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Equivalences and Differences in Conic Relaxations of Combinatorial Quadratic Optimization Problems

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Abstract. Various conic relaxations of quadratic optimization problems in nonnegative variables for combinatorial optimization problems, such as the binary integer quadratic problem, quadratic assignment problem (QAP), and maximum stable set problem have been proposed over the years. The binary and complementarity conditions of the combinatorial optimization problems can be expressed in several ways, each of which results in different conic relaxations. For the completely positive, doubly nonnegative and semidefinite relaxations of the combinatorial optimization problems, we prove the equivalences and differences among the relaxations by investigating the feasible regions obtained from different representations of the combinatorial condition, a generalization of the binary and complementarity condition. We also study theoretically the issue of the primal and dual nondegeneracy, the existence of an interior solution and the size of the relaxations, as a result of different representations of the combinatorial condition. These characteristics of the conic relaxations affect the numerical efficiency and stability of the solver used to solve them. We illustrate the theoretical results with numerical results on QAP instances solved by SDPT3, SDPNAL+ and the bisection and projection method.

Key words. Combinatorial optimization problems, binary and complementarity condition, completely positive relaxations, doubly nonnegative relaxations, semidefinite relaxations, equivalence of feasible regions, nondegeneracy.

AMS Classification. 90C20, 90C22, 90C25, 90C26.

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1 Introduction

We consider a general nonconvex quadratic optimization problem (abbreviated as QOP) with quadratic equality and/or inequality constraints. As the quadratic equalities can model the binary and complementarity conditions, QOPs have been studied for formulating various combinatorial optimization problems. More precisely, the binary condition $x_i \in \{a, b\}$ can be represented as a quadratic equality $(x_i - a)(b - x_i) = 0$ and the complementarity condition, which requires one of two variables x_i and x_j to be zero, can be written as $x_i x_j = 0$. As a result, various combinatorial optimization problems arising from important applications, such as the binary integer quadratic problem (BIQP), maximum stable set problem, max-cut problem, and quadratic assignment problem (QAP) have been dealt with in the framework of QOPs.

The optimal value of the nonconvex QOP in general cannot be exactly found efficiently by a computational method. Instead, it is approximated by a lower bound obtained from a conic relaxation of the problem. A conic relaxation problem is a (linear) conic optimization problem (COP) that minimizes a linear function in a symmetric matrix variable subject to linear equalities and inequalities and a cone constraint. When the cone consists of the positive semidefinite matrices, the doubly nonnegative matrices or the completely positive matrices, the COP is called a semidefinite programming (SDP) problem, a doubly nonnegative programming (DNN) problem or a completely positive programming (CPP) problem, respectively.

Two fundamental approaches have been known for constructing a conic relaxation problem from a given QOP. Shor [25] derived an SDP relaxation problem from the Lagrangian dual of a QOP. The other primal-based approach is called the lifting (procedure). Based on these two approaches, Poljak, Rendl and Wolkowicz [19] presented a general method for constructing an SDP relaxation problem from a given QOP. The lifting and the Lagrangian dual may be regarded as the most fundamental principles in the conic relaxation.

Recent developments on CPP reformulations of QOPs, which have shown to attain the exact optimal values of QOPs under rather mild assumptions, have given rise to extensive studies of CPP reformulations of combinatorial problems. In particular, QOPs over the standard simplex [7, 8], maximum stable set problems [10], graph partitioning problems [21], and quadratic assignment problems [22] are equivalently expressed as CPPs. Burer's CPP reformulations [9] of a class of linearly constrained QOP in continuous and binary variables generalizes the problems considered in [7, 8, 10, 21, 22]. However, these CPP reformulations of QOPs are mainly of theoretical interest as CPPs are known to be numerically intractable [18].

Numerically tractable DNN relaxations are preferred over SDP relaxations as the former can generally provide a much higher lower bound. But the computational burden of solving a DNN problem increases very rapidly with the matrix size. Numerical methods to mitigate this difficulty have been proposed, for instance, the bisection and projection method [6, 15, 16] and SDPNAL+ [27]. They both reported numerical results on large-scale DNN problems that could not be solved by primal-dual interior-point methods.

While many conic relaxations have been proposed for various QOP instances over the years, their equivalences and differences are not clearly understood for most of the conic relaxations. In theory, two conic relaxations are determined to be equivalent if they have a common optimal value. This theoretical equivalence, however, never guarantees their computational equivalence. That is, when they are solved by a numerical method, one conic relaxation problem may provide a better approximate optimal value in less execution time than the other. This difference is often caused by the difference in the size of the problem, the existence of primal/dual interior feasible solutions and the primal/dual nondegeneracy. Thus two conic relaxations can be equivalent in some aspects but different in some others.

The purpose of this paper is, first, to present a theoretical basis that reveals the underlying connections of various conic relaxations of QOPs arising from combinatorial optimization problems, in particular, their equivalences and differences with respect to the optimal value, the sizes of the conic relaxation problems, the existence of primal/dual interior feasible solutions, and the primal/dual nondegeneracy. Second, to further investigate the simplification technique by Burer [9] for reducing the size of matrix variable for numerical efficiency. We show that the technique has the potential to reduce possible primal/dual degeneracy.

