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Spatial stochastic models for analysis of heterogeneous cellular networks with repulsively deployed base stations

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Abstract

We consider spatial stochastic models of downlink heterogeneous cellular networks (HCNs) with multiple tiers, where the base stations (BSs) of each tier have a particular spatial density, transmission power and path-loss exponent. Prior works on such spatial models of HCNs assume, due to its tractability, that the BSs are deployed according to homogeneous Poisson point processes. This means that the BSs are located independently of each other and their spatial correlation is ignored. In the current paper, we propose two spatial models for the analysis of downlink HCNs, in both of which the BSs are deployed according to α-Ginibre point processes. The α-Ginibre point processes constitute a class of determinantal point processes and account for the repulsion between the BSs. Besides, the degree of repulsion can be adjusted according to the value of $\alpha \in (0, 1]$. For such proposed models, we derive computable representations for the coverage probability of a typical user—the probability that the downlink signal-to-interference-plus-noise ratio for the typical user achieves a target threshold. We exhibit the results of some numerical experiments and compare the proposed models and the Poisson based model.

Keywords: Heterogeneous cellular networks; spatial stochastic models; determinantal point processes; α-Ginibre point processes; signal-to-interference-plus-noise ratio, coverage probability.

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1 Introduction

Due to the increasing variety and popularity of mobile applications, modern cellular networks have been becoming complex. In fact, one of the key features in 3G or LTE cellular networks is the heterogeneity (see, e.g., [1, 12]). A heterogeneous cellular network (HCN) consists of multiple tiers of wireless base stations (BSs), where the BSs of each tier have a particular spatial density, a transmission power, a path-loss exponent and so on, so that a conventional macrocell based cellular network is overlaid with diverse kinds of small cells such as microcells, picocells and femtocells. On the other hand, the statistics of signal-to-interference-plus-noise ratio (SINR) over a cellular network critically depend on the configuration of the BSs. Thus, to capture the irregularity of BS locations, many researchers have considered and analyzed spatial stochastic models of the HCNs with multiple tiers (see, e.g., [4, 15, 10, 13, 5, 6]), where the BS locations are modeled by spatial point processes and the analysis is based on the theory of point processes and stochastic geometry (see, e.g., [3, 8]).

Such prior works are inherently extensions of the downlink model with a single tier by Andrews et al. [2] and assume that the BSs of each tier are deployed according to a homogeneous Poisson point process. This means that the BSs are located independently of each other and their spatial correlation is ignored. While this assumption of Poisson processes indeed makes the analysis tractable, a real cellular network could be designed such that the BSs are not too close to each other. In fact, the numerical experiments of [2] shows that the coverage probability of a typical user—the probability that the downlink SINR for the typical user achieves a target threshold—for the spatial model using the real data of actual BS deployments lies between those of the Poisson based model and the square lattice model, where the BSs are located at the grids of square lattice. This observation suggests that the actual BSs would be deployed more regularly than Poisson point processes. Recently, Miyoshi and Shirai [14] proposed and analyzed a downlink cellular network model with a single tier such that the BSs are deployed according to the Ginibre point process. The Ginibre point process is one of the determinantal point processes and accounts for the repulsion between the BSs (see, e.g., Hough *et al.* [9], Shirai & Takahashi [16] and Soshnikov [17] for the Ginibre and general determinantal point processes). We can find in the numerical experiments of [14] that the coverage probability for the Ginibre based model shows the similar feature to that using the real data of the actual BS deployments given in [2].

In the current paper, we extend the model of [14] and propose the spatial models for the analysis of the downlink HCNs, where the BSs are deployed according to α-Ginibre point processes. The α-Ginibre point processes are also determinantal point processes and are introduced by Goldman [7] for constituting an intermediate class between the Poisson and Ginibre point processes. The usual Ginibre process is just the one with $\alpha = 1$ and the α-Ginibre point process converges in distribution to a homogeneous Poisson process as $\alpha \to 0$. That is, the degree of repulsion is tunable by the value of $\alpha \in (0, 1]$. We propose two distinct models using the α-Ginibre point processes. In one model, the BSs of different tiers are deployed according to mutually independent α-Ginibre processes, where the α can take different values in the different tiers. This model can express the HCN, where the BSs of the high-power tier generating the macrocells are deployed more regularly while the BSs of the low-power tier are deployed rather close to a Poisson process. In this model, however, we can not account for the repulsion between a BS of one tier and another. In the other model, all the BSs are deployed according to an α-Ginibre point process and are classified into multiple tiers by mutually independent marks. In this model, all the BSs repel with each other while we can not alter the degree of repulsion in different tiers (it should be noted that for the Poisson based models, these two classifications yield an identical model). For both of such models, we derive numerically computable representations for the coverage probability of a typical user. We then compare the proposed models and also the Poisson based model through numerical experiments.

The rest of the paper is organized as follows. In the next section, we make a brief review on the α-Ginibre point processes, where we present the definition and some fundamental properties. In section 3, we first consider the downlink cellular network model with a single tier, where the BSs are deployed according to an α-Ginibre point process. We derive a numerically computable representation for the coverage probability and investigate, through a numerical experiment, how the value of α has an impact on the coverage probability. The multitier HCN models are then proposed in section 4, where two distinct models are considered and the computable representation of the coverage probability is derived for each model. The results of numerical experiments are exhibited in section 5, where we compare the proposed models and the Poisson based model.

2 α-Ginibre point processes

As is the usual Ginibre point process, the α-Ginibre processes are determinantal point processes on the complex plain $\mathbb{C}$ defined as follows. Let Φ denote a simple point process on $\mathbb{C}$ and $\rho_n \colon \mathbb{C}^n \to \mathbb{R}_+$, $n \in \mathbb{N}$, denote its joint intensities with respect to some locally finite measure μ

on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$; that is, for any disjoint $C_1, C_2, \dots, C_n \in \mathcal{B}(\mathbb{C})$,

$$\mathsf{E}(\Phi(C_1)\,\Phi(C_2)\cdots\Phi(C_n)) = \int_{C_1\times C_2\times\cdots\times C_n} \rho_n(z_1, z_2, \dots, z_n)\,\mu(\mathrm{d}z_1)\,\mu(\mathrm{d}z_2)\cdots\mu(\mathrm{d}z_n),$$

and $\rho_n(z_1, z_2, \dots, z_n) = 0$ when $z_i = z_j$ for $i \neq j$ (see e.g., [9, 16, 17]). The point process Φ is said to be a determinantal point process with kernel $K\colon \mathbb{C}^2 \to \mathbb{C}$ with respect to the reference measure μ if ρ_n, $n \in \mathbb{N}$, satisfy

$$\rho_n(z_1, z_2, \dots, z_n) = \det(K(z_i, z_j))_{1\le i,j\le n}, \quad z_1, z_2, \dots, z_n \in \mathbb{C},$$

