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Long-tailed degree distribution of a random geometric graph constructed by the Boolean model with spherical grains

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Shorttitle: Long-tailed degree distribution of a random geometric graph

Abstract

We consider a random geometric graph constructed by the homogeneous Boolean model with spherical grains in $\mathbb{R}^d$, $d \geq 2$; that is, a node of the graph corresponds to a germ of the Boolean model and there is an edge between two nodes when their grains intersect with each other. We show that, when the radius distribution of grains is long-tailed, so is the degree distribution of the graph. Our result includes as special cases that, if the radius distribution is regularly varying with index $-\alpha$ with $\alpha > d$, then the degree distribution is regularly varying with index $-\alpha/d$ and, in the case of $d = 2$, if the radius distribution is long-tailed with the second moment, then the degree distribution is square-root insensitive. In the proof, a subclass of long-tailed distributions — called x^{1-p}-insensitive distributions with $p \in (0, 1)$ — plays a key role.

Keywords. Boolean models with spherical grains; random geometric graphs; degree distributions; long-tailed distributions; x^{1-p}-insensitive distributions.

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1 Introduction and the result

We consider a homogeneous Boolean model with spherical grains in $\mathbb{R}^d$, $d \in \{2, 3, \dots\}$. Let $\Psi = \{X_i\}_{i\in\mathbb{N}}$ denote a stationary Poisson point process on $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$ and let $\{R_i\}_{i\in\mathbb{N}}$ denote a sequence of i.i.d. random variables on $\mathbb{R}_+$, which is also independent of the Poisson process Ψ. We can see that $\Psi_F = \{(X_i, R_i)\}_{i\in\mathbb{N}}$ forms a marked point process on $\mathbb{R}^d$ with mark space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, where F denotes the distribution of R_i, $i \in \mathbb{N}$. The homogeneous Boolean model with random spherical grains are then given by

$$\Xi = \bigcup_{i\in\mathbb{N}} (X_i + B_0(R_i)),$$

where $B_0(r)$ denotes a closed ball on $\mathbb{R}^d$ centered at $0 \in \mathbb{R}^d$ with radius $r > 0$, and $x+C = \{x+y : y \in C\}$ for $x \in \mathbb{R}^d$ and $C \in \mathcal{B}(\mathbb{R}^d)$. The points of Ψ are also called germs and the balls $B_0(R_i)$, $i \in \mathbb{N}$, are called grains. The Boolean model is known as one of the most important and simplest examples of stochastic geometry and has been studied thoroughly in the literature (see, e.g., Stoyan, Kendall & Mecke [11, Chapter 3] or Baccelli & Błaszczyszyn [3, Chapter 3] for more general Boolean models). We assume $\mathrm{E}({R_1}^d) < \infty$, so that the number of grains $X_i + B_0(R_i)$, $i \in \mathbb{N}$, intersecting with a given compact set in $\mathbb{R}^d$ is almost surely finite (see [3, Example 3.1.3] or Heinrich [8]).

In this short note, we are interested in the connectivity of grains and consider a random geometric graph G_Ξ constructed by the Boolean model Ξ, where node $i \in \mathbb{N}$ of G_Ξ corresponds to the point X_i of Ψ and there is an edge between two nodes i and j $(i \neq j)$ if $(X_i + B_0(R_i)) \cap (X_j + B_0(R_j)) \neq \emptyset$; that is, two grains intersect with each other. We consider the distribution of degrees (the numbers of edges incident to respective nodes) of graph G_Ξ when the radius distribution F of grains is heavy-tailed and we show that, when F is long-tailed, then so is the degree distribution. Our result gives an example of models generating so-called scale-free networks. The class of long-tailed distributions forms the largest operational class of heavy-tailed distributions and is defined as follows (see, e.g., Foss et al. [7, Chapter 2] for more details).

Definition 1 A nonnegative random variable X and its distribution are said to be long-tailed if, for any fixed $a \in \mathbb{R}$,

$$\mathrm{P}(X > x + a) \sim \mathrm{P}(X > x) \quad \text{as } x \to \infty. \tag{1}$$

Here and throughout this note, we use the standard notation $f(x) \sim g(x)$ as $x \to \infty$ for any two real functions f and g on $\mathbb{R}$ satisfying $\lim_{x\to\infty} f(x)/g(x) = 1$. Clearly, if (1) holds for $a > 0$, then so does for $a < 0$, and vice versa.

