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Abstract

Recently, stochastic geometry models for wireless communication networks have been attracting much attention. This is because the performance of such networks critically depends on the spatial configuration of wireless nodes and the irregularity of node configuration in a real network can be captured by a spatial point process. However, most analyses of such stochastic geometry models for wireless networks assume, due to its tractability, that the wireless nodes are located according to homogeneous Poisson point processes. This means that the wireless nodes are located independently with each other and their spatial correlation is ignored. In this work, we propose a stochastic geometry model of cellular networks such that the wireless base stations are located according to the Ginibre point process. The Ginibre point process is one of the determinantal point processes and accounts for the repulsion between the base stations. For the proposed model, we derive a computable representation for the coverage probability—the probability that the signal-to-interference-plus-noise ratio (SINR) for a mobile user achieves a target threshold. To capture its qualitative property, we further investigate the asymptotics of coverage probability as the SINR threshold becomes large in a special case. The results of numerical experiments are also exhibited.

Keywords: Stochastic geometry models; wireless communication networks; cellular networks; determinantal point processes; Ginibre point process; signal-to-interference-plus-noise ratio, coverage probability.

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1 Introduction

Recently, stochastic geometry models for wireless communication networks have been attracting much attention (see, e.g., introductory articles by Andrews *et al.* [2] and Haenggi *et al.* [8], and a monograph by Baccelli & Błaszczyszyn [3]). This is because the performance of such networks critically depends on the spatial configuration of wireless nodes and the irregularity of node configuration in a real network can be captured by a spatial point process. For cellular networks, some works have also proposed and analyzed the stochastic geometry models, where the wireless base stations and mobile users are located randomly on the Euclidean plane, and various performance indices such as the coverage probability—the probability that the signal-to-interference-plus-noise ratio (SINR) for a mobile user achieves a target threshold—have been evaluated (see, e.g., [1, 5, 6] which are reviewed briefly in the next section).

Most analyses of such stochastic geometry models for wireless networks, however, assume that the wireless nodes are located according to homogeneous Poisson point processes though the modeling is possible using general spatial point processes. While this assumption makes the models tractable, it means that the wireless nodes are located independently with each other and their spatial correlation is ignored. Since real networks can be designed such that two wireless nodes are not too close, the models accounting for repulsion between the nodes must be required. A few works have so far allowed the non-Poisson configurated wireless nodes except the classical grid models for cellular networks. For example, Błaszczyszyn & Yogeshwaran [3] studied the connectivity of sub-Poisson SINR graphs and Ganti *et al.* [7] developed the series expansion for functions of interference using the factorial moment expansion.

In this work, we propose a stochastic geometry model of cellular networks such that the wireless base stations are located according to the Ginibre point process. The Ginibre point process is one of the determinantal point processes, which are used to model fermions in quantum mechanics and account for the repulsion between the particles, and has been studied well since it has several desirable features (see, e.g., Hough *et al.* [9], Shirai & Takahashi [11] and Soshnikov [12]). For the proposed model, we derive a computable representation for the coverage probability. Furthermore, to capture its qualitative property, we investigate the asymptotics of coverage probability as the SINR threshold becomes large in the interference-limited case. Though we here focus on the coverage probability in a basic model, it would be possible to extend our results to more practical problems developed in [1, 5, 6].

The rest of paper is organized as follows. In the next section, we describe our stochastic geometry model of cellular networks by following [1] and make a brief review on some related works with Poisson configurated base stations. We also derive a basic formula for the coverage probability, which plays a key role in our analysis. In Section 3, we give the definition and some useful properties of the Ginibre point process, where we further define a scaled version of that process. The computable integral representation for the coverage probability is derived in Section 4. The effect of random frequency reuse is also considered there. In Section 5, we investigate the asymptotic property of coverage probability as the SINR threshold becomes large in the interference-limited case. The results of numerical experiments are exhibited in Section 6. Finally, concluding remarks are made in Section 7.

2 Stochastic geometry model of cellular networks

We here describe a stochastic geometry model of cellular wireless networks, which mainly follows [1] though some notations are altered for convenience. Let Φ denote a point process on $\mathbb{R}^2$ and X_i, $i \in \mathbb{N}$, denote the points of Φ, where the order of $X_1, X_2, \dots$ is arbitrary. The point process Φ represents the configuration of wireless base stations and we refer to the base station located at X_i as station i. We assume that Φ is simple and locally finite a.s. and also motion-invariant (stationary and isotropic). The transmission power of each base station is constant at $1/\mu$, $\mu > 0$. Each mobile user is associated with the closest base station; that is, the mobile users in the Voronoi cell of a base station are associated with that station. Thus, due to the motion invariance of point process and the homogeneity of base stations, we can focus on a typical mobile user located at the origin $o = (0, 0)$. We assume the Rayleigh fading for the random effect of fading/shadowing from each base station to a user, so that the transmission power multiplied by the fading effect from station i to the typical user at the origin, denoted by F_i, is an exponential random variable with mean $1/\mu$, where F_i, $i \in \mathbb{N}$, are mutually independent and also independent of point process Φ. The path-loss function ℓ representing the attenuation of signals with the distance is given by $\ell(r) = \alpha\, r^{-2\beta}$, $r > 0$, for some $\alpha > 0$ and $\beta > 1$.

In the setting described above, the cumulative interference received by the typical user is given by

$$I_o = \sum_{j \in \mathbb{N} \setminus \{B_o\}} F_j\, \ell(|X_j|), \tag{1}$$

where B_o denotes the base station associated with the typical user. The SINR of typical user from the associated base station is then expressed as

$$\mathsf{SINR}_o = \frac{F_{B_o}\, \ell(|X_{B_o}|)}{W_o + I_o}, \tag{2}$$

where W_o denotes the thermal noise at the origin and is independent of $\Phi_F = \{(X_i, F_i)\}_{i \in \mathbb{N}}$. We assume that the Laplace-Stieltjes transform of W_o is known to be computable. We consider the coverage probability as the performance index, which is defined as $p(\theta, \beta) = \mathsf{P}(\mathsf{SINR}_o > \theta)$; the probability that the SINR of typical mobile user achieves a predefined threshold $\theta > 0$.