where det denotes the determinant. The determinantal point process $\Phi^{*\alpha}$ is said to be an α-Ginibre process with $\alpha \in (0, 1]$ when the kernel is given as $K^{*\alpha}(z, w) = e^{z\overline{w}/\alpha}$, $z, w \in \mathbb{C}$, with respect to the (scaled) Gaussian measure $\mu^{*\alpha}(\mathrm{d}z) = \pi^{-1}\, e^{-|z|^2/\alpha}\, m(\mathrm{d}z)$, where $\overline{w}$ denotes the complex conjugate of $w \in \mathbb{C}$ and m denotes the Lebesgue measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. The choice of pair $(K^{*\alpha}, \mu^{*\alpha})$ is not unique and the determinantal point process with kernel $\tilde{K}^{*\alpha}(z, w) = \pi^{-1}\, e^{-(|z|^2+|w|^2)/(2\alpha)}\, e^{z\overline{w}/\alpha}$ with respect to the Lebesgue measure m defines the same process as $\Phi^{*\alpha}$ (see [9, Section 4.2]). The usual Ginibre point process is just the one with $\alpha = 1$ and it can be shown that $\Phi^{*\alpha}$ converges in distribution to a homogeneous Poisson point process as $\alpha \to 0$ (see [7]). That is, the α-Ginibre processes constitute an intermediate class between the Poisson and Ginibre point processes by adjusting the value of $\alpha \in (0, 1]$. As is the usual Ginibre point process, it can be verified that the α-Ginibre processes are motion-invariant (stationary and isotropic) and their intensities are equal to π^{-1}; that is, $\mathsf{E}\Phi^{*\alpha}(C) = \pi^{-1}\, m(C)$, $C \in \mathcal{B}(\mathbb{R})$, for $\alpha \in (0, 1]$. Thus, to make it have the intensity parameter $\lambda > 0$, we consider the scaled process $\Phi^{*\alpha}_\lambda$ which has the kernel

$$K^{*\alpha}_\lambda(z, w) = e^{\pi\lambda z\overline{w}/\alpha}, \tag{1}$$

with respect to

$$\mu^{*\alpha}_\lambda(\mathrm{d}z) = \lambda\, e^{-\pi\lambda|z|^2/\alpha}\, m(\mathrm{d}z). \tag{2}$$

Or equivalently, $\tilde{K}^{*\alpha}_\lambda(z, w) = \lambda\, e^{-\pi\lambda(|z|^2+|w|^2)/(2\alpha)}\, e^{\pi\lambda z\overline{w}/\alpha}$ with respect to the Lebesgue measure. Due to the radial symmetry of α-Ginibre point processes, we can apply Theorem 4.7.1 of [9] to (1) and (2) above, and we obtain the following proposition, which is a generalization of Kostlan's result [11] for the usual Ginibre point process.

Proposition 1 *Let X_i, $i \in \mathbb{N}$, denote the points of the α-Ginibre point process with intensity λ. Then, the set $\{|X_i|^2\}_{i\in\mathbb{N}}$ has the same distribution as $\check{\boldsymbol{Y}} = \{\check{Y}_i\}_{i\in\mathbb{N}}$, which is constructed from $\boldsymbol{Y} = \{Y_i\}_{i\in\mathbb{N}}$ such that Y_i, $i \in \mathbb{N}$, are mutually independent and each Y_i follows the ith Erlang* *distribution with rate parameter* $\pi\,\lambda/\alpha$ *(*$Y_i \sim \mathrm{Gamma}(i, \pi\,\lambda/\alpha)$*) and it is included in* $\check{\boldsymbol{Y}}$ *with probability* α *independently of others.*

According to Proposition 1, we can construct the α-Ginibre point process $\Phi_\lambda^{*\alpha}$ with intensity λ from the usual Ginibre point process $\Phi_{\lambda/\alpha}^{*1} = \{\overline{X}_i\}_{i\in\mathbb{N}}$ with intensity λ/α by independent α-thinning; that is, by deleting each point $\overline{X}_i$, $i \in \mathbb{N}$, of $\Phi_{\lambda/\alpha}^{*1}$ with probability $1-\alpha$ independently. Note that, by Proposition 1, the set $\{|\overline{X}_i|^2\}_{i\in\mathbb{N}}$ has the same distribution as $\boldsymbol{Y} = \{Y_i\}_{i\in\mathbb{N}}$ such that $Y_i \sim \mathrm{Gamma}(i, \pi\,\lambda/\alpha)$, $i \in \mathbb{N}$, are mutually independent. Let $\{\xi_i\}_{i\in\mathbb{N}}$ denote the set of marks of $\Phi_{\lambda/\alpha}^{*1}$ such that ξ_i, $i \in \mathbb{N}$, are mutually independent and identically distributed as $\mathsf{P}(\xi_i = 1) = \alpha$ and $\mathsf{P}(\xi_i = 0) = 1-\alpha$. Then, $\Phi_\lambda^{*\alpha}$ is obtained by

$$\Phi_\lambda^{*\alpha}(C) = \sum_{i\in\mathbb{N}} \xi_i\,\mathbf{1}_C(\overline{X}_i), \quad C \in \mathcal{B}(\mathbb{C}). \tag{3}$$

which we use as the building block of our analysis in the following sections.

3 Downlink network model with a single tier

We here consider the downlink cellular network model with a single tier (see, e.g., [2, 14]) and investigate the impact of the α-Ginibre point processes on the performance. Let $\Phi = \{X_i\}_{i\in\mathbb{N}}$ denote a point process on $\mathbb{R}^2$, where the order of $X_1, X_2, \dots$ is arbitrary. We assume that Φ is almost surely (a.s.) simple and locally finite, and also stationary with intensity $\lambda > 0$. The point process Φ represents the configuration of the BSs and we refer to the BS located at X_i as BS i. The transmission power of each BS is constant at $p > 0$. We assume that each user is associated with the closest BS; that is, the users in the Voronoi cell of a BS are associated with that BS. Due to the stationarity of Φ and the equality of BSs, we can focus on a typical user located at the origin $o = (0,0)$. We assume the Rayleigh fading for the random effect of fading/shadowing from each BS to a user, so that the fading effect F_i from BS i to the typical user is an exponential random variable with unit mean; $F_i \sim \mathrm{Exp}(1)$, where F_i, $i \in \mathbb{N}$, are mutually independent and also independent of Φ. The path-loss function representing the attenuation of signals with the distance is given by $\ell(r) = c\,r^{-\beta}$, $r > 0$, for some $c > 0$ and $\beta > 2$, where c and β are called respectively the path-loss coefficient and path-loss exponent. The SINR of the typical user from the associated BS is then expressed as

$$\mathsf{SINR}_o = \frac{p\,F_{B_o}\,\ell(|X_{B_o}|)}{W_o + I_o(B_o)}, \tag{4}$$

where B_o denotes the index of the BS associated with the typical user; that is, $\{B_o = i\} = \{|X_i| \le |X_j|, j \in \mathbb{N}\}$, W_o denotes a random variable representing the thermal noise at the origin and $I_o(i) = p \sum_{j\in\mathbb{N}\setminus\{i\}} F_j\, \ell(|X_j|)$ represents the cumulative interference signal from all the BSs except i. We assume that W_o is independent of $\{(X_i, F_i)\}_{i\in\mathbb{N}}$ and the Laplace-Stieltjes transform (LST) of W_o is known to be computable. We consider the coverage probability as a performance index, which is the probability $\mathsf{P}(\mathsf{SINR}_o > \theta)$ that the SINR of the typical user achieves a predefined threshold $\theta > 0$.

