2(t_c)$.

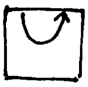
Cor 8.22

$\tilde{Z}^Y : \mathcal{I}(F_g) \to \{\text{units of } (\mathcal{A}^Y(\lfloor g \rceil^+ \cup \lfloor g \rceil^-), \star)\}$ is injective.

∵ $\underbrace{\bigcap_{n=0}^{\infty} \pi_1 F_g[n]}_{\substack{\text{lower}\\ \text{central}\\ \text{series}}} \underset{\text{known}}{=} \{1\} \sim \bigcap_{n=0}^{\infty} \mathrm{Ker}\,\tau_n = \{id\}$. //

For $M \in \mathcal{C}y\ell(F_g)$ with $w_t(M) = w_b(M) = (\cdots((\bullet\bullet)\bullet)\cdots\bullet)$,

denote $\overline{M} = \varepsilon^{\otimes g} \circ M \circ \eta^{\otimes g} \in \mathcal{L}Cob(\emptyset, \emptyset)$

where $\varepsilon =$  $\in \mathcal{L}Cob(\bullet, \emptyset)$, $\eta =$  $\in \mathcal{L}Cob(\emptyset, \bullet)$.

Thm 8.24

(1) $\widetilde{Z}(\overline{M}) = \left(\widetilde{Z}^Y(M) / i^+ \mapsto 0, i^- \mapsto 0\right) \in \mathcal{A}(\emptyset)$

and Θ-coordinates of $\widetilde{Z}(\overline{M})$ are equal to $\frac{\lambda(\overline{M})}{2}$. (Casson inv.)

(2) For $M, N \in \mathcal{C}y\ell(F_g)$,

$$\lambda(\overline{M \circ N}) = \lambda(\overline{M}) + \lambda(\overline{N}) + 2\,(\text{the } \Theta\text{-coordinate in } \tau_1(M) \star \tau_1(N)).$$

Cor 8.6

$M \in \mathcal{C}y\ell(F_g)$, s.t.

$M \overset{Y_k}{\sim} (F_g \times [-1,1], id \times \{-1\}, id \times \{1\})$

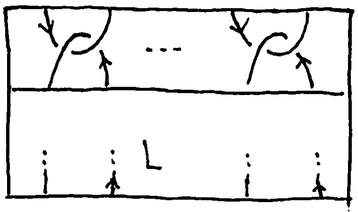
Then, $\widetilde{Z}^Y(M) = \emptyset + (i\text{-deg} \geq k)$.

Cor

For $\varphi \in \mathcal{I}(F_g)[k]$, $\widetilde{Z}^Y(\varphi) = \emptyset + (i\text{-deg} \geq k)$.

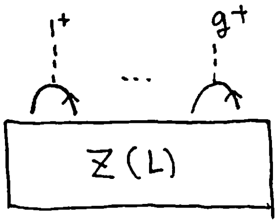
Lemma 5.5

$M \in \mathcal{L}Cob_q(w, v)$ which is presented as follows:

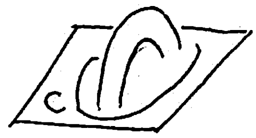
$M =$ 

, where L is a tangle in $[-1,1]^3$ with $w_b(L) = (v/\bullet \mapsto (+-))$, $w_t(L) = (w/\bullet \mapsto (+-))$.

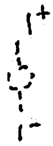
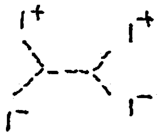
Then,

$\widetilde{Z}(M) = \chi^{-1}\Big($  $\Big)$.