Some works have so far considered similar cellular network models, where the base stations are located according to homogeneous Poisson point processes. Andrews *et al.* [1] dealt with more general fading distributions and evaluated the coverage probability and the mean achievable rate defined as $\tau(\beta) = \mathsf{E} \ln(1 + \mathsf{SINR}_o)$. Decreusefond *et al.* [5] proposed the model incorporating the time-invariant shadowing and the time-variant fading, and evaluated the handover probability under the assumption that the associated base stations are altered when the SINR from the current associated station continues to be lower than the threshold. Dhillon *et al.* [6] extended the model to that with multi-tiers of heterogeneous base stations, which generates the macro, pico or femto cells.

In this paper, we adopt the Ginibre point process (or its scaled version) as the point process Φ representing the configuration of base stations. Samples of Poisson and Ginibre point processes with the same intensity are found in Fig. 1, where we can see that the points of Ginibre process are distributed

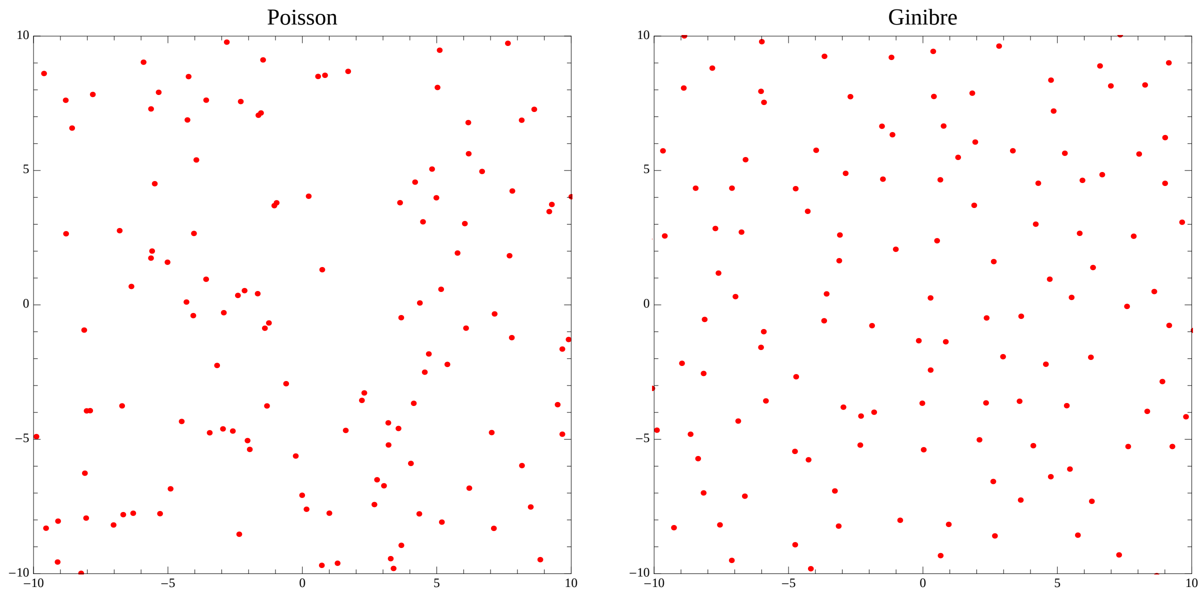

Figure 1: Samples of Poisson point process (left) and Ginibre point process (right).

more evenly. Also, comparing it with Fig. 2 in [1], we can find that the point configuration of Ginibre process is relatively closer to a real base station deployment by a major service provider in a relatively flat urban area than that of a Poisson process. Before proceeding to the description of Ginibre point process, we give a basic formula for the coverage probability, which plays a key role in our analysis.

Lemma 1 *For the cellular network model described above, the coverage probability for a typical mobile user satisfies*

$$p(\theta,\beta) = \mathsf{E}\Bigl(\mathcal{L}_W\Bigl(\frac{\mu\,\theta\,|X_{B_0}|^{2\beta}}{\alpha}\Bigr) \prod_{j\in\mathbb{N}\setminus\{B_0\}} \Bigl(1+\theta\,\Bigl|\frac{X_{B_0}}{X_j}\Bigr|^{2\beta}\Bigr)^{-1}\Bigr), \tag{3}$$

where $\mathcal{L}_W$ denotes the Laplace-Stieltjes transform of W_o.

Proof: We have from (2) that

$$\begin{aligned}\mathsf{P}(\mathsf{SINR}_o > \theta) &= \sum_{i=1}^{\infty} \mathsf{P}(\mathsf{SINR}_o > \theta, B_o = i) \\ &= \sum_{i=1}^{\infty} \mathsf{P}\Bigl(F_i > \frac{\theta\,(W_o+I_o)}{\ell(|X_i|)}, |X_i| \le |X_j|, j\in\mathbb{N}\Bigr). \end{aligned}\tag{4}$$

Since F_i is exponentially distributed with mean $1/\mu$, and W_o and I_o are mutually independent, conditioning yields

$$\begin{aligned}\mathsf{P}\Bigl(F_i > \frac{\theta\,(W_o+I_o)}{\ell(|X_i|)}, |X_i| \le |X_j|, j\in\mathbb{N}\Bigr) &= \mathsf{E}\bigl(e^{-\mu\theta W_o/\ell(|X_i|)}\, e^{-\mu\theta I_o/\ell(|X_i|)}\, \mathbf{1}_{\{|X_i|\le|X_j|,j\in\mathbb{N}\}}\bigr) \\ &= \mathsf{E}\Bigl(\mathcal{L}_W\Bigl(\frac{\mu\,\theta}{\ell(|X_i|)}\Bigr)\, \mathsf{E}\bigl(e^{-\mu\theta I_o/\ell(|X_i|)} \mid \Phi\bigr)\mathbf{1}_{\{|X_i|\le|X_j|,j\in\mathbb{N}\}}\Bigr), \end{aligned}\tag{5}$$