In this setting, suppose that $\Phi = \Phi_\lambda^{*\alpha}$; that is, the BSs are deployed according to the α-Ginibre point process with intensity λ (where a point $z = x + i\, y \in \mathbb{C}$ is identified as $(x, y) \in \mathbb{R}^2$). We then have the following.

Theorem 1 *Consider the cellular network model with a single tier such that the BSs are deployed according to the α-Ginibre point process with intensity λ. Then, the downlink coverage probability of a typical user is given by*

$$\mathsf{P}(\mathsf{SINR}_o > \theta) = \alpha \int_0^\infty e^{-s}\, \mathcal{L}_W\Bigl(\frac{\theta}{p\, c}\Bigl(\frac{\alpha\, s}{\pi\, \lambda}\Bigr)^{\beta/2}\Bigr)\, M(s,\theta)\, S(s,\theta)\, \mathrm{d}s, \tag{5}$$

where $\mathcal{L}_W$ denotes the LST of W_o and

$$M(s,\theta) = \prod_{j=0}^\infty \Bigl(1 - \alpha + \frac{\alpha}{j!} \int_s^\infty \frac{t^j\, e^{-t}}{1 + \theta\, (s/t)^{\beta/2}}\, \mathrm{d}t\Bigr), \tag{6}$$

$$S(s,\theta) = \sum_{i=0}^\infty s^i \left((1-\alpha)\, i! + \alpha \int_s^\infty \frac{t^i\, e^{-t}}{1 + \theta\, (s/t)^{\beta/2}}\, \mathrm{d}t\right)^{-1}. \tag{7}$$

Proof: The proof intimately follows the same line as that of Theorem 1 in [14]. The main difference is that we here construct the α-Ginibre point process Φ by (3) from the usual Ginibre point process $\overline{\Phi} = \Phi_{\lambda/\alpha}^{*1} = \{\overline{X}_i\}_{i\in\mathbb{N}}$ with intensity λ/α. Recall that $\{\xi_i\}_{i\in\mathbb{N}}$ in (3) is the set of independent marks of $\overline{\Phi}$ such that $\mathsf{P}(\xi_i = 1) = \alpha$ and $\mathsf{P}(\xi_i = 0) = 1 - \alpha$. Note that a BS really exists at $\overline{X}_i$ only when $\xi_i = 1$, $i \in \mathbb{N}$, so that $\{B_o = i\} = \{\xi_i = 1\} \cap \mathcal{A}_i$, where $\mathcal{A}_i = \{|\overline{X}_i| < |\overline{X}_j| \text{ for } j \in \mathbb{N}_\xi \setminus \{i\}\}$ with random subset $\mathbb{N}_\xi = \{j \in \mathbb{N} \mid \xi_j = 1\}$ of $\mathbb{N}$. Thus, we have from (4) that

$$\begin{aligned}\mathsf{P}(\mathsf{SINR}_o > \theta) &= \sum_{i\in\mathbb{N}} \mathsf{P}(\mathsf{SINR}_o > \theta,\, B_o = i) \\ &= \alpha \sum_{i\in\mathbb{N}} \mathsf{P}\Bigl(F_i > \frac{\theta\, (W_o + I_o(i))}{p\, \ell(|\overline{X}_i|)},\, \mathcal{A}_i\Bigr),\end{aligned} \tag{8}$$

where we use that ξ_i is independent of others with $\mathsf{P}(\xi_i = 1) = \alpha$ in the second equality. Note also that the cumulative interference $I_o(i)$, given $\xi_i = 1$, is now reduced to

$$I_o(i) = p \sum_{j\in\mathbb{N}\setminus\{i\}} \xi_j\, F_j\, \ell(|\overline{X}_j|). \tag{9}$$

Since $F_i \sim \mathrm{Exp}(1)$ is independent of others, and also W_o and $I_o(i)$ are independent of each other, conditioning the inside of P in (8) yields

$$\begin{aligned}
&\mathsf{P}\Bigl(F_i > \frac{\theta\,(W_o + I_o(i))}{p\,\ell(|\overline{X}_i|)},\, \mathcal{A}_i\Bigr) \\
&= \mathsf{E}\Bigl(e^{-\theta W_o/(p\ell(|\overline{X}_i|))}\, e^{-\theta I_o(i)/(p\ell(|\overline{X}_i|))}\, \mathbf{1}_{\mathcal{A}_i}\Bigr) \\
&= \mathsf{E}\Bigl(\mathcal{L}_W\Bigl(\frac{\theta}{p\,\ell(|\overline{X}_i|)}\Bigr)\, \mathsf{E}\Bigl(e^{-\theta I_o(i)/(p\ell(|\overline{X}_i|))} \Bigm| \overline{\Phi},\, \{\xi_j\}_{j\in\mathbb{N}\setminus\{i\}}\Bigr)\, \mathbf{1}_{\mathcal{A}_i}\Bigr),
\end{aligned} \tag{10}$$

where $\mathbf{1}_A$ denotes the indicator for a set A. Furthermore, since $F_j \sim \mathrm{Exp}(1)$, $j \in \mathbb{N}$, are mutually independent, the expression (9) of interference $I_o(i)$ leads to

$$\begin{aligned}
\mathsf{E}\Bigl(e^{-\theta I_o(i)/(p\ell(|\overline{X}_i|))} \Bigm| \overline{\Phi},\, \{\xi_j\}_{j\in\mathbb{N}\setminus\{i\}}\Bigr) &= \prod_{j\in\mathbb{N}\setminus\{i\}} \mathsf{E}\Bigl(e^{-\theta\xi_j F_j \ell(|\overline{X}_j|)/\ell(|\overline{X}_i|)} \bigm| \overline{\Phi},\, \{\xi_j\}_{j\in\mathbb{N}\setminus\{i\}}\Bigr) \\
&= \prod_{j\in\mathbb{N}\setminus\{i\}} \Bigl(1 + \theta\,\xi_j\, \frac{\ell(|\overline{X}_j|)}{\ell(|\overline{X}_i|)}\Bigr)^{-1},
\end{aligned} \tag{11}$$