We briefly explain how various conic relaxations can be generated in a systematic manner from a QOP with a combinatorial condition. Our QOP model is

$$\zeta^* = \min \left\{ f_0(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n_+, \ f_j(\boldsymbol{x}) = 0 \ (1 \leq j \leq \ell) \right\}, \tag{1}$$

where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $f_j : \mathbb{R}^n \to \mathbb{R}$ are defined as $f_j(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q}_j \boldsymbol{x} + 2\boldsymbol{c}_j^T \boldsymbol{x} - d_j$, $\boldsymbol{Q}_j$ is a symmetric matrix, $\boldsymbol{c}_j \in \mathbb{R}^n$, $d_j \in \mathbb{R}$ $(0 \leq j \leq \ell)$. As a combinatorial condition, we consider:

(G) Exactly one of x_j $(j \in J_p)$ is 1 and all others are 0 $(1 \leq p \leq m)$,

where each J_p $(1 \leq p \leq m)$ is a nonempty subset of $\{1, \ldots, n\}$. Condition (G) can be represented as quadratic equalities, thus, (G) can be embedded as the equalities $f_j(\boldsymbol{x}) = 0$ $(j = 1, \ldots, \ell)$ of QOP (1). It is important to note that the representation is not unique.

Condition (G) is an extension of the 0-1 condition $x \in \{0, 1\}$ with a slack variable: $x + u = 1$ and $x, u \in \{0, 1\}$, which can be written as $(x + u - 1)^2 = 0$, $x(1 - x) = 0$ and $u(1 - u) = 0$. This is the well-known quadratic equality representation of 0-1 variable. Alternatively, the first equality can be replaced by the two equalities $(x + u)^2 = 1$ and $x + u = 1$, and the last two by the complementarity condition $x, u \geq 0$ and $xu = 0$. Consequently, at least four different combinations of quadratic equality constraints can be obtained in this simple example. The number of quadratic equality representations increases in general cases. Thus, we consider (G) separately from the equalities as

$$\zeta^*(G) = \min \left\{ f_0(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n_+, \ f_j(\boldsymbol{x}) = 0 \ (1 \leq j \leq \ell_0), \ (\mathrm{G}) \right\}, \tag{2}$$

where $0 \leq \ell_0 \leq \ell$ and $f_j(\boldsymbol{x}) = 0$ $(1 \leq j \leq \ell_0)$ are regarded as given (non-combinatorial) quadratic equality constraints.

For constructing a conic relaxation of QOP (2), we apply the following two procedures to (2):

(I) Replace Condition (G) with an equivalent set of quadratic equalities $f_j(\boldsymbol{x}) = 0$ $(\ell_0 + 1 \leq \ell_0 + \ell_1)$ for some ℓ_1 to obtain a QOP of the form (1) with $\ell = \ell_0 + \ell_1$.

(II) Apply the standard lifting to QOP (1) with a closed convex cone, chosen among the SDP, DNN and CPP cones.

As a result of Procedure (II) applied to QOP (1), we obtain a COP, which depends on the choices of quadratic equalities to represent Condition (G) in (I) and the closed convex cone in (II). Although some of the COPs with a common closed convex cone, say the DNN cone, have an equivalent optimal value, they may be different in other issues related to the numerical efficiency and stability.

This paper is organized as follows: In Section 2, we explain the lifting in Procedure (II) for QOP (1), and describe the simplification technique to reduce the size of the matrix variable based on Burer's idea [9]. The main results of this paper are dealt with in Sections 3, 4 and 5. Specifically, Section 3 is devoted to Procedure (I) in detail. We also show fundamental relations among the liftings of quadratic equality constraints for representing Condition (G), and how Burer's simplification technique [9] can be applied to the lifted constraints. In Section 4, we investigate the differences in the issues related to the numerical efficiency and stability among the conic relaxation problems of QOP (2) generated in Section 3. Section 5 discusses the effectiveness of Burer's simplification to resolve the primal and dual degeneracy in the conic relaxation. In Section 6, our main theoretical results in Section 3 are applied to two classes of QOP instances, the BIQP and the QAP. Relations among the existing conic relaxations of these instances and the newly presented conic relaxations in Section 3 are shown. In Section 7, numerical results on QAP instances are presented to partially illustrate the theoretical results in Sections 3, 4 and 5.

2 Preliminaries

2.1 Notation and symbols

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. We assume that each $\boldsymbol{x} \in \mathbb{R}^n$ is a column vector of element x_i $(1 \leq i \leq n)$, and $\boldsymbol{x}^T$ denotes the transpose of $\boldsymbol{x} \in \mathbb{R}^n$. Let $\mathbb{R}^n_+$ denote the nonnegative orthant $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ of $\mathbb{R}^n$. Let $\mathbb{S}^m$ denote the linear space of $m \times m$ real symmetric matrices. We introduce cones of matrices in $\mathbb{S}^m$ as follows:

$$
\begin{aligned}
\mathbb{S}^m_+ &= \left\{\boldsymbol{A} \in \mathbb{S}^m : \boldsymbol{z}^T\boldsymbol{A}\boldsymbol{z} \geq 0 \text{ for all } \boldsymbol{z} \in \mathbb{R}^m\right\} \\
&= \text{the conic hull of } \left\{\boldsymbol{z}\boldsymbol{z}^T : \boldsymbol{z} \in \mathbb{R}^m\right\} \quad \text{(the positive semidefinite (SDP) cone)}, \\
\Gamma^m &= \left\{\boldsymbol{z}\boldsymbol{z}^T : \boldsymbol{z} \in \mathbb{R}^m_+\right\}, \\
\mathbb{C}^m_{\rm PP} &= \left\{\boldsymbol{B} \in \mathbb{S}^m : \begin{array}{l} \boldsymbol{B} = \sum_{i=1}^r \boldsymbol{z}_i\boldsymbol{z}_i^T,\ \boldsymbol{z}_i \in \mathbb{R}^m_+ \ (1 \leq i \leq r) \\ \text{for some positive integer } r \end{array}\right\} \\
&= \text{the conic hull of } \Gamma^m \text{ (the completely positive (CPP) cone)}, \\
\mathbb{N}^m &= \{\boldsymbol{A} \in \mathbb{S}^m : A_{ij} \geq 0 \ (1 \leq i \leq m,\ 1 \leq j \leq m)\} \\
&\quad \text{(the cone of nonnegative matrices)}, \\
\mathbb{D}^m_{\rm NN} &= \mathbb{S}^m_+ \cap \mathbb{N}^m \quad \text{(the doubly nonnegative (DNN) cone)}.
\end{aligned}
$$

Note that Γ^m is a closed nonconvex cone in $\mathbb{S}^m$ if $m \geq 2$ and that all others are closed convex cones in $\mathbb{S}^m$.