Due to the stationarity, we can focus on the degree distribution of one node and we consider the Palm version of marked point process Ψ_F; that is, we add a point at the origin with mark R_0 which follows the distribution F independently from Ψ_F (see, e.g., [3, Remark 2.1.7] or Daley & Vere-Jones [4, Example 13.4(a)]). We refer to the node corresponding to the marked point $(0, R_0)$ as node 0 and let D_0 denote the degree of the node 0. Our result in this note is then as follows.

Theorem 1 *If R_0 is long-tailed with $\mathrm{E}({R_0}^d) < \infty$, then*

$$\mathrm{P}(D_0 > k) \sim \mathrm{P}(\lambda\, \pi_d\, {R_0}^d > k) = \overline{F}\Big(\Big(\frac{k}{\lambda\, \pi_d}\Big)^{1/d}\Big) \quad \text{as } k \to \infty,$$

where $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}_+$ and $\pi_d = \pi^{d/2}/\Gamma(d/2+1)$ with the Gamma function Γ; that is, $\pi_d\, r^d$ represents the volume of a d-dimensional ball with radius r.

We note that Theorem 1 includes as special cases that, if the radius distribution F is regularly varying with index $-\alpha$ for $\alpha > d$, then the degree distribution is regularly varying with index $-\alpha/d$; that is, the graph G_Ξ is scale-free in the sense that the degree distribution follows the power-law. Furthermore, in the case of $d = 2$, if F is long-tailed with the second moment, then the degree distribution is square-root insensitive (see, e.g., Jelenković et al. [9] for the definition) since ${R_0}^2$ is square-root insensitive if and only if R_0 is long-tailed. This result can also be thought as an extension of Theorem 1 in Miyoshi et al. [10] for one-dimensional interval graphs to high dimensional spaces.

We prove Theorem 1 in Section 3, where a subclass of long-tailed distributions — called x^{1-p}-insensitive distributions with $p \in (0,1)$ — plays a key role. This class is a generalization of the square-root insensitive distributions studied in [9] (see also Asmussen et al. [2] and Foss & Korshunov [5]) and also a subclass of h-insensitive distributions in Foss et al. [6, 7]. Thus, before providing the proof of Theorem 1, we study this class of distributions in the next section.

2 x^{1-p}-insensitive distributions

Definition 2 For $p \in (0,1)$, a nonnegative random variable X and its distribution are said to be x^{1-p}-insensitive if

$$\mathrm{P}(X > x + x^{1-p}) \sim \mathrm{P}(X > x) \quad \text{as } x \to \infty.$$

This is a subclass of h-insensitive distributions in [6, 7] with $h(x) = x^{1-p}$, $p \in (0,1)$, and a generalization of square-root insensitive distributions, where $p = 1/2$. We can see that, this class of distributions is close to the slowly varying distributions (see, e.g., [7] for the definition) when p is close to 0 , while it is close to long-tailed distributions when p is close to 1. The following lemma characterizes the x^{1-p}-insensitive distributions.

Lemma 1 *For any $p \in (0,1)$ and a nonnegative random variable X, the following are equivalent.*

(i) X is x^{1-p}-insensitive.

(ii) For any fixed $a \in \mathbb{R}$, $\mathrm{P}(X > x + a\,x^{1-p}) \sim \mathrm{P}(X > x)$ as $x \to \infty$.

(iii) X^p is long-tailed.

Proof of (i) ⇔ (ii): Since (ii) ⇒ (i) is obvious from the definition, we verify (i) ⇒ (ii) below. We first show the case of $a > 0$. Since there exists a nonnegative integer k such that $k \le a < k+1$ for any $a > 0$, it suffices to show that, for any positive integer k,

$$\mathrm{P}(X > x + k\,x^{1-p}) \sim \mathrm{P}(X > x) \quad \text{as } x \to \infty. \tag{2}$$