where the second equality follows from the LST $\mathcal{L}_F(s) = (1+s)^{-1}$ of $F_j \sim \mathrm{Exp}(1)$. On the other hand, note that

$$\mathbf{1}_{\mathcal{A}_i} = \prod_{j\in\mathbb{N}\setminus\{i\}} \mathbf{1}_{\{\xi_j=1, |\overline{X}_j|>|\overline{X}_i|\}\cup\{\xi_j=0\}},$$

and ξ_j, $j \in \mathbb{N}$, are mutually independent. Thus, conditioning on $\overline{\Phi}$, we have from (8), (10) and (11) that

$$\begin{aligned}
\mathsf{P}(\mathsf{SINR}_o > \theta) &= \alpha \sum_{i\in\mathbb{N}} \mathsf{E}\Bigl(\mathcal{L}_W\Bigl(\frac{\theta}{p\,\ell(|\overline{X}_i|)}\Bigr) \prod_{j\in\mathbb{N}\setminus\{i\}} \Bigl(1 + \theta\,\xi_j\, \frac{\ell(|\overline{X}_j|)}{\ell(|\overline{X}_i|)}\Bigr)^{-1} \mathbf{1}_{\{\xi_j=1, |\overline{X}_j|>|\overline{X}_i|\}\cup\{\xi_j=0\}}\Bigr) \\
&= \alpha \sum_{i\in\mathbb{N}} \mathsf{E}\Bigl(\mathcal{L}_W\Bigl(\frac{\theta}{p\,\ell(|\overline{X}_i|)}\Bigr) \prod_{j\in\mathbb{N}\setminus\{i\}} \Bigl[1 - \alpha + \alpha\,\Bigl(1 + \theta\, \frac{\ell(|\overline{X}_j|)}{\ell(|\overline{X}_i|)}\Bigr)^{-1} \mathbf{1}_{\{|\overline{X}_j|>|\overline{X}_i|\}}\Bigr]\Bigr).
\end{aligned} \tag{12}$$

Now, we apply Proposition 1 again to (12); that is, $\{|\overline{X}_i|^2\}_{i\in\mathbb{N}} =_d \{Y_i\}_{i\in\mathbb{N}}$ with $Y_i = \alpha\, Z_i/(\pi\,\lambda)$ such that $Z_i \sim \mathrm{Gamma}(i,1)$, $i \in \mathbb{N}$, are mutually independent. Noting that $\{Y_j > Y_i\} = \{Z_j > Z_i\}$, $j \in \mathbb{N}\setminus\{i\}$, are conditionally independent given Z_i and recalling $\ell(r) = c\,r^{-\beta}$, $r > 0$, we have

$$\mathsf{P}(\mathsf{SINR}_o > \theta)$$

$$= \alpha \sum_{i\in\mathbb{N}} \mathsf{E}\biggl(\mathcal{L}_W\Bigl(\frac{\theta}{p\,c} \Bigl(\frac{\alpha\, Z_i}{\pi\,\lambda} \Bigr)^{\beta/2} \Bigr) \prod_{j\in\mathbb{N}\setminus\{i\}} \Bigl[1 - \alpha + \alpha\, \mathsf{E}\Bigl(\Bigl(1 + \theta \Bigl(\frac{Z_i}{Z_j} \Bigr)^{\beta/2} \Bigr)^{-1} \mathbf{1}_{\{Z_j > Z_i\}} \Bigm| Z_i \Bigr) \Bigr] \biggr).$$

Finally, applying the density function of $Z_i \sim \mathrm{Gamma}(i,1)$, $i \in \mathbb{N}$, to the above, we obtain (5) after some manipulations. □

Remark 1 We find in (5)–(7) that parameters λ, p and c only appear in $\mathcal{L}_W$, which implies that, in the noise-free (interference-limited) case, the coverage probability does not depend on the spatial density of BSs, transmission power and path-loss coefficient.

Remark 2 Similar to Theorem 2 in [14], we can verify that, in the noise-free case, the coverage probability has the asymptotic property as

$$\lim_{\theta\to\infty} \theta^{2/\beta}\, \mathsf{P}(\mathsf{SINR}_o > \theta) = \alpha \int_0^\infty \prod_{j=2}^\infty \Bigl[1 - \alpha + \frac{\alpha}{(j-1)!} \int_0^\infty \frac{t^{j-1}\, e^{-t}}{1 + (s/t)^{\beta/2}}\, \mathrm{d}t \Bigr]\, \mathrm{d}s.$$

That is, the distribution of the SINR for the typical user has the Pareto tail with exponent $-2/\beta$. This asymptotic property is, however, due to the unboundedness of the path-loss function ℓ at the origin. If ℓ is bounded, we can show that the coverage probability decays faster than any polynomials as $\theta \to \infty$ (see [14] for the detail).

In Figure 1, we compare the coverage probability with the different values of α. Each plot gives the coverage probability for a given value of θ in the case of $W_o \equiv 0$ (noise-free) and $c = 1$, $\beta = 4$ ($\ell(r) = r^{-4}$). We find that the coverage probability is increasing in α. The larger value of α indicates that the BSs are deployed more repulsively, so that we have a conjecture that the coverage probability is increasing, or equivalently, the SINR is stochastically increasing, in the repulsion of BS deployments. The rigorous proof of this monotonicity is still open.

4 Downlink network models with multiple tiers

We extend the single tier model in the preceding section to the multitier HCN models. Let K denote a positive integer and let $\mathcal{K} = \{1, 2, \dots, K\}$. The model consists of K tiers of BSs, where the BSs of tier $k \in \mathcal{K}$ have the specific spatial density λ_k, transmission power p_k, path-loss function $\ell_k(r) = c_k\, r^{-\beta_k}$, $r > 0$, given $c_k > 0$ and $\beta_k > 2$, and the SINR target threshold θ_k. The BSs of tier k are deployed according to a stationary and a.s. simple point process Φ_k on $\mathbb{R}^2$ with intensity λ_k. Due to the stationarity of Φ_k, $i \in \mathcal{K}$, we can also focus on a typical user located at the origin $o = (0, 0)$. As in the single tier model, we assume mutually independent