Given matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{S}^m$, $\boldsymbol{A} \bullet \boldsymbol{B}$ stands for the inner product $\sum_{i=1}^m \sum_{j=1}^m A_{ij}B_{ij}$. Given $\boldsymbol{Q} \in \mathbb{S}^m$, we often write the quadratic form $\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ in a vector variable $\boldsymbol{x} \in \mathbb{R}^m$ as $\boldsymbol{Q} \bullet \boldsymbol{x}\boldsymbol{x}^T$, which is relaxed into a linear form $\boldsymbol{Q} \bullet \boldsymbol{Y}$ in the matrix variable $\boldsymbol{Y} \in \mathbb{S}^m_+$ by replacing

$\boldsymbol{x}\boldsymbol{x}^T \in \mathbb{S}^m_+$ with $\boldsymbol{Y} \in \mathbb{S}^m_+$. The notation $\mathrm{diag}(\boldsymbol{Y})$ means the m-dimensional column vector consisting of the diagonal elements Y_{ii} $(1 \leq i \leq m)$ of $\boldsymbol{Y} \in \mathbb{S}^m$. With this notation, the 0-1 condition $x_i \in \{0, 1\}$ $(1 \leq i \leq m)$ for $\boldsymbol{x} \in \mathbb{R}^m$ can be written as $\boldsymbol{x} = \mathrm{diag}(\boldsymbol{x}\boldsymbol{x}^T)$.

2.2 A general QOP and its conic relaxations

To derive various conic relaxations of QOP (1) in a systematic manner, we first introduce constant and variable matrices. Let

$$
\begin{aligned}
\boldsymbol{H}_0 &= \text{the } (1+n) \times (1+n) \text{ matrix with 1 at the upper-left corner and 0 elsewhere,} \\
\boldsymbol{F}_j &= \begin{pmatrix} -d_j & \boldsymbol{c}_j^T \\ \boldsymbol{c}_j & \boldsymbol{Q}_j \end{pmatrix} \in \mathbb{S}^{1+n} \ (0 \leq j \leq \ell), \ \boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \in \mathbb{S}^{1+n}.
\end{aligned}
$$

Each quadratic function f_j is written as $f_j(\boldsymbol{x}) = \boldsymbol{F}_j \bullet \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix}$, and the function f_j in $\boldsymbol{x} \in \mathbb{R}^n$ is *lifted* to a linear function $\bar{f}_j$ in $\boldsymbol{Z} \in \mathbb{S}^{1+n}$ as follows: $\bar{f}_j(\boldsymbol{Z}) = \boldsymbol{F}_j \bullet \boldsymbol{Z}$ with $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = x_0 = 1$. If we compare $f_j(\boldsymbol{x})$ with $\bar{f}_j(\boldsymbol{Z})$, we see that $\boldsymbol{x}\boldsymbol{x}^T$ is replaced by $\boldsymbol{Y} \in \mathbb{S}^n$.

We now introduce a general conic optimization problem (COP), denoted by $\mathrm{P}(\mathbb{K}^{1+n}, L)$:

$$
\zeta_{\mathrm{p}}(\mathbb{K}^{1+n}, L) \;=\; \min_{\boldsymbol{Z}} \left\{ \boldsymbol{F}_0 \bullet \boldsymbol{Z} : \ \boldsymbol{Z} \in \mathbb{K}^{1+n}, \ \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1, \ \boldsymbol{Z} \in L \ \right\}.
$$

Here $\mathbb{K}^{1+n} \subseteq \mathbb{S}^{1+n}$ is a closed (not necessarily convex) cone and L a linear subspace of $\mathbb{S}^{1+n}$. Notice that three types of constraints exist in $\mathrm{P}(\mathbb{K}^{1+n}, L)$: a cone constraint $\boldsymbol{Z} \in \mathbb{K}^{1+n}$, a single inhomogeneous linear equality $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$, and a linear space constraint $\boldsymbol{Z} \in L$. In the subsequent discussion, we frequently represent a QOP and its conic relaxations in terms of $\mathrm{P}(\mathbb{K}^{1+n}, L)$ with some closed cone $\mathbb{K}^{1+n} \supset \Gamma^{1+n}$ and some linear subspace L. The linear space L is usually described as a finite number of linear equalities.