The case of $k = 1$ is just the definition of x^{1-p}-insensitivity and we assume (2) for some $k > 0$. Then,

$$\begin{aligned} 1 &\ge \frac{\mathrm{P}(X > x + (k+1)\,x^{1-p})}{\mathrm{P}(X > x)} \\ &\ge \frac{\mathrm{P}(X > x + x^{1-p} + k\,(x + x^{1-p})^{1-p})}{\mathrm{P}(X > x + x^{1-p})}\,\frac{\mathrm{P}(X > x + x^{1-p})}{\mathrm{P}(X > x)} \to 1 \quad \text{as } x \to \infty, \end{aligned}$$

so that, the induction leads to (2) for any positive integer k. We next show (ii) for $a < 0$. Let $y = x + a\,x^{1-p}$. Then, since $y/x \to 1$ as $x \to \infty$, for any $b > -a > 0$, there exists an $x_0 > 0$ such that $-a\,x^{1-p} \le b\,y^{1-p}$ for $x \ge x_0$. Hence, for $x \ge x_0$,

$$1 \le \frac{\mathrm{P}(X > x + a\,x^{1-p})}{\mathrm{P}(X > x)} \le \frac{\mathrm{P}(X > y)}{\mathrm{P}(X > y + b\,y^{1-p})} \to 1 \quad \text{as } x \to \infty.$$

□

Proof of (ii) ⇔ (iii): We first assume (ii). Then, for any $a > 0$ and a sufficiently large x, we have

$$1 \le \frac{\mathrm{P}(X^p > x - a)}{\mathrm{P}(X^p > x)} = \frac{\mathrm{P}(X > (x-a)^{1/p})}{\mathrm{P}(X > x^{1/p})} \le \frac{\mathrm{P}(X > x^{1/p} - (a/p)\, x^{1/p-1})}{\mathrm{P}(X > x^{1/p})} \to 1 \quad \text{as } x \to \infty.$$

We next assume (iii). Then, for any $a > 0$,

$$1 \ge \frac{\mathrm{P}(X > x + a\, x^{1-p})}{\mathrm{P}(X > x)} \ge \frac{\mathrm{P}\big(X > (x^p + p\, a)^{1/p}\big)}{\mathrm{P}(X > x)} = \frac{\mathrm{P}(X^p > x^p + p\, a)}{\mathrm{P}(X^p > x^p)} \to 1 \quad \text{as } x \to \infty,$$

which completes the proof. □

We can see from Lemma 1(i)⇔(iii) that Theorem 1 states that the degree distribution of graph G_Ξ is $x^{1-1/d}$-insensitive when the radius distribution F of grains is long-tailed with dth moment. The next lemma gives an implication property of x^{1-p}-insensitive distributions in $p \in (0,1)$ and also it says that, for $p \in (0,1)$, an x^{1-p}-insensitive distribution has a heavier tail than the Weibull tail e^{-ax^p}, $a > 0$.

Lemma 2 *If a nonnegative random variable X is x^{1-p}-insensitive for $p \in (0,1)$, then the following hold.*

(i) X is x^{1-q}-insensitive for $p < q < 1$.

(ii) $e^{ax^p}\, \mathrm{P}(X > x) \to \infty$ as $x \to \infty$ for any $a > 0$.

Proof: (i) It is obvious from

$$1 \ge \frac{\mathrm{P}(X > x + x^{1-q})}{\mathrm{P}(X > x)} \ge \frac{\mathrm{P}(X > x + x^{1-p})}{\mathrm{P}(X > x)} \to 1 \quad \text{as } x \to \infty.$$

(ii) Since Lemma 1 says that X^p is long-tailed, we have $e^{ay}\, \mathrm{P}(X^p > y) \to \infty$ as $y \to \infty$ for any $a > 0$ (see, e.g., [7, Lemma 2.17]). □

We conclude this section with the lemma which provides a tool for verifying Theorem 1.

Lemma 3 *If a nonnegative random variable X is x^{1-p}-insensitive for $p \in (0,1)$, then for any $a > 0$,*

$$\mathrm{P}(X + a\, X^{1-p} > x) \sim \mathrm{P}(X > x) \quad \textit{as } x \to \infty.$$

Proof: Since $\mathrm{P}(X + a\, X^{1-p} > x) = \mathrm{P}(X > x) + \mathrm{P}(X + a\, X^{1-p} > x \ge X)$, we have

$$1 \le \frac{\mathrm{P}(X + a\, X^{1-p} > x)}{\mathrm{P}(X > x)} \le 1 + \frac{\mathrm{P}(x \ge X > x - a\, x^{1-p})}{\mathrm{P}(X > x)}.$$

Here, since $\mathrm{P}(x \ge X > x - a\, x^{1-p}) = \mathrm{P}(X > x - a\, x^{1-p}) - \mathrm{P}(X > x)$, we obtain the result from Lemma 1. □

3 Proof of Theorem 1

In this section, we provide the proof of Theorem 1. To do so, we first give the following lemma.