where $\mathbf{1}_A$ denotes the indicator for a set A. Furthermore, since F_j, $j \in \mathbb{N}$, are mutually independent, applying (1) under the condition that $|X_i| \le |X_j|$, $j \in \mathbb{N}$, we have

$$\begin{aligned}\mathsf{E}\big(e^{-\mu\theta I_o/\ell(|X_i|)} \mid \Phi\big) &= \prod_{j\in\mathbb{N}\setminus\{i\}} \mathsf{E}\big(e^{-\mu\theta F_j \ell(|X_j|)/\ell(|X_i|)} \mid \Phi\big) \\ &= \prod_{j\in\mathbb{N}\setminus\{i\}} \Big(1+\theta\,\Big|\frac{X_i}{X_j}\Big|^{2\beta}\Big)^{-1},\end{aligned} \tag{6}$$

where the Laplace-Stieltjes transform $\mathcal{L}_F(s) = \mu/(\mu+s)$ of F_j and $\ell(r) = \alpha\, r^{-2\,\beta}$, $r > 0$, are applied in the second equality. Hence, applying (5) and (6) to (4), we obtain (3). □

3 Ginibre point process

In this section, we give the definition of Ginibre point process and make a brief review on its useful properties (see, e.g., [9, 11, 12] for details). The Ginibre point process is one of the determinantal point processes on the complex plane $\mathbb{C}$ defined as follows. Let Φ denote a simple point process on $\mathbb{C}$ and ρ_n: $\mathbb{C}^n \to \mathbb{R}_+$, $n \in \mathbb{N}$, denote its correlation functions (joint intensities) with respect to some Radon measure ν on $\mathbb{C}$; that is, for any disjoint $C_1, C_2, \dots, C_n \in \mathcal{B}(\mathbb{C})$,

$$\mathsf{E}\big(\Phi(C_1)\,\Phi(C_2)\cdots\Phi(C_n)\big) = \int_{C_1\times C_2\times\cdots\times C_n} \rho_n(z_1, z_2, \dots, z_n)\,\nu(\mathrm{d}z_1)\,\nu(\mathrm{d}z_2)\cdots\nu(\mathrm{d}z_n), \tag{7}$$

and $\rho_n(z_1, z_2, \dots, z_n) = 0$ when $z_i = z_j$ for $i \ne j$. The point process Φ is said to be a determinantal point process with kernel K: $\mathbb{C}^2 \to \mathbb{C}$ with respect to ν if ρ_n, $n \in \mathbb{N}$, satisfy

$$\rho_n(z_1, z_2, \dots, z_n) = \det\big(K(z_i, z_j)\big)_{1\le i,j\le n}, \quad z_1, z_2, \dots, z_n \in \mathbb{C}, n \in \mathbb{N}. \tag{8}$$

Furthermore, Φ is said to be the Ginibre point process when the kernel K is given by $K(z, w) = e^{z\overline{w}}$, $z, w \in \mathbb{C}$, with respect to $\nu(\mathrm{d}z) = \pi^{-1}\, e^{-|z|^2}\, m(\mathrm{d}z)$, where $\overline{w}$ denotes the conjugate of $w \in \mathbb{C}$ and m denotes the Lebesgue measure on $\mathbb{C}$. It is also equivalent that $K(z, w) = \pi^{-1}\, e^{-(|z|^2+|w|^2)/2}\, e^{z\overline{w}}$ with respect to $\nu(\mathrm{d}z) = m(\mathrm{d}z)$. The Ginibre point process is known to be motion-invariant. One of its useful properties comes from the radial symmetricity and is described as follows (see, e.g., Kostlan [10] or [9, Section 4.7]).

Proposition 1 (Kostlan [10]) *Let X_i, $i \in \mathbb{N}$, denote the points of Ginibre point process. Then, the set $\{|X_i|\}_{i\in\mathbb{N}}$ has the same distribution as $\{\sqrt{Y_i}\}_{i\in\mathbb{N}}$, where Y_i, $i \in \mathbb{N}$, are mutually independent and each Y_i follows the ith Erlang distribution with the unit rate parameter, denoted by $Y_i \sim \mathrm{Gamma}(i, 1)$, $i \in \mathbb{N}$.*

By the definition of Ginibre point process, we see $\mathsf{E}\Phi(C) = \pi^{-1}\, m(C)$ for $C \in \mathcal{B}(\mathbb{C})$; that is, the (first order) intensity is equal to π^{-1} with respect to the Lebesgue measure. To make it possible to control the intensity, we consider a scaled version Φ_c of Ginibre point process with the scaling parameter $c > 0$. That is given by the kernel $K_c(z, w) = c\, e^{cz\overline{w}}$ with respect to the reference measure $\nu_c(\mathrm{d}z) = \pi^{-1}\, e^{-c|z|^2}\, m(\mathrm{d}z)$, or equivalently, $K_c(z, w) = (c/\pi)\, e^{-c(|z|^2+|w|^2)/2}\, e^{cz\overline{w}}$ with respect to the Lebesgue measure. The scaled

Ginibre point process Φ_c has the intensity c/π and, for the points X_i, $i \in \mathbb{N}$, of Φ_c, the set $\{|X_i|\}_{i\in\mathbb{N}}$ has the same distribution as $\{\sqrt{Y_i}\}_{i\in\mathbb{N}}$, where Y_i, $i \in \mathbb{N}$, are mutually independent and each Y_i follows the ith Erlang distribution with rate parameter c, denoted by $Y_i \sim \mathrm{Gamma}(i, c)$, $i \in \mathbb{N}$.

4 Performance analysis

We adopt the scaled Ginibre point process Φ_c given in Section 3 as the configuration of base stations in the cellular network model described in Section 2, where a point $z = x + i\,y \in \mathbb{C}$ is identified as $(x, y) \in \mathbb{R}^2$.