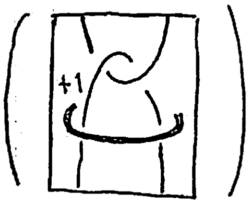
Example

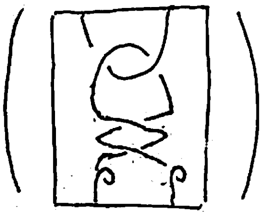


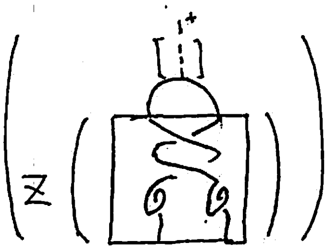
$= F_1.$

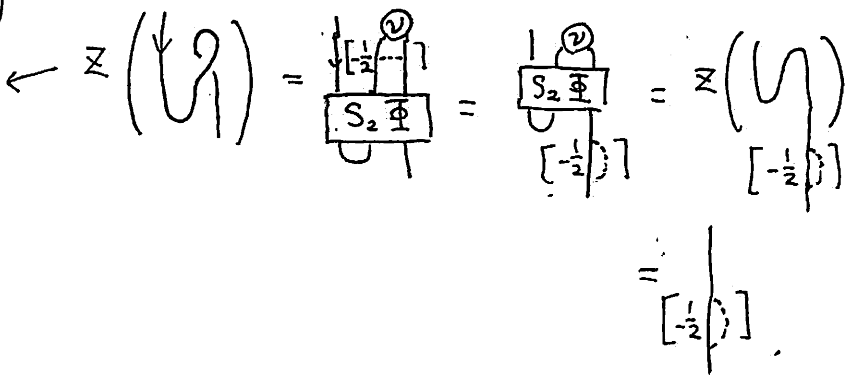
$\tilde{Z}^{Y}(t_c) = \emptyset - \frac{1}{2}$  $+ \frac{1}{2}$  $+ \, (i\text{-deg} \geq 3)$.

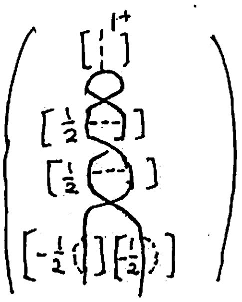
(∵) Step 1

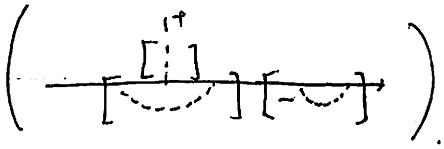
$\tilde{Z}(t_c) = \tilde{Z}$ 

$= \tilde{Z}$ 

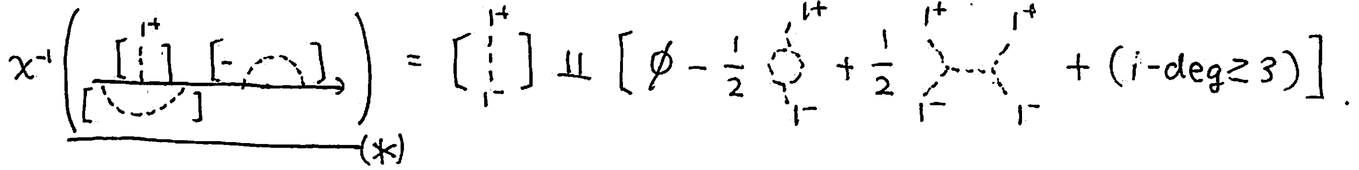
$= \chi^{-1}$ 



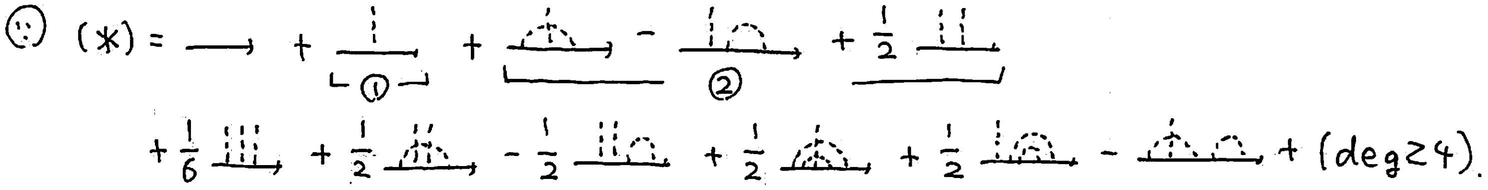
$= \chi^{-1}$ 

$= \chi^{-1}$ 

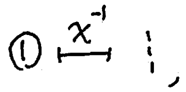
Step2



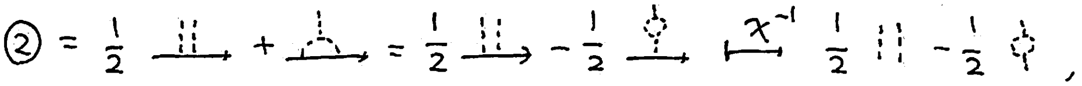
χ^{-1}
1+
1−
(∗)
$\sqcup$
$\emptyset - \frac{1}{2}$
$+ \frac{1}{2}$
$+ (i\text{-deg} \geq 3)$