Lemma 4 *Let N_μ denote a Poisson random variable with mean $\mu > 0$ and let Λ denote a nonnegative random variable independent of N_μ. If Λ is square-root insensitive, then the mixed-Poisson random variable N_Λ satisfies*

$$\mathrm{P}(N_\Lambda > k) \sim \mathrm{P}(\Lambda > k) \quad \textit{as } k \to \infty.$$

Although Lemma 4 is a special case of Theorem 3 in [9], for completeness of the note, we give its proof in Appendix. Using this Lemma, we can give the proof of Theorem 1.

Proof of Theorem 1: Since

$$\mathrm{P}(D_0 > k) = \int_0^\infty \mathrm{P}(D_0 > k \mid R_0 = y)\,\mathrm{d}F(y), \quad k \in \mathbb{Z}_+,$$

we first consider the conditional degree distribution given $R_0 = y$. Since $D_0\big|_{R_0=y}$ is equal to the number of grains $X_i + B_0(R_i)$, $i \in \mathbb{N}$, intersecting with the ball $B_0(y)$, Lemma 3.1.5 of [3] implies that it is a Poisson random variable with mean $\lambda\,\mathrm{E}\big(|B_0(y+R_1)|\big) = \lambda\,\pi_d\,\mathrm{E}\big((y+R_1)^d\big)$, where $|C|$ stands for the volume (Lebesgue measure) of $C \in \mathcal{B}(\mathbb{R}^d)$. Therefore, we have

$$\mathrm{P}(D_0 > k) = \mathrm{P}\big(N_{\lambda\pi_d\mathrm{E}((R_0+R_1)^d|R_0)} > k\big), \quad k \in \mathbb{Z}_+,$$

where N_μ denotes a Poisson distributed random variable with mean $\mu > 0$. We now check that $\mathrm{E}\big((R_0 + R_1)^d \mid R_0\big)$ meets the condition of Lemma 4; that is, it is square-root insensitive. Note from Lemma 1 that ${R_0}^d$ is $x^{1-1/d}$-insensitive since R_0 is long-tailed. Letting $r^{(m)} = \mathrm{E}({R_0}^m)$, we have $\mathrm{E}\big((y+R_1)^d\big) = \sum_{i=0}^d \binom{d}{i}\, r^{(d-i)}\, y^i = y^d + d\, r^{(1)}\, y^{d-1} + o(y^{d-1})$ as $y \to \infty$, so that, Lemma 3 ensures that $\mathrm{E}\big((R_0+R_1)^d \mid R_0\big)$ is also $x^{1-1/d}$-insensitive, and thus, it is square-root insensitive by Lemma 2(i). Hence, Lemmas 3 and 4 yield that

$$\mathrm{P}(D_0 > k) \sim \mathrm{P}\big(\lambda\,\pi_d\,\mathrm{E}\big((R_0+R_1)^d \mid R_0\big) > k\big) \sim \mathrm{P}\big(\lambda\,\pi_d\,{R_0}^d > k\big) \quad \text{as } k \to \infty.$$

□

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A Proof of Lemma 4

To prove Lemma 4, we use the following (see also [9, Lemma 6]).

Lemma 5 *Let N denote a non-delayed renewal process with inter-renewal sequence $\{\tau_i\}_{i\in\mathbb{N}}$ satisfying $\mathrm{E}({\tau_1}^2)<\infty$. Then, for any $\delta>0$, there exists a constant $c_\delta>0$ such that*

$$\mathrm{P}\Big(N((0,t])-\frac{t}{\mathrm{E}\tau_1}>u\Big)\le e^{-c_\delta u^2/t},\quad t>0,\ 0\le u\le\delta\, t.$$

Proof: Markov's inequality yields that, for $s>0$,

$$\begin{aligned}\mathrm{P}\Big(N((0,t])-\frac{t}{\mathrm{E}\tau_1}>u\Big)&=\mathrm{P}\Big(N((0,t])\ge\Big\lfloor u+\frac{t}{\mathrm{E}\tau_1}\Big\rfloor+1\Big)\\&=\mathrm{P}\Big(\sum_{i=1}^{\lfloor u+t/\mathrm{E}\tau_1\rfloor+1}\tau_i\le t\Big)\le e^{st}\,(\mathrm{E}e^{-s\tau_1})^{u+t/\mathrm{E}\tau_1},\end{aligned}$$

where $\lfloor x\rfloor=\max\{n\in\mathbb{Z}:n\le x\}$; the maximal integer not greater than $x\in\mathbb{R}$. Applying $e^{-x}\le 1-x+x^2$, $x\in\mathbb{R}$, and then $1+y\le e^y$, $y\in\mathbb{R}$, into the last expression above, we have

$$\mathrm{P}\Big(N((0,t])-\frac{t}{\mathrm{E}\tau_1}>u\Big)\le\exp\Big\{-s\,u\,\mathrm{E}\tau_1+s^2\,\mathrm{E}({\tau_1}^2)\Big(u+\frac{t}{\mathrm{E}\tau_1}\Big)\Big\}.$$