4.1 Integral representation of coverage probability

Theorem 1 *Consider the cellular network model described in Section 2 with the base stations located according to the scaled Ginibre point process Φ_c defined in Section 3. Then, the coverage probability of a typical mobile user is given by*

$$p(\theta, \beta) = \int_0^\infty e^{-v}\, \mathcal{L}_W\Bigl(\frac{\mu\,\theta}{\alpha}\Bigl(\frac{v}{c}\Bigr)^\beta\Bigr)\, M(v, \theta, \beta)\, S(v, \theta, \beta)\, \mathrm{d}v, \tag{9}$$

where

$$M(v, \theta, \beta) = \prod_{j=1}^\infty \frac{1}{(j-1)!} \int_v^\infty \frac{s^{j-1}\, e^{-s}}{1 + \theta\,(v/s)^\beta}\, \mathrm{d}s, \tag{10}$$

$$S(v, \theta, \beta) = \sum_{i=1}^\infty v^{i-1} \Bigl(\int_v^\infty \frac{s^{i-1}\, e^{-s}}{1 + \theta\,(v/s)^\beta}\, \mathrm{d}s\Bigr)^{-1}. \tag{11}$$

Note that the coverage probability $p(\theta, \beta)$ given in (9)–(11) is not closed-form but computable by numerical integration provided the Laplace-Stieltjes transform $\mathcal{L}_W$ of W_o.

Proof: Let $Y_j \sim \mathrm{Gamma}(j, c)$, $j \in \mathbb{N}$, be mutually independent. For the points X_i, $i \in \mathbb{N}$, of point process Φ_c, $\{|X_i|\}_{i\in\mathbb{N}}$ has the same distribution as $\{\sqrt{Y_i}\}_{i\in\mathbb{N}}$ by the arguments in the preceding section. Thus, from (3) in Lemma 1 and the conditional independence of $\mathbf{1}_{\{Y_j \ge Y_i\}}$, $j \in \mathbb{N} \setminus \{i\}$, given Y_i, we have

$$\begin{aligned}
\mathsf{P}(\mathsf{SINR}_o > \theta) &= \sum_{i=1}^\infty \mathsf{E}\Bigl(\mathcal{L}_W\Bigl(\frac{\mu\,\theta\,{Y_i}^\beta}{\alpha}\Bigr) \prod_{j\in\mathbb{N}\setminus\{i\}} \Bigl(1 + \theta\Bigl(\frac{Y_i}{Y_j}\Bigr)^\beta\Bigr)^{-1} \mathbf{1}_{\{Y_j \ge Y_i\}}\Bigr) \\
&= \sum_{i=1}^\infty \int_0^\infty \frac{c^i\, u^{i-1}\, e^{-c\,u}}{(i-1)!}\, \mathcal{L}_W\Bigl(\frac{\mu\,\theta\,u^\beta}{\alpha}\Bigr) \prod_{j\in\mathbb{N}\setminus\{i\}} \int_u^\infty \frac{c^j\, y^{j-1}\, e^{-c\,y}}{(j-1)!} \Bigl(1 + \theta\Bigl(\frac{u}{y}\Bigr)^\beta\Bigr)^{-1} \mathrm{d}y\, \mathrm{d}u,
\end{aligned} \tag{12}$$

where the second equality follows from applying the density functions of Y_j, $j \in \mathbb{N}$. Hence, changing the variables to $s = c\,y$ and $v = c\,u$, we obtain (9) after some manipulations. □

Remark 1 We can see from (9) that, in the interference-limited case ($W_o \equiv 0$), the coverage probability $p(\theta, \beta)$ is irrelevant to the parameters c, α and μ. This is also the case where the base stations are located according to a homogeneous Poisson point process. In such a case, following Theorem 2 of [1] (or applying the Laplace functional of Poisson point process to (3)), the coverage probability is given by

$$p^{(\mathrm{Poi})}(\theta,\beta) = \int_0^\infty \mathcal{L}_W\Bigl(\frac{\mu\,\theta}{\alpha}\Bigl(\frac{v}{\pi\,\lambda}\Bigr)^\beta\Bigr)\,\exp\bigl\{-v\,\bigl(1+\rho(\theta,\beta)\bigr)\bigr\}\,\mathrm{d}v, \tag{13}$$

where $\lambda > 0$ denotes the intensity of Poisson point process and

$$\rho(\theta,\beta) = \frac{\theta^{1/\beta}}{\beta}\int_{1/\theta}^\infty \frac{u^{-1+1/\beta}}{u+1}\,\mathrm{d}u. \tag{14}$$

Remark 2 As in [1], it is not difficult to generalize the distribution of fading/shadowing from the interfering base stations while it remains experiencing the Rayleigh fading from the associated base station. Provided that $|X_i| \le |X_j|$ for all $j \in \mathbb{N}$, we assume that F_i is still exponentially distributed with mean μ^{-1}, but $F_j = \mu^{-1}\,G_j$ for $j \in \mathbb{N}\setminus\{i\}$ where G_j's are mutually independent and identically distributed nonnegative random variables with the unit mean, and also independent of $\Phi_c = \{X_i\}_{i\in\mathbb{N}}$, F_i and W_0. Let $\mathcal{L}_G$ denote the Laplace-Stieltjes transform of G_j. Then, the coverage probability is obtained similar to Theorem 1 by replacing $(1+\theta\,(v/s)^\beta)^{-1}$ in (10) and (11) as $\mathcal{L}_G(\theta\,(v/s)^\beta)$. In this case, the coverage probability is still computable whenever so is $\mathcal{L}_G$ (e.g., the probability density function of G_j is available).