We take $L = \{\boldsymbol{Z} \in \mathbb{S}^{1+n} : \boldsymbol{F}_j \bullet \boldsymbol{Z} = 0 \ (1 \leq j \leq \ell)\}$. Since $\boldsymbol{Z} \in \Gamma^{1+n}$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$ if and only if $\boldsymbol{x} \in \mathbb{R}^n_+$ and $\boldsymbol{Y} = \boldsymbol{x}\boldsymbol{x}^T$, $\mathrm{P}(\Gamma^{1+n}, L)$ coincides with QOP (1). By letting $\mathbb{K}^{1+n} = \mathbb{S}^{1+n}_+$, $\mathbb{D}^{1+n}_{\mathrm{NN}}$ and $\mathbb{C}^{1+n}_{\mathrm{PP}}$, we get the SDP, DNN, and CPP relaxations of QOP (1), respectively. Since $\mathbb{S}^{1+n}_+ \supset \mathbb{D}^{1+n}_{\mathrm{NN}} \supset \mathbb{C}^{1+n}_{\mathrm{PP}}$ (the conic hull of Γ^{1+n}) $\supset \Gamma^{1+n}$, it follows that $\zeta_{\mathrm{p}}(\mathbb{S}^{1+n}_+, L) \leq \zeta_{\mathrm{p}}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L) \leq \zeta_{\mathrm{p}}(\mathbb{C}^{1+n}_{\mathrm{PP}}, L) \leq \zeta_{\mathrm{p}}(\Gamma^{1+n}, L) = \zeta^*$. It is known that some CPP relaxations attain the exact optimal value, i.e., $\zeta_{\mathrm{p}}(\mathbb{C}^{1+n}_{\mathrm{PP}}, L) = \zeta^*$, for certain classes of QOPs [3, 5, 9, 10, 20, 22]. Among such QOPs, we are particularly interested in BIQPs and QAPs, which are considered in Section 6.1 and Section 6.2, respectively.

2.3 Eliminating $\boldsymbol{x}$ from the conic relaxation problem $\mathrm{P}(\mathbb{K}^{1+n}, L)$

Burer [9] presented a simplification technique to eliminate the variable $\boldsymbol{x}$ of a QOP in binary and nonnegative variables from its CPP reformulation under the assumption of the existence of a special valid constraint, *i.e.*, an equality $\boldsymbol{h}^T\boldsymbol{x} = 1$ with $\boldsymbol{h} \geq \boldsymbol{0}$ satisfied by every feasible solution of the QOP. The same technique can be applied to $\mathrm{P}(\mathbb{K}^{1+n}, L)$ with $\mathbb{K}^{1+n} \in \{\mathbb{S}^{1+n}_+, \mathbb{D}^{1+n}_{\mathrm{NN}}, \mathbb{C}^{1+n}_{\mathrm{PP}}\}$. Suppose that $\boldsymbol{h} \geq \boldsymbol{0}$ and that $\boldsymbol{h}^T\boldsymbol{x} = 1$ is valid for every feasible solution $\boldsymbol{x}$ of QOP (1). Then the equalities $\boldsymbol{x} = \boldsymbol{x}\boldsymbol{x}^T\boldsymbol{h}$ and $\boldsymbol{h}^T\boldsymbol{x}\boldsymbol{x}^T\boldsymbol{h} = 1$ are valid constraints for QOP (1). As a result, their lifted equalities $\boldsymbol{x} = \boldsymbol{Y}\boldsymbol{h}$ and $\boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} = 1$ can be added to the constraints of $\mathrm{P}(\mathbb{K}^{1+n}, L)$ or L can be replaced with $L \cap L_{\mathrm{S}}$, where

$$L_{\mathrm{S}} = \left\{ \boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} : \boldsymbol{h}^T \boldsymbol{Y} \boldsymbol{h} = x_0, \ \boldsymbol{x} = \boldsymbol{Y}\boldsymbol{h} \right\}. \tag{3}$$

Let $\mathbb{K}^k$ denote $\mathbb{S}^k_+$, $\mathbb{D}^k_{\mathrm{NN}}$, or $\mathbb{C}^k_{\mathrm{PP}}$ ($k \in \{n, 1+n\}$). For simplicity of notation, we introduce linear maps $\boldsymbol{\Phi} : \mathbb{S}^n \to \mathbb{S}^{1+n}$, $\boldsymbol{\Psi} : \mathbb{S}^{1+n} \to \mathbb{S}^n$ and $\boldsymbol{\Pi} : \mathbb{S}^{1+n} \to \mathbb{S}^n$ such that

$$\begin{aligned}
\boldsymbol{\Phi}(\boldsymbol{Y}) &= \begin{pmatrix} \boldsymbol{h}^T \\ \boldsymbol{I} \end{pmatrix} \boldsymbol{Y} \begin{pmatrix} \boldsymbol{h} & \boldsymbol{I} \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} & \boldsymbol{h}^T\boldsymbol{Y} \\ \boldsymbol{Y}\boldsymbol{h} & \boldsymbol{Y} \end{pmatrix} \in \mathbb{S}^{1+n} \text{ for every } \boldsymbol{Y} \in \mathbb{S}^n, \\
\boldsymbol{\Psi}(\boldsymbol{Z}) &= \begin{pmatrix} \boldsymbol{h} & \boldsymbol{I} \end{pmatrix} \boldsymbol{Z} \begin{pmatrix} \boldsymbol{h}^T \\ \boldsymbol{I} \end{pmatrix} \in \mathbb{S}^n \text{ for every } \boldsymbol{Z} \in \mathbb{S}^{1+n}, \\
\boldsymbol{\Pi}(\boldsymbol{Z}) &= \boldsymbol{Y} \in \mathbb{S}^n \text{ for every } \boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \in \mathbb{S}^{1+n},
\end{aligned}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. Then L_{S} can be rewritten as $L_{\mathrm{S}} = \boldsymbol{\Phi}(\mathbb{S}^n)$. Some properties of these mappings are listed as follows:

$$\boldsymbol{A} \bullet \boldsymbol{\Phi}(\boldsymbol{B}) = \boldsymbol{\Psi}(\boldsymbol{A}) \bullet \boldsymbol{B} \text{ for every } \boldsymbol{A} \in \mathbb{S}^{1+n} \text{ and } \boldsymbol{B} \in \mathbb{S}^n,$$

$$\boldsymbol{\Phi}^{-1}(\boldsymbol{Z}) = \boldsymbol{\Pi}(\boldsymbol{Z}) = \boldsymbol{Y} \text{ for every } \boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \in \boldsymbol{\Phi}(\mathbb{S}^n), \tag{4}$$

$$\boldsymbol{\Psi}(\boldsymbol{Z}) = \boldsymbol{\Pi}(\boldsymbol{Z}) = \boldsymbol{Y} \text{ for every } \boldsymbol{Z} = \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{Y} \end{pmatrix} \in \mathbb{S}^{1+n}, \tag{5}$$

$$\boldsymbol{\Phi}(\mathbb{K}^n) \subset \mathbb{K}^{1+n}, \ \boldsymbol{\Psi}\left(\mathbb{K}^{1+n}\right) = \mathbb{K}^n \text{ and } \boldsymbol{\Pi}\left(\mathbb{K}^{1+n}\right) = \mathbb{K}^n. \tag{6}$$