Now, we choose $s=u\,(\mathrm{E}\tau_1)^2/[2\,t\,(1+\delta\,\mathrm{E}\tau_1)\,\mathrm{E}({\tau_1}^2)]$. Then, the inside of the braces on the right-hand side above leads to

$$-\frac{(\mathrm{E}\tau_1)^3}{4\,(1+\delta\,\mathrm{E}\tau_1)\,\mathrm{E}({\tau_1}^2)}\Big(2-\frac{1+(u/t)\,\mathrm{E}\tau_1}{1+\delta\,\mathrm{E}\tau_1}\Big)\frac{u^2}{t}\le-\frac{(\mathrm{E}\tau_1)^3}{4\,(1+\delta\,\mathrm{E}\tau_1)\,\mathrm{E}({\tau_1}^2)}\,\frac{u^2}{t},$$

where the inequality follows from $u/t\le\delta$. □

In the proof below, we use the standard notations $f(x)\lesssim g(x)$ and $f(x)\gtrsim g(x)$ as $x\to\infty$ which stand for $\limsup_{x\to\infty}f(x)/g(x)\le 1$ and $\liminf_{x\to\infty}f(x)/g(x)\ge 1$ respectively.

Proof of Lemma 4: We first show the asymptotic upper bound,

$$\mathrm{P}(N_\Lambda>k)\lesssim\mathrm{P}(\Lambda>k)\quad\text{as }k\to\infty.\tag{3}$$

Let a and b denote constants such that $a>0$ and $0<b<1$. Then, for $k>a^2/(1-b)^2$,

$$\mathrm{P}(N_\Lambda>k)\le\mathrm{P}(\Lambda>k-a\sqrt{k})+\mathrm{P}(N_\Lambda>k,\ b\,k<\Lambda\le k-a\sqrt{k})+\mathrm{P}(N_{bk}>k),\tag{4}$$

where the third term on the right-hand side follows since Poisson random variables are stochastically monotone in their means. Since Λ is square-root insensitive, the first term on the right-hand side above leads to $\mathrm{P}(\Lambda>k-a\sqrt{k})\sim\mathrm{P}(\Lambda>k)$ as $k\to\infty$. Thus, one needs to show that the last two terms on the right-hand side of (4) are $o\big(\mathrm{P}(\Lambda>k)\big)$ as $k\to\infty$. We first consider the third term on the right-hand side of (4). We can consider N_{bk} the number of points in $(0,b\,k]$ of a homogeneous Poisson process with unit intensity. Since $b\in(0,1)$, there exists a $\delta\ge(1-b)/b>0$, so that Lemma 5 implies that

$$\mathrm{P}\big(N_{bk}-b\,k>(1-b)\,k\big)\le e^{-c_\delta(1-b)^2k/b}=o\big(\mathrm{P}(\Lambda>k)\big)\quad\text{as }k\to\infty.$$

Next, we consider the second term on the right-hand side (RHS) of (4). Since $k-\lambda<(1/b-1)\,\lambda$ for $\lambda>b\,k$, Lemma 5 with $\delta=(1/b-1)$ implies that

$$(\text{2nd term on RHS of (4)})=\int_{b\,k}^{k-a\sqrt{k}}\mathrm{P}(N_\lambda>k)\,\mathrm{P}(\Lambda\in\mathrm{d}\lambda)$$

$$\leq \int_0^{k-a\sqrt{k}} e^{-c_\delta\,(k-\lambda)^2/\lambda}\,\mathrm{P}(\Lambda \in \mathrm{d}\lambda).$$