Remark 3 Besides the coverage probability, [1] evaluated the mean achievable rate $\tau(\beta) = \mathsf{E}\ln(1+\mathsf{SINR}_o)$ of a typical mobile user, which comes from Shannon's channel capacity $B\log_2(1+\mathsf{SNR})$ with bandwidth B and signal-to-noise ratio SNR. We can also derive the numerically computable representation for the mean achievable rate from Theorem 1. Since $\ln(1+\mathsf{SINR}_o) > 0$ a.s.,

$$\tau(\beta) = \int_0^\infty \mathsf{P}(\ln(1+\mathsf{SINR}_o) > t)\,\mathrm{d}t = \int_0^\infty \mathsf{P}(\mathsf{SINR}_o > e^t - 1)\,\mathrm{d}t = \int_0^\infty p(e^t-1,\beta)\,\mathrm{d}t.$$

4.2 Frequency reuse

The frequency reuse is one of the ways to increase the coverage probability by reducing the number of interfering base stations. In this section, we follow [1] and consider the per-cell random frequency reuse technique. The reuse factor $\delta \in \mathbb{N}$ determines the number of different frequency bands used by the network; that is, the total frequency band is divided into δ subbands and each base station chooses one of the δ subbands uniformly at random for the use of its own cell. The interfering base stations for the typical user are then those using the same frequency band as his/her associated base station. Let R_i denote the frequency band of station i, where R_i, $i \in \mathbb{N}$, are mutually independent and distributed as $\mathsf{P}(R_i = k) = 1/\delta$, $k = 1,2,\dots,\delta$, and also independent of $\Phi_F = \{(X_i,F_i)\}_{i\in\mathbb{N}}$ and W_o. The SINR of typical user from the associated base station is given by

$$\mathsf{SINR}_o^{(\mathrm{FR})} = \frac{F_{B_o}\,\ell(|X_{B_o}|)}{W_o + I_o^{(FR)}},$$

where

$$I_o^{(\mathrm{FR})} = \sum_{j\in\mathbb{N}\setminus\{B_o\}} F_j\,\ell(|X_j|)\,\mathbf{1}_{\{R_j=R_{B_o}\}}.$$

The coverage probability then reduces to $p(\theta,\beta,\delta) = \mathsf{P}(\mathsf{SINR}_o^{(\mathrm{FR})} > \theta)$.

Corollary 1 *Consider the cellular network model as in Theorem 1 but applying the random frequency reuse such that δ frequency bands are randomly allocated to the cells. The coverage probability is then given by*

$$p(\theta,\beta,\delta) = \int_0^\infty e^{-v}\,\mathcal{L}\Bigl(\frac{\mu\theta}{\alpha}\Bigl(\frac{v}{c}\Bigr)^\beta\Bigr)\,M(v,\theta,\beta,\delta)\,S(v,\theta,\beta,\delta)\,\mathrm{d}v, \tag{15}$$

where

$$M(v,\theta,\beta,\delta) = \prod_{j=1}^\infty \frac{1}{(j-1)!}\int_v^\infty s^{j-1}e^{-s}\Bigl\{1-\frac{1}{\delta}\Bigl[1-\Bigl(1+\theta\Bigl(\frac{v}{s}\Bigr)^\beta\Bigr)^{-1}\Bigr]\Bigr\}\,\mathrm{d}s,$$
$$S(v,\theta,\beta,\delta) = \sum_{i=1}^\infty v^{i-1}\Bigl(\int_v^\infty s^{i-1}e^{-s}\Bigl\{1-\frac{1}{\delta}\Bigl[1-\Bigl(1+\theta\Bigl(\frac{v}{s}\Bigr)^\beta\Bigr)^{-1}\Bigr]\Bigr\}\,\mathrm{d}s\Bigr)^{-1}.$$

Proof: In this case, formula (3) in Lemma 1 reduces to

$$p(\theta,\beta,\delta) = \mathsf{E}\Bigl(\mathcal{L}_W\Bigl(\frac{\mu\,\theta\,|X_{B_0}|^{2\beta}}{\alpha}\Bigr)\prod_{j\in\mathbb{N}\setminus\{B_0\}}\Bigl\{1-\frac{1}{\delta}\Bigl[1-\Bigl(1+\theta\Bigl|\frac{X_{B_0}}{X_j}\Bigr|^{2\beta}\Bigr)^{-1}\Bigr]\Bigr\}\Bigr). \tag{16}$$

The remaining procedures are the same as those for Theorem 1 and are omitted. □

Remark 4 It is clear from (16) that the coverage probability is increasing in $\delta = 1, 2, \dots$ for general stationary point process $\Phi = \{X_i\}_{i\in\mathbb{N}}$. Since the frequency band is divided by δ, the mean achievable rate considered in Remark 3 is now given by

$$\tau(\beta,\delta) = \delta^{-1}\,\mathsf{E}\ln(1+\mathsf{SINR}_o^{(\mathrm{FR})}) = \frac{1}{\delta}\int_0^\infty p(e^t-1,\beta,\delta)\,\mathrm{d}t.$$

5 Asymptotic analysis in interference limited case

By Theorem 1, we can evaluate the coverage probability $p(\theta,\beta)$ numerically. In this section, we investigate its asymptotic property as $\theta\to\infty$ in the interference-limited case.

Theorem 2 *In the interference-limited case, the coverage probability derived in Theorem 1 satisfies*

$$\lim_{\theta\to\infty}\theta^{1/\beta}\,p(\theta,\beta) = \int_0^\infty\prod_{j=2}^\infty\frac{1}{(j-1)!}\int_0^\infty\frac{y^{j-1}\,e^{-y}}{1+(v/y)^\beta}\,\mathrm{d}y\,\mathrm{d}v. \tag{17}$$

The right-hand side of (17) is finite and also computable by numerical integration.

Proof: In the interference-limited case, since $\mathcal{L}_W(\cdot) = 1$, (12) reduces to

$$\begin{aligned} p(\theta,\beta) &= \sum_{i=1}^\infty \mathsf{E}\Bigl(\prod_{j\in\mathbb{N}\setminus\{i\}}\Bigl(1+\theta\Bigl(\frac{Y_i}{Y_j}\Bigr)^\beta\Bigr)^{-1}\mathbf{1}_{\{Y_j\ge Y_i\}}\Bigr) \\ &= \mathsf{E}\Bigl(\prod_{j=2}^\infty\Bigl(1+\theta\Bigl(\frac{Y_1}{Y_j}\Bigr)^\beta\Bigr)^{-1}\mathbf{1}_{\{Y_j\ge Y_1\}}\Bigr) + \sum_{i=2}^\infty\mathsf{E}\Bigl(\prod_{j\in\mathbb{N}\setminus\{i\}}\Bigl(1+\theta\Bigl(\frac{Y_i}{Y_j}\Bigr)^\beta\Bigr)^{-1}\mathbf{1}_{\{Y_j\ge Y_i\}}\Bigr), \end{aligned} \tag{18}$$