The first three identities are straightforward. For the last property, recall $\mathbb{K}^k \in \{\mathbb{C}^k_{\mathrm{PP}}, \mathbb{D}^k_{\mathrm{NN}}, \mathbb{S}^k_+\}$ ($k \in \{n, 1+n\}$). The identity (4) means that $\boldsymbol{\Pi}$ is an extension of $\boldsymbol{\Phi}^{-1} : \boldsymbol{\Phi}(\mathbb{S}^n) \to \mathbb{S}^n$ to the entire space $\mathbb{S}^{1+n}$.

Under the assumption that $\mathbb{K}^{1+n} \cap L \subset L_{\mathrm{S}}$, we show that $\mathrm{P}(\mathbb{K}^{1+n}, L)$ is equivalent to a simplified COP, denoted as $\mathrm{P}'(\mathbb{K}^n, L')$:

$$\zeta'_{\mathrm{p}}(\mathbb{K}^n, L') = \min_{\boldsymbol{Y}} \left\{ \boldsymbol{F}'_0 \bullet \boldsymbol{Y} : \ \boldsymbol{Y} \in \mathbb{K}^n, \quad \boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = 1, \ \boldsymbol{Y} \in L' \ \right\},$$

where $L' = \boldsymbol{\Phi}^{-1}(L) = \{\boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{\Phi}(\boldsymbol{Y}) \in L\}$ and $\boldsymbol{F}'_0 = \boldsymbol{\Psi}(\boldsymbol{F}_0)$. Throughout the paper, we use the prime symbol $'$ for functions, linear subspaces of $\mathbb{S}^n$, feasible regions of COPs, matrices in $\mathbb{S}^n$, and the conditions induced from the original ones in $\mathbb{S}^{1+n}$ by the simplification.

Lemma 2.1. *Assume that $\mathbb{K}^{1+n} \cap L \subset L_{\mathrm{S}}$.*

(i) Suppose that $\boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix}$ is a feasible solution of $P(\mathbb{K}^{1+n}, L)$. Then, $\boldsymbol{Y} = \boldsymbol{\Pi}(\boldsymbol{Z})$ is a feasible of $P'(\mathbb{K}^n, L')$ and $\boldsymbol{F}'_0 \bullet \boldsymbol{Y} = \boldsymbol{F}_0 \bullet \boldsymbol{Z}$.

(ii) Suppose that $\boldsymbol{Y}$ is a feasible solution of $P'(\mathbb{K}^n, L')$. Then $\boldsymbol{Z} = \boldsymbol{\Phi}(\boldsymbol{Y})$ is a feasible solution of $P(\mathbb{K}^{1+n}, L)$ and $\boldsymbol{F}_0 \bullet \boldsymbol{Z} = \boldsymbol{F}'_0 \bullet \boldsymbol{Y}$.

Proof. (i) By the assumption, we know that $1 = \boldsymbol{H}_0 \bullet \boldsymbol{Z} = x_0 = \boldsymbol{h}^T \boldsymbol{Y} \boldsymbol{h} = \boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y}$ and

$$\mathbb{K}^{1+n} \cap L \ni \boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}^T \boldsymbol{Y} \boldsymbol{h} & (\boldsymbol{Y}\boldsymbol{h})^T \\ \boldsymbol{Y}\boldsymbol{h} & \boldsymbol{Y} \end{pmatrix} = \boldsymbol{\Phi}(\boldsymbol{Y}).$$

By (6) and the definition of L', $\boldsymbol{Y}$ is a feasible solution of $\mathrm{P}'(\mathbb{K}^n, L')$. By the definition of $\boldsymbol{F}_0'$, we also see that $\boldsymbol{F}_0 \bullet \boldsymbol{Z} = \boldsymbol{F}_0 \bullet \boldsymbol{\Phi}(\boldsymbol{Y}) = \boldsymbol{\Psi}(\boldsymbol{F}_0) \bullet \boldsymbol{Y} = \boldsymbol{F}_0' \bullet \boldsymbol{Y}$.

(ii) By (6), $\boldsymbol{Z} = \boldsymbol{\Phi}(\boldsymbol{Y}) \in \mathbb{K}^{1+n}$. By $\boldsymbol{Y} \in L'$ and the definition of L', we see that $\boldsymbol{Z} \in L$. Furthermore, $1 = \boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = \boldsymbol{\Psi}(\boldsymbol{H}_0) \bullet \boldsymbol{Y} = \boldsymbol{H}_0 \bullet \boldsymbol{\Phi}(\boldsymbol{Y}) = \boldsymbol{H}_0 \bullet \boldsymbol{Z}$ holds. Hence $\boldsymbol{Z}$ is a feasible solution of $\mathrm{P}(\mathbb{K}^{1+n}, L)$. Similarly, the equality $\boldsymbol{F}_0 \bullet \boldsymbol{Z} = \boldsymbol{F}_0' \bullet \boldsymbol{Y}$ follows as in (i). □

When L is represented in homogeneous linear equations $\boldsymbol{F}_j \bullet \boldsymbol{Z} = 0$ $(1 \leq j \leq \ell)$ as in the previous section, $L' = \{\boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{F}_j' \bullet \boldsymbol{Y} = 0 \ (1 \leq j \leq \ell)\}$, where $\boldsymbol{F}_j' = \boldsymbol{\Psi}(\boldsymbol{F}_j)$ $(1 \leq j \leq \ell)$.