Note here that, for any $\lambda \in (0, k - a\sqrt{k}]$, we have $e^{-c_\delta\,(k-\lambda)^2/\lambda} \leq e^{-c_\delta\,(k-\lambda)^2/k}$, so that integration by parts and change of variables to $y = (k-\lambda)/\sqrt{k}$ result in

$$\begin{aligned}(\text{2nd term on RHS of (4)}) &\leq e^{-c_\delta\,k} + \frac{2\,c_\delta}{k}\int_0^{k-a\sqrt{k}} (k-\lambda)\,e^{-c_\delta\,(k-\lambda)^2/k}\,\mathrm{P}(\Lambda > \lambda)\,\mathrm{d}\lambda \\ &= e^{-c_\delta\,k} + 2\,c_\delta \int_a^{\sqrt{k}} y\,e^{-c_\delta\,y^2}\,\mathrm{P}(\Lambda > k - y\sqrt{k})\,\mathrm{d}y. \qquad (5)\end{aligned}$$

The first term on the right-hand side above is clearly $o\big(\mathrm{P}(\Lambda > k)\big)$ as $k \to \infty$. For the integrand above, since $\sqrt{\Lambda}$ is long-tailed when Λ is square-root insensitive, for any $\epsilon > 0$, there exists a $c_\epsilon > 0$ such that, for $y \leq \sqrt{k}$ and sufficiently large k,

$$\mathrm{P}(\Lambda > k - y\sqrt{k}) \leq \mathrm{P}(\sqrt{\Lambda} > \sqrt{k} - y) \leq c_\epsilon\,e^{\epsilon\,y}\,\mathrm{P}(\Lambda > k),$$

where the first inequality follows from $\sqrt{k - y\sqrt{k}} \geq \sqrt{k} - y$ for $y \leq \sqrt{k}$ and the last inequality immediately follows from the definition of long-tailed distributions; that is, for any long-tailed random variable X and any $\epsilon > 0$, there exists a $c_\epsilon > 0$ and $x_\epsilon > 0$ such that $\mathrm{P}(X > x - u) \leq c_\epsilon\,e^{\epsilon\,u}\,\mathrm{P}(X > x)$ for all $x - u > x_\epsilon$. Thus, we obtain

$$(\text{2nd term on RHS of (5)}) \leq c_\epsilon\,\mathrm{P}(\Lambda > k)\int_a^\infty e^{\epsilon\,y}\,(2\,c_\delta\,y)\,e^{-c_\delta\,y^2}\,\mathrm{d}y = c_\epsilon\,\mathrm{P}(\Lambda > k)\,\mathrm{E}(e^{\epsilon Y}\,1_{\{Y>a\}}),$$

where Y denotes a random variable according to Weibull distribution $\mathrm{P}(Y > y) = e^{-c_\delta\,y^2}$, $y \geq 0$. Since there exists an $\epsilon_0 > 0$ such that $\mathrm{E}e^{\epsilon Y} < \infty$ for $\epsilon < \epsilon_0$, there also exists an $a_\epsilon > 0$ such that $\mathrm{E}(e^{\epsilon Y}\,1_{\{Y>a\}}) < \epsilon$ for $a \geq a_\epsilon$; that is, the second term on the right-hand side of (5) is $o\big(\mathrm{P}(\Lambda > k)\big)$ as $k \to \infty$, which leads to (3).

We next show the asymptotic lower bound,

$$\mathrm{P}(N_\Lambda > k) \gtrsim \mathrm{P}(\Lambda > k) \quad \text{as } k \to \infty. \qquad (6)$$

We have for $a > 0$,

$$\mathrm{P}(N_\Lambda > k) \geq \mathrm{P}\big(N_\Lambda > k,\ \Lambda > k + a\sqrt{k}\big) \geq \mathrm{P}(N_{k+a\sqrt{k}} > k)\,\mathrm{P}(\Lambda > k + a\sqrt{k}).$$

Here, we obtain that, for $k > a^2$,

$$\mathrm{P}(N_{k+a\sqrt{k}} > k) \geq \mathrm{P}\Big(\frac{N_{k+a\sqrt{k}} - (k + a\sqrt{k})}{\sqrt{k + a\sqrt{k}}} > -\frac{a}{\sqrt{2}}\Big).$$

Hence, the square-root insensitivity of Λ and the central limit theorem for renewal process (see, e.g., Asmussen [1, Chap. V, Theorem 6.3]) result in, for an appropriate $\sigma > 0$,

$$\mathrm{P}(N_\Lambda > k) \gtrsim \Phi\Big(\frac{a}{\sigma\sqrt{2}}\Big)\,\mathrm{P}(\Lambda > k) \quad \text{as } k \to \infty,$$

where Φ denotes the standard normal distribution. Finally, letting $a \to \infty$ leads to (6). □