where $Y_j \sim \mathrm{Gamma}(j,1)$, $j \in \mathbb{N}$, are mutually independent. We now evaluate the two terms on the right-hand side (RHS) of (18) separately. First, since Y_1 is exponentially distributed with the unit mean,

$$
\begin{aligned}
\text{(1st term on RHS of (18))} &= \int_0^\infty e^{-u} \prod_{j=2}^\infty \mathsf{E}\Big(\Big(1+\theta\Big(\frac{u}{Y_j}\Big)^\beta\Big)^{-1} \mathbf{1}_{\{Y_j \geq u\}}\Big)\,\mathrm{d}u \\
&= \theta^{-1/\beta} \int_0^\infty e^{-\theta^{-1/\beta} v} \prod_{j=2}^\infty \mathsf{E}\Big(\Big(1+\Big(\frac{v}{Y_j}\Big)^\beta\Big)^{-1} \mathbf{1}_{\{Y_j \geq \theta^{-1/\beta} v\}}\Big)\,\mathrm{d}v, \qquad (19)
\end{aligned}
$$

where the second equality follows from changing the variable to $v = \theta^{1/\beta}\, u$. The right-hand side of (19) multiplied by $\theta^{1/\beta}$ converges to that of (17) as $\theta \to \infty$ by the monotone convergence theorem with $e^{-\theta^{-1/\beta} v} \uparrow 1$ and $\mathbf{1}_{\{Y_j \geq \theta^{-1/\beta} v\}} \uparrow 1$ and by applying the density functions of $Y_j \sim \mathrm{Gamma}(j,1)$, $j = 2, 3, \ldots$.

It remains to show that the second term on the right-hand side of (18) is $o(\theta^{-1/\beta})$ as $\theta \to \infty$. Since $\big(1+\theta\,(Y_i/Y_j)^\beta\big)^{-1} \leq 1$,

$$
\text{(2nd term on RHS of (18))} \leq \sum_{i=2}^\infty \mathsf{E}\Big(\Big(1+\theta\Big(\frac{Y_i}{Y_1}\Big)^\beta\Big)^{-1} \mathbf{1}_{\{Y_1 \geq Y_i\}}\Big). \qquad (20)
$$

Applying the density functions of $Y_1 \sim \mathrm{Gamma}(1,1)$ and $Y_i \sim \mathrm{Gamma}(i,1)$, $i = 2, 3, \ldots$, to the summand of (20), we have

$$
\begin{aligned}
\mathsf{E}\Big(\Big(1+\theta\Big(\frac{Y_i}{Y_1}\Big)^\beta\Big)^{-1} \mathbf{1}_{\{Y_1 \geq Y_i\}}\Big) &= \int_0^\infty \frac{u^{i-1}\, e^{-u}}{(i-1)!} \int_u^\infty e^{-y} \Big(1+\theta\Big(\frac{u}{y}\Big)^\beta\Big)^{-1} \mathrm{d}y\,\mathrm{d}u \\
&= \frac{1}{(i-1)!} \int_1^\infty \Big(1+\frac{\theta}{s^\beta}\Big)^{-1} \int_0^\infty u^i\, e^{-(s+1)u}\,\mathrm{d}u\,\mathrm{d}s \\
&= i \int_1^\infty \frac{1}{(s+1)^{i+1}} \Big(1+\frac{\theta}{s^\beta}\Big)^{-1} \mathrm{d}s, \qquad (21)
\end{aligned}
$$

where the second equality follows from changing the variable to $s = y/u$ and the third equality holds by the definition of Gamma functions. Here, letting $\beta^* = \lfloor \beta + 1 \rfloor$ and summing up the right-hand side of (21) over $i = \beta^*, \beta^* + 1, \ldots$, we have

$$
\begin{aligned}
\sum_{i=\beta^*}^\infty i \int_1^\infty \frac{1}{(s+1)^{i+1}} \Big(1+\frac{\theta}{s^\beta}\Big)^{-1} \mathrm{d}s &\leq \theta^{-1} \sum_{i=\beta^*}^\infty i \int_1^\infty \frac{s^\beta}{(s+1)^{i+1}}\,\mathrm{d}s \\
&= \theta^{-1} \int_1^\infty \frac{(\beta^*\, s+1)\, s^{\beta-2}}{(s+1)^{\beta^*}}\,\mathrm{d}s,
\end{aligned}
$$

where we use $(1+\theta/s^\beta)^{-1} \leq s^\beta/\theta$ in the inequality. The last integrand is $O(s^{-\beta^*-1+\beta})$ as $s \to \infty$, so that the integration is finite; that is, the last expression is $O(\theta^{-1})$ as $\theta \to \infty$. On the other hand, for $i = 2, 3, \ldots, \beta^* - 1 \leq \beta$, the right-hand side of (21) satisfies

$$
\begin{aligned}
i \int_1^\infty \frac{1}{(s+1)^{i+1}} \Big(1+\frac{\theta}{s^\beta}\Big)^{-1} \mathrm{d}s &\leq i \int_1^\infty s^{-i-1} \Big(1+\frac{\theta}{s^\beta}\Big)^{-1} \mathrm{d}s \\
&= \frac{i\,\theta^{-i/\beta}}{\beta} \int_{1/\theta}^\infty \frac{t^{-i/\beta}}{t+1}\,\mathrm{d}t \\
&\leq \frac{i\,\theta^{-i/\beta}}{\beta} \Big[\int_{1/\theta}^1 t^{-i/\beta}\,\mathrm{d}t + \int_1^\infty t^{-i/\beta-1}\,\mathrm{d}t\Big]
\end{aligned}
$$

$$= \begin{cases} \dfrac{\beta}{\beta-i}\,\theta^{-i/\beta} - \dfrac{i}{\beta-i}\,\theta^{-1} \le \dfrac{\beta}{\beta-i}\,\theta^{-i/\beta} & \text{for } i<\beta, \\ \theta^{-1}\,(\ln\theta+1) & \text{for } i=\beta, \end{cases}$$

where the first equality follows from changing the variable to $t = s^\beta/\theta$ and the next inequality follows from $1/(t+1) \le \min(1, 1/t)$. The last expressions are $o(\theta^{-1/\beta})$ as $\theta \to \infty$ for both $i < \beta$ and $i = \beta$, which completes the proof. □