3 Lifted constraints of equivalent quadratic constraints

In this section, we discuss Procedure (I) in detail to derive QOP (1) from QOP (2) by representing Condition (G) in QOP (2) with a set of quadratic equalities. Many combinatorial optimization problems, such as BIQPs, QAPs, and maximum stable set problems, include the condition (G) as constraints. Notice that the equality $\sum_{i \in J_p} x_i = 1$ holds if the condition is satisfied. In particular, when J_p consists of two distinct i and j, x_j serves a slack variable to x_i and vice versa. This special case will be further studied in Section 6.1. A more general case where J_p includes more than two elements is applied to QAPs in Section 6.2.

Let $\boldsymbol{e}_{J_p} \in \mathbb{R}^n$ be a 0-1 vector whose ith element is 1 if $i \in J_p$ and 0 otherwise. Define

$$\boldsymbol{E}_{J_p} = \boldsymbol{e}_{J_p} \boldsymbol{e}_{J_p}^T \qquad \text{and} \qquad \boldsymbol{E}_{J_p}^+ = \begin{pmatrix} 1 & -\boldsymbol{e}_{J_p}^T \\ -\boldsymbol{e}_{J_p} & \boldsymbol{E}_{J_p} \end{pmatrix} \qquad (1 \leq p \leq m).$$

Condition (G) can be interpreted as two different set of constraints. First, linear equalities and 0-1 conditions:

$$\boldsymbol{e}_{J_p}^T \boldsymbol{x} - 1 = 0 \text{ and } x_j^2 - x_j = 0 \ (j \in J_p) \ \ (1 \leq p \leq m). \tag{7}$$

Second, linear equalities and complementarity conditions:

$$\boldsymbol{e}_{J_p}^T \boldsymbol{x} - 1 = 0 \text{ and } x_i x_j = 0 \ (i, j \in J_p, \ i \neq j) \ \ (1 \leq p \leq m). \tag{8}$$

Moreover, several reformulations of the linear equality constraint $\boldsymbol{e}_{J_p}^T \boldsymbol{x} - 1 = 0$ as quadratic equalities can be induced. This motivates us to examine the differences and equivalences among them. In Section 3.1, we first state three equivalent reformulations of the linear constraints in quadratic equalities, and show the equivalence between their lifted constraints. Then the differences of the lifted constraints of the 0-1 and complementarity conditions are discussed in Section 3.2. In Section 3.3, we describe a general COP. In Section 3.4, we apply Lemma 2.1 to the lifted constraints and reduce their sizes.

3.1 Linear constraint

We present three different representations of linear constraints in quadratic equalities and their lifted equalities. First, $\boldsymbol{e}_{J_p}^T\boldsymbol{x} - 1 = 0$ is simply squared to obtain the quadratic equality $(\boldsymbol{e}_{J_p}^T\boldsymbol{x} - 1)^2 = 0$, which can be written as $\boldsymbol{E}_{J_p}^+ \bullet \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{x}\boldsymbol{x}^T \end{pmatrix} = 0$ and $x_0 = \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$. Thus, its lifted constraint is as follows:

(E1) $\boldsymbol{Z} \in L_{\rm E1} \equiv \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \boldsymbol{E}_{J_p}^+ \bullet \boldsymbol{Z} = 0 \ (1 \leq p \leq m) \right\}$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$.

In [3], a CPP reformulation of a class of QOPs using (E1) is proposed by Arima, Kim, and Kojima; See also [5].

Second, consider a pair of the linear constraint $\boldsymbol{e}_{J_p}^T\boldsymbol{x} = 1$ and the redundant quadratic constraint $\boldsymbol{x} = \boldsymbol{x}\boldsymbol{x}^T\boldsymbol{e}_{J_p}$, which are lifted to the following constraints:

(E2) $\boldsymbol{Z} \in L_{\rm E2} \equiv \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \boldsymbol{e}_{J_p}^T\boldsymbol{x} = x_0 \text{ and } \boldsymbol{x} = \boldsymbol{Y}\boldsymbol{e}_{J_p} \ (1 \leq p \leq m) \right\}$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$.

Condition (E2) was used in [23] for SDP and DNN relaxations of QAPs (denoted by QAP_{R_0}, QAP_{R_2} and QAP_{R_3}). These relaxations are presented in detail in Section 6.2.1.

Finally, by lifting the linear constraint $\boldsymbol{e}_{J_p}^T\boldsymbol{x} = 1$ and the redundant quadratic constraint $(\boldsymbol{e}_{J_p}^T\boldsymbol{x})^2 = 1$, we obtain the following condition:

(E3) $\boldsymbol{Z} \in L_{\rm E3} \equiv \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \boldsymbol{e}_{J_p}^T\boldsymbol{x} = x_0 \text{ and } \boldsymbol{e}_{J_p}^T\boldsymbol{Y}\boldsymbol{e}_{J_p} = x_0 \ (1 \leq p \leq m) \right\}$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$.

As we know that $x_0 = \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$ in Conditions (E1), (E2) and (E3), the equality $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$ could be eliminated by replacing $\boldsymbol{Z}$ with $\boldsymbol{Z} = \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix}$ and x_0 with 1. This elimination, however, would render $L_{\rm E1}$, $L_{\rm E2}$ and $L_{\rm E3}$ to be affine spaces instead of linear subspaces. For convenience, the equality and linear subspaces will remain as they are.