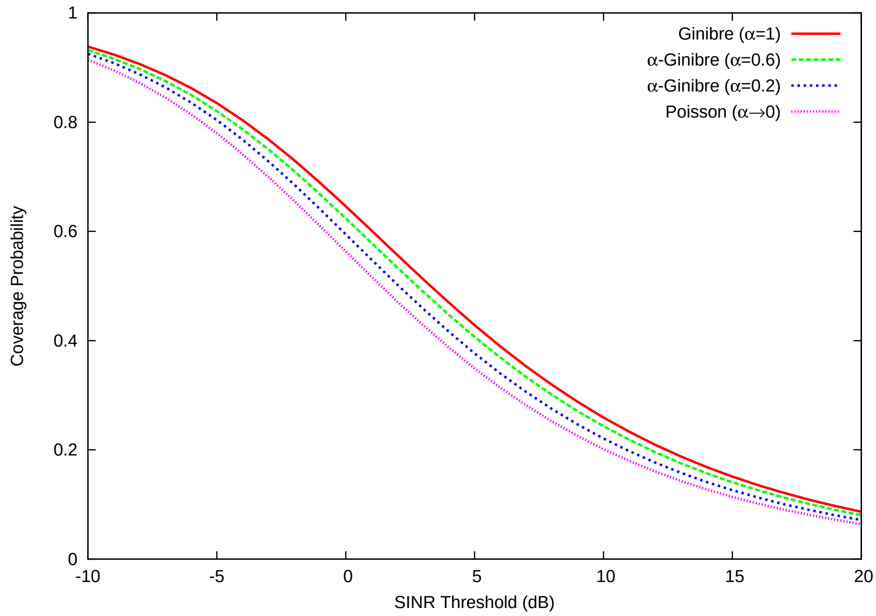

Figure 1: Comparison of coverage probability in terms of α in the single tier model ($\ell(r) = r^{-4}$, no noise).

Rayleigh fading from all the BSs and also an independent thermal noise at the origin. We here consider the unbiased cell association; that is, each user is associated with the BS that offers the strongest average received power. That is, the typical user at the origin is associated with the ith BS of tier k, which is deployed at $X_{k,i}$, when

$$p_k\, \ell_k(|X_{k,i}|) \ge p_m\, \ell_m(|X_{m,j}|) \quad \text{for all } (m,j) \in \mathcal{K} \times \mathbb{N}, \tag{13}$$

where $p_k\, \ell_k(|X_{k,i}|)$ represents the average received power at the origin from the ith BS of tier k since the fading effect is averaged out. Note that the modification to biased association is easy by introducing the bias factor (see [10]). We propose two distinct models below. In Model 1, the BSs of the respective tiers are deployed according to mutually independent α-Ginibre point processes, where the values of α can take different values for the different tiers. In Model 2, on the other hand, all the BSs are deployed according to an α-Ginibre point process and they are classified into multiple tiers by mutually independent marks. Note here that, for the Poisson based models, these two classifications of tiers result in an identical model since the superposition of mutually independent Poisson processes forms a Poisson process and the independent thinning of a Poisson process also yields another Poisson process (see e.g., [3, Section 1.3]).

4.1 Model 1: Mutually independent tiers

The BSs of tier k are deployed according to the α_k-Ginibre point process $\Phi_k = \{X_{k,i}\}_{i\in\mathbb{N}}$, where Φ_k, $k \in \mathcal{K}$, are mutually independent and we refer to the BS at $X_{k,i}$ as BS (k,i). Let $B_o = (\zeta_o, \eta_o)$ represent the index of the BS to which the typical user is associated; that is, ζ_o denotes the tier with which the typical user is associated and $\{B_o = (k,i)\} = \{p_k\,\ell_k(|X_{k,i}|) \ge p_m\,\ell_m(|X_{m,j}|),\ (m,j) \in \mathcal{K}\times\mathbb{N}\}$ by the unbiased association (13). The SINR of the typical user is then expresses as

$$\mathsf{SINR}_o^{(1)} = \frac{p_{\zeta_o}\,F_{B_o}\,\ell_{\zeta_o}(|X_{B_o}|)}{W_o + I_o^{(1)}(B_o)},$$

where $F_{k,i} \sim \mathrm{Exp}(1)$, $(k,i) \in \mathcal{K}\times\mathbb{N}$, represent the fading effect from BS (k,i) to the typical user and the interference I_o is given by

$$I_o^{(1)}(k,i) = \sum_{(m,j)\in\mathcal{K}\times\mathbb{N}\setminus\{(k,i)\}} p_m\,F_{m,j}\,\ell_m(|X_{m,j}|). \tag{14}$$

Theorem 2 *Consider the HCN model with K tiers, where the tiers are mutually independent and the BSs of tier k are deployed according to the α_k-Ginibre point process with intensity λ_k, $k \in \mathcal{K}$. Then, the downlink coverage probability of a typical user is given by*

$$\mathsf{P}(\mathsf{SINR}_o > \theta_{\zeta_o}) = \sum_{k=1}^{K} \alpha_k \int_0^\infty e^{-s}\,\mathcal{L}_W\Bigl(\frac{\theta_k}{p_k\,c_k}\Bigl(\frac{\alpha_k\,s}{\pi\,\lambda_k}\Bigr)^{\beta_k/2}\Bigr)\,M_k^{(1)}(s,\theta_k)\,S_k^{(1)}(s,\theta_k)\,\mathrm{d}s, \tag{15}$$

where

$$M_k^{(1)}(s,\theta) = \prod_{m=1}^{K}\prod_{j=0}^{\infty}\Biggl(1-\alpha_m+\frac{\alpha_m}{j!}\int_{C_{k,m}^{(1)}(s)}^\infty \frac{t^j\,e^{-t}}{1+\theta\,\bigl(C_{k,m}^{(1)}(s)/t\bigr)^{\beta_m/2}}\,\mathrm{d}t\Biggr),$$

$$S_k^{(1)}(s,\theta) = \sum_{i=0}^{\infty} s^i\,\Biggl((1-\alpha_k)\,i! + \alpha_k\int_s^\infty \frac{t^i\,e^{-t}}{1+\theta\,(s/t)^{\beta_k/2}}\,\mathrm{d}t\Biggr)^{-1},$$

and

$$C_{k,m}^{(1)}(s) = \frac{\pi\,\lambda_m}{\alpha_m}\Bigl(\frac{p_m\,c_m}{p_k\,c_k}\Bigr)^{2/\beta_m}\Bigl(\frac{\alpha_k\,s}{\pi\,\lambda_k}\Bigr)^{\beta_k/\beta_m},\quad s \ge 0. \tag{16}$$

Proof: The proof is similar to that of Theorem 1 but is slightly complicated due to the multiple tiers. For $k \in \mathcal{K}$, let $\overline{\Phi}_k = \{\overline{X}_{k,i}\}_{i\in\mathbb{N}}$ denote the usual Ginibre point process with intensity λ_k/α_k and let $\{\xi_{k,i}\}_{i\in\mathbb{N}}$ denote the sequence of independent marks of $\overline{\Phi}_k$ such that $\mathsf{P}(\xi_{k,i} = 1) = \alpha_k$ and $\mathsf{P}(\xi_{k,i} = 0) = 1-\alpha_k$. The α_k-Ginibre point process Φ_k is then constructed by

$$\Phi_k(C) = \sum_{i\in\mathbb{N}} \xi_{k,i}\,\mathbf{1}_C(\overline{X}_{k,i}),\quad C \in \mathcal{B}(\mathbb{C}),$$

and the interference (14) is reduced to

$$I_o^{(1)}(k,i) = \sum_{(m,j)\in\mathcal{K}\times\mathbb{N}\setminus\{(k,i)\}} p_m\,\xi_{m,j}\,F_{m,j}\,\ell_m(|\overline{X}_{m,j}|). \tag{17}$$