Remark 5 Theorem 2 states that, in the interference-limited case, the distribution of the SINR of typical mobile user has the Pareto tail without the mean. This result is also the case where the base stations are located according to a homogeneous Poisson point process. In the interference-limited case of the Poisson base station model, (13) reduces to

$$p^{(\mathrm{Poi})}(\theta,\beta) = \frac{1}{1+\rho(\theta,\beta)}.$$

Here, we have from (14) that

$$\theta^{-1/\beta}\,\rho(\theta,\beta) = \frac{1}{\beta}\int_{1/\theta}^{\infty} \frac{u^{-1+1/\beta}}{u+1}\,\mathrm{d}u \to \frac{\pi}{\beta}\csc\frac{\pi}{\beta} \quad \text{as } \theta\to\infty,$$

which implies that

$$\lim_{\theta\to\infty} \theta^{1/\beta}\,p^{(\mathrm{Poi})}(\theta,\beta) = \frac{\beta}{\pi}\sin\frac{\pi}{\beta}. \tag{22}$$

6 Numerical experiments

In this section, we show the results of some numerical experiments for computing the coverage probabilities. We first compare the results of computing (9) for the Ginibre base station model with those of (13) for the corresponding Poisson model. In Fig. 2, each plot shows the coverage probability for a given value of θ, where the intensity $\lambda = c/\pi$ is set at $1/\pi$ for both the point processes and the thermal noise is given as a constant such that $\mathsf{SNR} = (\mu\,W_o)^{-1}$ ($\mathsf{SNR} = \infty$ stands for no noise). The coefficient α of path-loss function is set at 1 and two cases of $\beta = 1.25$ and $\beta = 2.0$ (that is, $\ell(r) = r^{-2.5}$ and $\ell(r) = r^{-4}$) are computed. For both the Ginibre and Poisson models, the gaps between the cases of $\mathsf{SNR} = 10$ and $\mathsf{SNR} = \infty$ are small, particularly for small value of β, which implies that the thermal noise is not a very important consideration. Furthermore, comparing Fig. 2 with Fig. 4 in [1], we can find that the coverage probability for the Ginibre model is very close to that for the corresponding model with a real base station deployment by a major service provider in a relatively flat urban area. This confirms that the Ginibre base station model is considerable as a good approximative model for real cellular networks.

Next, to see the effect of frequency reuse on the coverage probability, the results of computing (15) are exhibited in Fig. 3, where the curves for $\delta = 1$ are the same as those for the Ginibre model with $\mathsf{SNR} = \infty$ in Fig. 2. We can see that the coverage probability is much improved by the frequency reuse.

In the third and final experiment, we compare the coverage probability with the corresponding asymptotics in the interference-limited case. In Fig. 4, the comparison results between (9) and (17) for the

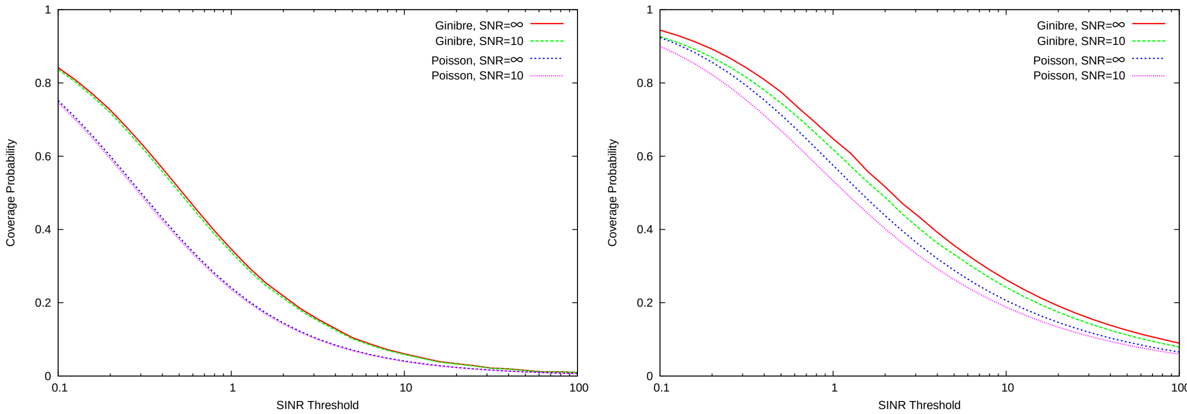

Figure 2: Comparison of coverage probability between the Ginibre base station model and the corresponding Poisson model for $\beta = 1.25$ (left) and $\beta = 2.0$ (right).

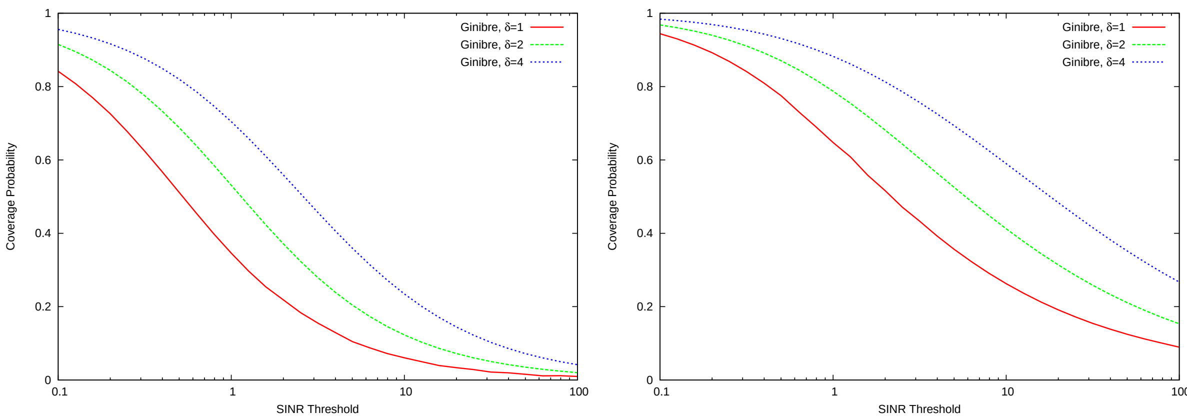

Figure 3: Effect of frequency reuse on the coverage probability for $\beta = 1.25$ (left) and $\beta = 2.0$ (right).