The following lemma shows that (E1), (E2), and (E3) are equivalent under the semidefinite constraints $\boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \in \mathbb{S}_+^{1+n}$, but the equivalence does not hold in general.

Lemma 3.1.

(i) $L_{\rm E2} \subset L_{\rm E3} \subset L_{\rm E1}$.

(ii) $L_{\rm E2} \cap \mathbb{S}_+^{1+n} = L_{\rm E3} \cap \mathbb{S}_+^{1+n} = L_{\rm E1} \cap \mathbb{S}_+^{1+n}$.

Proof. (i) Let $\boldsymbol{Z} \in L_{\rm E2}$. Then $\boldsymbol{e}_{J_p}^T\boldsymbol{x} = x_0$ and $\boldsymbol{x} = \boldsymbol{Y}\boldsymbol{e}_{J_p}$ $(1 \leq p \leq m)$. By multiplying $\boldsymbol{e}_{J_p}^T$ to $\boldsymbol{x} = \boldsymbol{Y}\boldsymbol{e}_{J_p}$ $(1 \leq p \leq m)$, we obtain $\boldsymbol{e}_{J_p}^T\boldsymbol{Y}\boldsymbol{e}_{J_p} = x_0$ $(1 \leq p \leq m)$. Thus, $\boldsymbol{Z} \in L_{\rm E3}$. To show $L_{\rm E3} \subset L_{\rm E1}$, we rewrite the two equalities (with each p) characterizing $L_{\rm E3}$ as

$$\begin{pmatrix} 2 & -\boldsymbol{e}_{J_p}^T \\ -\boldsymbol{e}_{J_p} & \boldsymbol{O} \end{pmatrix} \bullet \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} -1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{e}_{J_p}\boldsymbol{e}_{J_p}^T \end{pmatrix} \bullet \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} = 0,$$

respectively. Then, we obtain $\boldsymbol{E}_{J_p} \bullet \boldsymbol{Z} = 0$ by adding these equalities. As a result, $L_{\rm E3} \subset L_{\rm E1}$.

(ii) In view of (i), it suffices to show that $L_{\rm E1} \cap \mathbb{S}_+^{1+n} \subset L_{\rm E2} \cap \mathbb{S}_+^{1+n}$. The equality (for each p) characterizing $L_{\rm E1}$ is equivalent to $\begin{pmatrix} 1 & -\boldsymbol{e}_{J_p}^T \end{pmatrix} \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \begin{pmatrix} 1 \\ -\boldsymbol{e}_{J_p} \end{pmatrix} = 0$, which implies $\begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \begin{pmatrix} 1 \\ -\boldsymbol{e}_{J_p} \end{pmatrix} = \boldsymbol{0}$ if $\begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \in \mathbb{S}_+^{1+n}$. □

Remark 3.1. The discussions in this section are valid for the lifting of any system of linear equalities $b_p - \boldsymbol{a}_p^T \boldsymbol{x} = 0$ $(1 \le p \le m)$ with a slight modification, where $b_p \in \mathbb{R}$ and $\boldsymbol{a}_p \in \mathbb{R}^n$ $(1 \le p \le m)$. In this general case, Conditions (E1), (E2) and (E3) turn out to be

($\bar{\text{E}}$1) $\boldsymbol{Z} \in \bar{L}_{\text{E1}} \equiv \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \begin{pmatrix} b_p^2 & -b_p \boldsymbol{a}_p^T \\ -\boldsymbol{a}_p b_p & \boldsymbol{a}_p \boldsymbol{a}_p \end{pmatrix} \bullet \boldsymbol{Z} = 0 \ (1 \le p \le m) \right\}$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$.

($\bar{\text{E}}$2) $\boldsymbol{Z} \in \bar{L}_{\text{E2}} \equiv \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \boldsymbol{a}_p^T \boldsymbol{x} = b_p x_0, \ b_p \boldsymbol{x} = \boldsymbol{Y} \boldsymbol{a}_p \ (1 \le p \le m) \right\}$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$.

($\bar{\text{E}}$3) $\boldsymbol{Z} \in \bar{L}_{\text{E3}} \equiv \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \boldsymbol{a}_p^T \boldsymbol{x} = b_p x_0, \ b_p^2 x_0 = \boldsymbol{a}_p^T \boldsymbol{Y} \boldsymbol{a}_p \ (1 \le p \le m) \right\}$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$.

Lemma 3.1 remains valid with this modification. Burer [9] employed Condition ($\bar{\text{E}}$3) to derive a CPP reformulation of a class of linearly constrained QOPs in binary and continuous variables.

3.2 Representing a combinatorial constraint with 0-1 or complementarity condition

Condition (G) has been expressed by using linear and 0-1 constraints as in (7). Alternatively, it has been expressed by using the linear and complementarity constraints as in (8). We now focus on the liftings of the 0-1 and the complementarity constraints to $\mathbb{S}^{1+n}$, which are stated as Conditions (Z) and (C) respectively as follows:

(Z) $\boldsymbol{Z} \in L_{\text{Z}} \equiv \{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : x_j = Y_{jj} \ (j \in J_p, \ 1 \le p \le m) \}$.

(C) $\boldsymbol{Z} \in L_{\text{C}} \equiv \{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : Y_{ij} = 0 \ (i, j \in J_p, \ \ i \ne j, \ 1 \le p \le m) \}$.

The following lemma shows the relation of (Z) and (C).

Lemma 3.2.