Note that $\{B_o = (k,i)\} = \{\xi_{k,i} = 1\} \cap \mathcal{A}_{k,i}^{(1)}$, where

$$\mathcal{A}_{k,i}^{(1)} = \{\ p_k\,\ell_k(|\overline{X}_{k,i}|) > p_m\,\ell_m(|\overline{X}_{m,j}|) \text{ for } (m,j) \in (\mathcal{K}\times\mathbb{N})_\xi \setminus \{(k,i)\}\ \},$$

with $(\mathcal{K}\times\mathbb{N})_\xi = \{(m,j)\in\mathcal{K}\times\mathbb{N} \mid \xi_{m,j} = 1\}$. Thus, we have

$$\begin{aligned}
\mathsf{P}(\mathsf{SINR}_o^{(1)} > \theta_{\zeta_o}) &= \sum_{(k,i)\in\mathcal{K}\times\mathbb{N}} \mathsf{P}\bigl(\mathsf{SINR}_o^{(1)} > \theta_k, B_o = (k,i)\bigr) \\
&= \sum_{(k,i)\in\mathcal{K}\times\mathbb{N}} \alpha_k\,\mathsf{P}\biggl(F_{k,i} > \frac{\theta_k\,(W_o + I_o^{(1)}(k,i))}{p_k\,\ell_k(|\overline{X}_{k,i}|)},\, \mathcal{A}_{k,i}^{(1)}\biggr),
\end{aligned} \tag{18}$$

where we use that $\xi_{k,i}$ is independent of others with $\mathsf{P}(\xi_{k,i} = 1) = \alpha_k$ in the second equality. Similar to deriving (10) and (11), we have by applying (17),

$$\begin{aligned}
&\mathsf{P}\biggl(F_{k,i} > \frac{\theta_k\,(W_o + I_o^{(1)}(k,i))}{p_k\,\ell_k(|\overline{X}_{k,i}|)},\, \mathcal{A}_{k,i}^{(1)}\biggr) \\
&= \mathsf{E}\biggl(\mathcal{L}_W\Bigl(\frac{\theta_k}{p_k\,\ell_k(|\overline{X}_{k,i}|)}\Bigr) \prod_{(m,j)\in\mathcal{K}\times\mathbb{N}\setminus\{(k,i)\}} \Bigl(1 + \theta_k\,\xi_{m,j}\,\frac{p_m\,\ell_m(|\overline{X}_{m,j}|)}{p_k\,\ell_k(|\overline{X}_{k,i}|)}\Bigr)^{-1} \mathbf{1}_{\mathcal{A}_{k,i}^{(1)}}\biggr).
\end{aligned} \tag{19}$$

Here, note that

$$\mathbf{1}_{\mathcal{A}_{k,i}^{(1)}} = \prod_{(m,j)\in\mathcal{K}\times\mathbb{N}\setminus\{(k,i)\}} \mathbf{1}_{\{\xi_{m,j}=1,\, p_m\,\ell_m(|\overline{X}_{m,j}|)<p_k\,\ell_k(|\overline{X}_{k,i}|)\}\cup\{\xi_{m,j}=0\}},$$

so that, applying this to (19) and conditioning on $\overline{\Phi}_k$, we have

$$\begin{aligned}
&\mathsf{P}(\mathsf{SINR}_o^{(1)} > \theta_{\zeta_o}) \\
&= \sum_{(k,i)\in\mathcal{K}\times\mathbb{N}} \alpha_k\,\mathsf{E}\biggl(\mathcal{L}_W\Bigl(\frac{\theta_k}{p_k\,\ell_k(|\overline{X}_{k,i}|)}\Bigr) \\
&\quad\times \prod_{(m,j)\in\mathcal{K}\times\mathbb{N}\setminus\{(k,i)\}} \Bigl[1 - \alpha_m + \alpha_m\,\Bigl(1 + \theta_k\,\frac{p_m\,\ell_m(|\overline{X}_{m,j}|)}{p_k\,\ell_k(|\overline{X}_{k,i}|)}\Bigr)^{-1} \mathbf{1}_{\{p_m\,\ell_m(|\overline{X}_{m,j}|)<p_k\,\ell_k(|\overline{X}_{k,i}|)\}}\Bigr]\biggr).
\end{aligned}$$

Thus, applying Proposition 1 such that $\{|\overline{X}_{k,i}|^2\}_{i\in\mathbb{N}} =_d \{Y_{k,i}\}_{i\in\mathbb{N}}$ with $Y_{k,i} = \alpha_k\,Z_{k,i}/(\pi\,\lambda_k)$ where $Z_{k,i} \sim \text{Gamma}(i,1)$ are mutually independent and recalling $\ell_k(r) = c_k\,r^{-\beta_k}$, $r > 0$, we have

$$\mathsf{P}(\mathsf{SINR}_o^{(1)} > \theta_{\zeta_o})$$

$$= \sum_{(k,i)\in\mathcal{K}\times\mathbb{N}} \alpha_k \, \mathsf{E}\biggl(\mathcal{L}_W\Bigl(\frac{\theta_k}{p_k\, c_k}\Bigl(\frac{\alpha_k\, Z_{k,i}}{\pi\,\lambda_k}\Bigr)^{\beta_k/2}\Bigr)$$

$$\times \prod_{(m,j)\in\mathcal{K}\times\mathbb{N}\setminus\{(k,i)\}} \biggl[1-\alpha_m+\alpha_m\, \mathsf{E}\biggl(\Bigl(1+\theta_k\Bigl(\frac{C_{k,m}^{(1)}(Z_{k,i})}{Z_{m,j}}\Bigr)^{\beta_m/2}\Bigr)^{-1} \mathbf{1}_{\{Z_{m,j}>C_{k,m}(Z_{k,i})\}} \biggm| Z_{k,i}\biggr)\biggr]\biggr),$$

where $C_{k,m}^{(1)}$ in (16) is applied. Finally, applying the density function of $Z_{k,i}$ to the above, we obtain (15) after some manipulations. □

Remark 3 We find in Theorem 2 that, in the noise-free case, the coverage probability does not depend on the values of p_k and c_k, $k\in\mathcal{K}$, but on the ratios $p_m\, c_m/(p_k\, c_k)$, $k,m\in\mathcal{K}$. In addition, if all β_k, $k\in\mathcal{K}$, are equal, it does not also depend on the values of λ_k but on the ratios λ_m/λ_k, $k,m\in\mathcal{K}$.