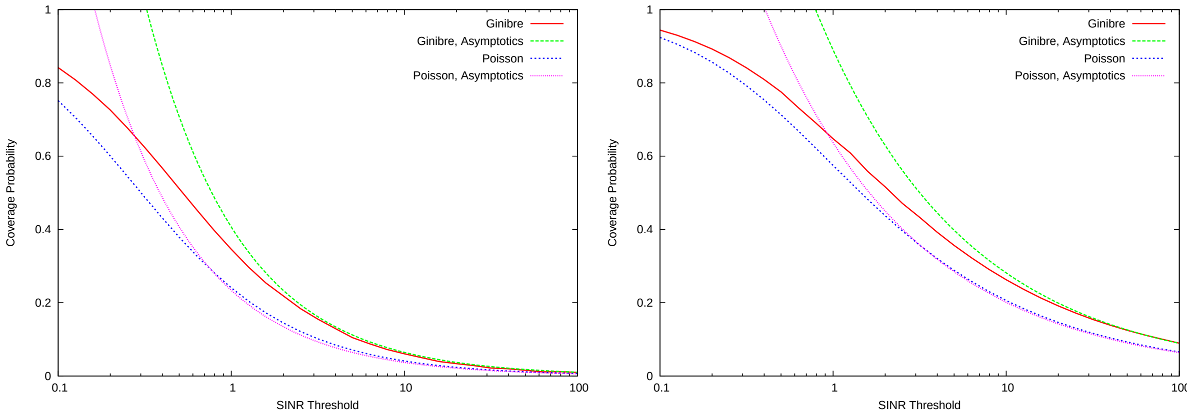

Figure 4: Comparison between the coverage probability and its asymptotic results for $\beta = 1.25$ (left) and $\beta = 2.0$ (right).

Ginibre model as well as those between (13) and (22) for the Poisson model are exhibited, where the curves for the coverage probability are the same as those for $\mathsf{SNR} = \infty$ in Fig. 2. Figure 4 shows that, in the Poisson model, the asymptotic results well agree with the coverage probability for relatively small values of θ and β. In the Ginibre model, however, the asymptotic results agree with the coverage probability only for large values of θ, particularly when the value of β is large. This implies that, to obtain a better approximation of the coverage probability in the Ginibre model, it is required to take not only the main term obtained by the asymptotic analysis but also some more terms of $o(\theta^{-1/\beta})$ as $\theta \to \infty$ into consideration.

7 Concluding remarks

We have considered a cellular network model such that the base stations are located according to the Ginibre point process and have derived the computable integral representation for the coverage probability. We have also investigated the asymptotic property of coverage probability in the interference-limited case. For future work, we can consider some problems in various directions. For example, although we have studied just a basic model, we could apply the Ginibre base station model to more practical problems such as those developed in [1, 5, 6]. Also, we could consider the applications to other wireless networks, where the percolation of SINR graphs might be attractive. As extensions of the model, we could consider more general stationary point processes. Since Proposition 1 comes from the radially symmetric property of Ginibre point process, the results obtained in the paper might be generalized to the models with other radially symmetric determinantal point processes. It might be also interesting to give a general condition that the coverage probability has the decay rate $\theta^{-1/\beta}$ as $\theta \to \infty$.

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References

[1] J. G. Andrews, F. Baccelli and R. K. Ganti (2011). A tractable approach to coverage and rate in cellular networks. *IEEE Trans. Commun.*, **59**, 3122–3134.

[2] J. G. Andrews, R. K. Ganti, M. Haenggi, N. Jindal and S. Weber (2010). A primer on spatial modeling and analysis in wireless networks. *IEEE Commun. Magazine*, **48**, 156–163.

[3] F. Baccelli and B. Błaszczyszyn (2009). Stochastic Geometry and Wireless Networks, Volume I: Theory/Volume II: Applications. *Foundations and Trends(R) in Networking*, **3**, 249–449/**4**, 1–312.

[4] B. Błaszczyszyn and D. Yogeshwaran (2010). Connectivity in sub-Poisson networks. *Proc. of 48th Annual Allerton Conf.*

[5] L. Decreusefond, P. Martins and T. T. Vu (2010). An analytical model for evaluating outage and handover probability of cellular wireless networks. *arXiv:1009.0193v1 [math.PR].*

[6] H. S. Dhillon, R. K. Ganti, F. Baccelli and J. G. Andrews (2012). Modeling and analysis of K-tier downlink heterogeneous cellular networks. *IEEE J. Select. Areas Commun.*, **30**, 550–560.

[7] R. K. Ganti, F. Baccelli and J. G. Andrews (2012). Series expansion for interference in wireless networks. *IEEE Trans. Inform. Theory*, **58**, 2194–2205.

[8] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse and M. Franceschetti (2009). Stochastic geometry and random graphs for the analysis and design of wireless networks. *IEEE J. Select. Areas Commun.*, **27**, 1029–1046.

[9] J. B. Hough, M. Krishnapur, Y. Peres and B. Virág (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes.* American Mathematical Society. Also available at `http://research.microsoft.com/en-us/um/people/peres/GAF_book.pdf`

[10] E. Kostlan (1992). On the spectra of Gaussian matrices. Directions in matrix theory (Auburn, AL, 1990). *Linear Algebra Appl.*, **162/164**, 385–388.

[11] T. Shirai and Y. Takahashi (2003). Random point fields associated with certain Fredholm determinants I: Fermion, Poisson and Boson processes. *J. Funct. Anal.*, **205**, 414–463.

[12] A. Soshnikov (2000). Determinantal random point fields. *Russian Math. Surveys*, **55**, 923–975.