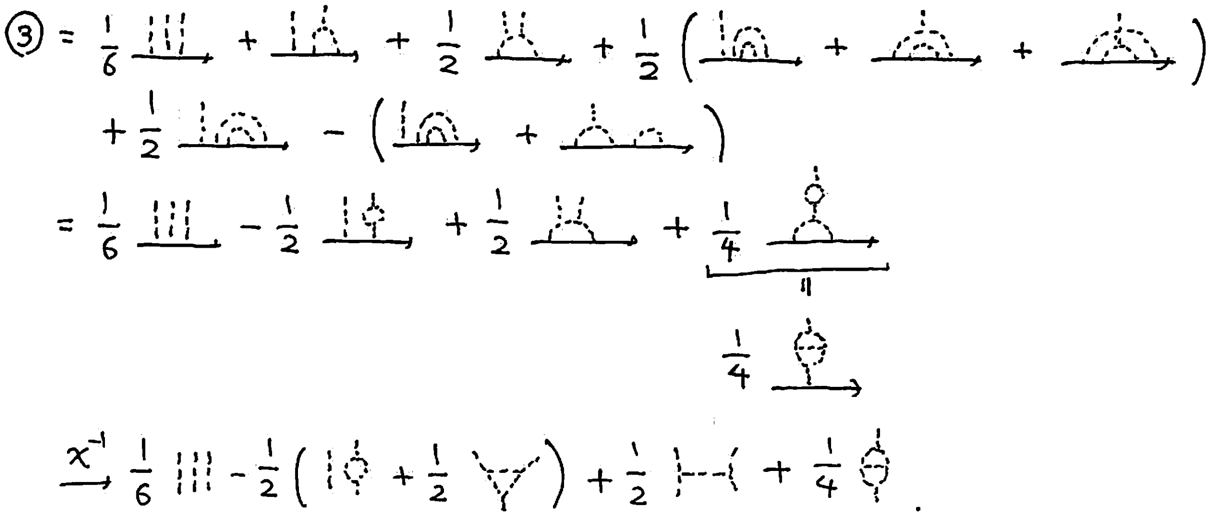
∵ (∗) =
①
②
$\frac{1}{2}$
$\frac{1}{6}$
$+ (\deg \geq 4)$.



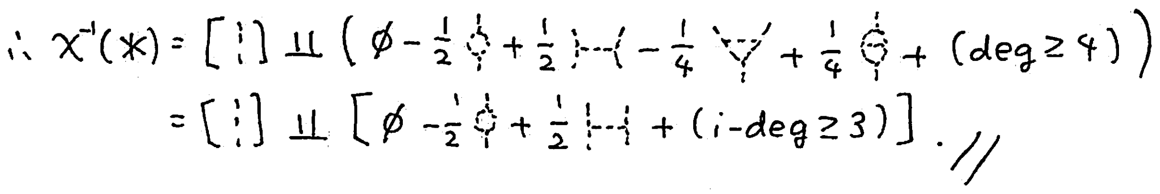
①
χ^{-1}



② =
$\frac{1}{2}$
χ^{-1}



③ =
$\frac{1}{6}$
$\frac{1}{2}$
$\frac{1}{4}$
χ^{-1}



∴ χ^{-1}(∗) =
$\sqcup$
$\emptyset - \frac{1}{2}$
$+ \frac{1}{2}$
$- \frac{1}{4}$
$+ \frac{1}{4}$
$+ (\deg \geq 4)$
$+ (i\text{-deg} \geq 3)$
//