4.2 Model 2: Classification by independent marks

In the second model, all the BSs are deployed according to an α-Ginibre point process $\Phi = \{X_i\}_{i\in\mathbb{N}}$ with intensity $\lambda = \sum_{k=1}^K \lambda_k$, where we refer to the BS at X_i as BS i. Let $\{\kappa_i\}_{i\in\mathbb{N}}$ denote a sequence of independent marks of Φ such that κ_i, $i\in\mathbb{N}$, are mutually independent and distributed as $\mathsf{P}(\kappa_i = k) = \lambda_k/\lambda$, $k\in\mathcal{K}$, $i\in\mathbb{N}$; that is, κ_i represents the tier of BS i. Let B_o denote the index of the BS to which the typical user is associated. Then, in this model, we have $\{B_o = i\} = \{p_{\kappa_i}\,\ell_{\kappa_i}(|X_i|) \ge p_{\kappa_j}\,\ell_{\kappa_j}(|X_j|),\ j\in\mathbb{N}\}$ by the unbiased association (13) and the SINR of the typical user is given by

$$\mathsf{SINR}_o^{(2)} = \frac{p_{\kappa_{B_o}}\, F_{B_o}\, \ell_{\kappa_{B_o}}(|X_{B_o}|)}{W_o + I_o^{(2)}(B_o)},$$

with mutually independent fading $F_i \sim \mathrm{Exp}(1)$, $i\in\mathbb{N}$, and the interference,

$$I_o^{(2)}(i) = \sum_{j\in\mathbb{N}\setminus\{i\}} p_{\kappa_j}\, F_j\, \ell_{\kappa_j}(|X_j|). \tag{20}$$

Theorem 3 *Consider the HCN model with K tiers, where the BSs are deployed according to the α-Ginibre point process with intensity λ and each BS belongs to the kth tier with probability λ_k/λ, $k\in\mathcal{K}$, independently of others. Then, the downlink coverage probability of a typical user is given by*

$$\mathsf{P}(\mathsf{SINR}_o^{(2)} > \theta_{\kappa_{B_o}}) = \alpha \sum_{k=1}^K \frac{\lambda_k}{\lambda} \int_0^\infty e^{-s}\, \mathcal{L}_W\Bigl(\frac{\theta_k}{p_k\, c_k}\Bigl(\frac{\alpha\, s}{\pi\,\lambda}\Bigr)^{\beta_k/2}\Bigr)\, M_k^{(2)}(s,\theta_k)\, S_k^{(2)}(s,\theta_k)\, \mathrm{d}s, \tag{21}$$

where

$$M_k^{(2)}(s,\theta) = \prod_{j=0}^{\infty}\biggl(1-\alpha+\frac{\alpha}{j!}\sum_{m=1}^{K}\frac{\lambda_m}{\lambda}\int_{C_{k,m}^{(2)}(s)}^{\infty}\frac{t^j\,e^{-t}}{1+\theta\,\bigl(C_{k,m}^{(2)}(s)/t\bigr)^{\beta_m/2}}\,\mathrm{d}t\biggr),$$

$$S_k^{(2)}(s,\theta) = \sum_{i=0}^{\infty} s^i\,\Biggl((1-\alpha)\,i!+\alpha\sum_{m=1}^{K}\frac{\lambda_m}{\lambda}\int_{C_{k,m}^{(2)}(s)}^{\infty}\frac{t^i\,e^{-t}}{1+\theta\,\bigl(C_{k,m}^{(2)}(s)/t\bigr)^{\beta_m/2}}\,\mathrm{d}t\Biggr)^{-1},$$

and

$$C_{k,m}^{(2)}(s) = \Bigl(\frac{\pi\,\lambda}{\alpha}\Bigr)^{1-\beta_k/\beta_m}\Bigl(\frac{p_m\,c_m}{p_k\,c_k}\Bigr)^{2/\beta_m} s^{\beta_k/\beta_m},\quad s\ge 0.$$

Proof: The proof is similar to those of Theorems 1 and 2, and we note only the differences. As in the proof of Theorem 1, let $\overline{\Phi} = \{\overline{X}_i\}_{i\in\mathbb{N}}$ denote the usual Ginibre point process with intensity λ/α, and let $\{\xi_i\}_{i\in\mathbb{N}}$ denote the set of independent marks of $\overline{\Phi}$ such that $\mathsf{P}(\xi_i = 1) = \alpha$ and $\mathsf{P}(\xi_i = 0) = 1-\alpha$. The interference (20) then reduces to

$$I_o^{(2)}(i) = \sum_{j\in\mathbb{N}\setminus\{i\}} p_{\kappa_j}\,\xi_j\,F_j\,\ell_{\kappa_j}(|\overline{X}_j|).$$

Instead of the event $\{B_o = i\}$ in the proofs of Theorems 1 and 2, we here use $\{B_o = i,\,\kappa_i = k\} = \{\xi_i = 1, \kappa_i = k\}\cap\mathcal{A}_{k,i}^{(2)}$, where

$$\mathcal{A}_{k,i}^{(2)} = \bigl\{p_k\,\ell_k(|X_i|) > p_{\kappa_j}\,\ell_{\kappa_j}(|X_j|)\text{ for } j\in\mathbb{N}_\xi\setminus\{i\}\bigr\},$$

with $\mathbb{N}_\xi = \{j\in\mathbb{N}\mid \xi_j = 1\}$. The remaining procedures are almost the same as those in the proofs of previous theorems except that we use $\mathsf{P}(\xi_i = 1, \kappa_i = k) = \alpha\,\lambda_k/\lambda$, $i\in\mathbb{N}$, $k\in\mathcal{K}$, and are omitted. □

Remark 4 As in Model 1, in the noise-free case, the coverage probability for Model 2 does not depend on the values of p_k and c_k, $k\in\mathcal{K}$, but on the ratios $p_m\,c_m/(p_k\,c_k)$, $k,m\in\mathcal{K}$. Furthermore, if all β_k, $k\in\mathcal{K}$, are equal, it does not depend on the values of λ_k but on the ratios λ_m/λ_k, $k,m\in\mathcal{K}$.

5 Numerical experiments

We compare the analytical results on the coverage probability for the two proposed models as well as that for the Poisson based model through some numerical experiments. Following [10], the coverage probability for the Poisson based model is given by

$$\mathsf{P}(\mathsf{SINR}_o^{(\mathrm{P})} > \theta_{\zeta_o}) = \sum_{k=1}^{K}\int_0^{\infty}\mathcal{L}_W\Bigl(\frac{\theta_k}{p_k\,c_k}\Bigl(\frac{v}{\pi\,\lambda_k}\Bigr)^{\beta_k/2}\Bigr)\,\exp\Bigl\{-\sum_{m=1}^{K}C_{k,m}^{(\mathrm{P})}(v)\,\Bigl(1+\rho(\theta_k,\beta_m)\Bigr)\Bigr\}\,\mathrm{d}v, \tag{22}$$

where $\mathsf{SINR}_o^{(\mathrm{P})}$ denotes the SINR of a typical user in the Poisson based model and

$$C_{k,m}^{(\mathrm{P})}(v) = \pi\,\lambda_m \left(\frac{p_m\,c_m}{p_k\,c_k}\right)^{2/\beta_m} \left(\frac{v}{\pi\,\lambda_k}\right)^{\beta_k/\beta_m}, \tag{23}$$

$$\rho(\theta,\beta) = \frac{2\,\theta^{2/\beta}}{\beta} \int_{1/\theta}^{\infty} \frac{u^{-1+2/\beta}}{1+u}\,\mathrm{d}u. \tag{24}$$

Throughout the experiments, we restrict ourselves to two-tier network models consisting of macrocells and picocells for simplicity and assume the noise-free case. We further set the path-loss coefficients and exponents at $c_1 = c_2 = 1$ and $\beta_1 = \beta_2 = 3$; that is, $\ell_1(r) = \ell_2(r) = r^{-3}$, $r > 0$. It is, of course, not difficult to extend these studies to more general settings. Since the path-loss functions are identical for the different tiers in the noise-free case, the coverage probability does not depend on the values of p_k or λ_k, $k = 1, 2$, but on the ratios p_1/p_2 and λ_1/λ_2 for both of the two proposed models (see Remarks 3 and 4). We can also find by (22)–(24) that this is the case for the Poisson based model. We set $p_1/p_2 = 100$ in the experiments below.

In the first experiment, we examine the impact of the values of (α_1, α_2) on the coverage probability for Model 1. Figure 2 shows the plots of the coverage probability for the given values of the SINR threshold $\theta = \theta_1 = \theta_2$, where the ratio of the intensities is set at $\lambda_1/\lambda_2 = 1/3$. We can see that the difference in the values of α_2 hardly has the influence on the coverage probability while the coverage probability decreases when the value of α_1 decreases. This implies that the repulsion in the low-power BSs has little effect on the coverage probability while that in the high-power BSs has the influence. As expected, the independent configuration (by the Poisson based model) shows the smallest coverage probability.

In the second experiment, we compare Models 1, 2 and the Poisson based model for varying ratio of the intensities λ_1/λ_2. Figure 3 plots the coverage probability for the given values of the SINR threshold $\theta = \theta_1 = \theta_2$, where we fix $\alpha_1 = \alpha_2 = \alpha = 1$ for Models 1 and 2. While the values of coverage probability for Model 2 and Poisson based model remain almost unchanged for the different values of λ_1/λ_2, that for Model 1 decreases when the intensity of the tier 2 increases. This result suggests that, if you would increase the number of low-power BSs, then you should take care of the repulsion between the different tiers.

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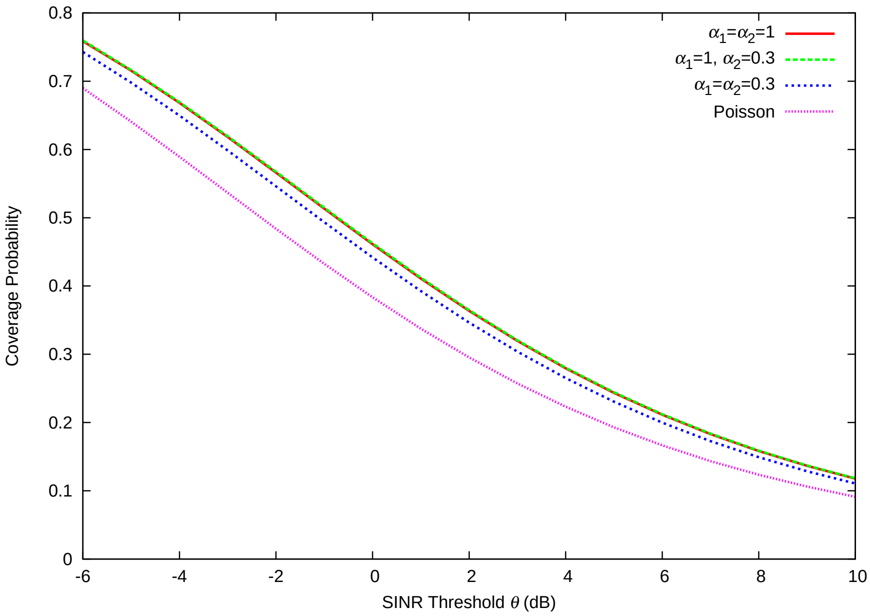

Figure 2: Coverage probability for Model 1 with different values of (α_1, α_2) and for the Poisson based model ($\lambda_1/\lambda_2 = 1/3$, $p_1/p_2 = 100$, $\ell_1(r) = \ell_2(r) = r^{-3}$, $r > 0$, $\theta = \theta_1 = \theta_2$).

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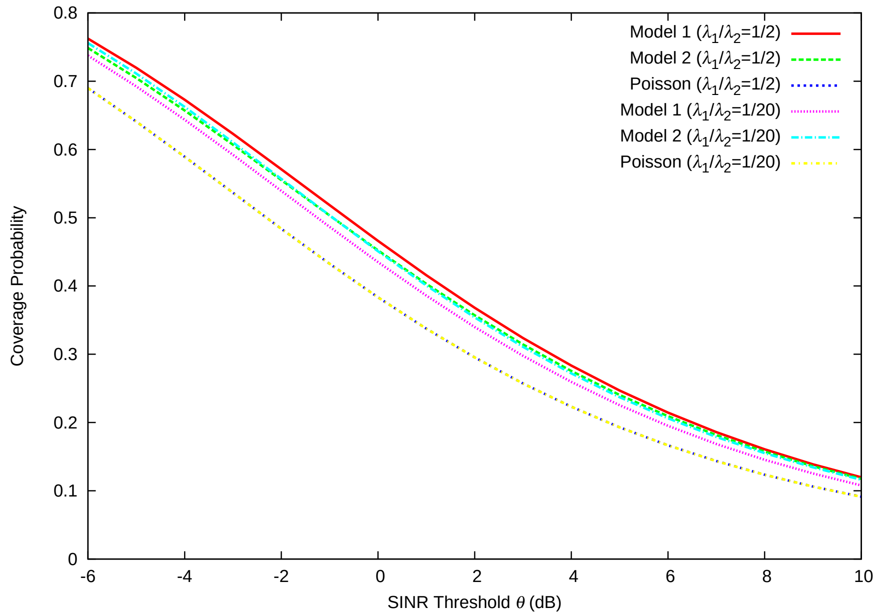

Figure 3: Coverage probability for Models 1, 2 and the Poisson based model with different values of λ_1/λ_2 ($p_1/p_2 = 100$, $\ell_1(r) = \ell_2(r) = r^{-3}$, $r > 0$, $\theta = \theta_1 = \theta_2$).

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