(i) $L_{\text{C}} \cap L_{\text{E2}} \subset L_{\text{Z}} \cap L_{\text{E2}}$.

(ii) $L_{\text{C}} \cap L_{\text{E2}} \cap \mathbb{N}^{1+n} = L_{\text{Z}} \cap L_{\text{E2}} \cap \mathbb{N}^{1+n}$.

(iii) If J_p consists of two elements i and j, then the three conditions $x_i = Y_{ii}$, $x_j = Y_{jj}$ and $Y_{ij} = 0$ are equivalent $(i, j \in J_p, \ i \ne j)$. Hence $L_{\text{C}} \cap L_{\text{E2}} = L_{\text{Z}} \cap L_{\text{E2}}$ in this case.

(iv) $L_{\text{C}} \cap L_{\text{E1}} \cap \mathbb{D}^{1+n}_{\text{NN}} = L_{\text{C}} \cap L_{\text{E2}} \cap \mathbb{D}^{1+n}_{\text{NN}} = L_{\text{C}} \cap L_{\text{E3}} \cap \mathbb{D}^{1+n}_{\text{NN}} = L_{\text{Z}} \cap L_{\text{E1}} \cap \mathbb{D}^{1+n}_{\text{NN}} = L_{\text{Z}} \cap L_{\text{E2}} \cap \mathbb{D}^{1+n}_{\text{NN}} = L_{\text{Z}} \cap L_{\text{E3}} \cap \mathbb{D}^{1+n}_{\text{NN}}$.

(v) If J_p consists of two elements i and j, then $L_{\text{C}} \cap L_{\text{E1}} \cap \mathbb{S}^{1+n}_+ = L_{\text{C}} \cap L_{\text{E2}} \cap \mathbb{S}^{1+n}_+ = L_{\text{C}} \cap L_{\text{E3}} \cap \mathbb{S}^{1+n}_+ = L_{\text{Z}} \cap L_{\text{E1}} \cap \mathbb{S}^{1+n}_+ = L_{\text{Z}} \cap L_{\text{E2}} \cap \mathbb{S}^{1+n}_+ = L_{\text{Z}} \cap L_{\text{E3}} \cap \mathbb{S}^{1+n}_+$.

Proof. To prove assertion (i), (ii) and (iii), assume that $\boldsymbol{Z} \in L_{\text{E2}}$. Then, $Y_{jj} + \sum_{k \in J_p \setminus \{j\}} Y_{jk} = x_j$ $(j \in J_p, \ 1 \le p \le m)$. If, in addition, $\boldsymbol{Z} \in L_{\text{C}}$, then $\sum_{k \in J_p \setminus \{j\}} Y_{jk} = 0$ $(j \in J_p, \ 1 \le p \le m)$. Thus, (i) follows. Now if $\boldsymbol{Z} \in \mathbb{N}^{1+n}$, then $\boldsymbol{Z} \in L_{\text{C}}$ iff $\sum_{k \in J_p \setminus \{j\}} Y_{jk} = 0$ $(j \in J_p, \ 1 \le p \le m)$. Hence (ii) holds. Next, if J_p consists of two elements i and j, then we have the identities $Y_{ii} + Y_{ij} = x_i$ and $Y_{jj} + Y_{ji} = x_j$ $(i, j \in J_p, \ i \ne j, \ 1 \le p \le m)$. Since $Y_{ij} = Y_{ji}$ $(1 \le i < j \le n)$, (iii) follows. Assertion (iv) follows from (ii) and Lemma 3.1, and (v) from (iii) and Lemma 3.1. □

3.3 Representation of COP

The linear subspace L in $\mathrm{P}(\mathbb{K}^{1+n}, L)$ is expressed as $L_0 \cap L_\mathrm{a} \cap L_\mathrm{b}$ using each $L_\mathrm{a} \in \{L_\mathrm{E1}, L_\mathrm{E2}, L_\mathrm{E3}\}$, each $L_\mathrm{b} \in \{L_\mathrm{C}, L_\mathrm{Z}\}$ and the set constructed by the remaining homogeneous linear equalities:

$$L_0 = \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \boldsymbol{F}_j \bullet \boldsymbol{Z} = 0 \ (1 \leq j \leq \ell_0) \right\}.$$

$\mathrm{P}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$ is rewritten as

$$\zeta_\mathrm{p}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b}) = \min \left\{ \boldsymbol{F}_0 \bullet \boldsymbol{Z} : \boldsymbol{Z} \in \mathrm{Feas}(\mathbb{K}^{1+n}, L_\mathrm{a}, L_\mathrm{b}) \right\},$$

where $\mathrm{Feas}(\mathbb{K}^{1+n}, L_\mathrm{a}, L_\mathrm{b}) = \left\{ \boldsymbol{Z} \in \mathbb{K}^{1+n} : \ \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1, \ \boldsymbol{Z} \in L_0 \cap L_\mathrm{a} \cap L_\mathrm{b} \ \right\}$. Since $\boldsymbol{Z} \in \Gamma^{1+n}$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$ iff $\boldsymbol{x} \in \mathbb{R}^n_+$ and $\boldsymbol{Y} = \boldsymbol{x}\boldsymbol{x}^T$, for any $L_\mathrm{a} \in \{L_\mathrm{E1}, L_\mathrm{E2}, L_\mathrm{E3}\}$, $\boldsymbol{Z} \in L_\mathrm{a} \cap L_\mathrm{Z} \cap \Gamma^{1+n}$ iff (7) holds, and $\boldsymbol{Z} \in L_\mathrm{a} \cap L_\mathrm{C} \cap \Gamma^{1+n}$ iff (8) holds. Therefore, for any $L_\mathrm{a} \in \{L_\mathrm{E1}, L_\mathrm{E2}, L_\mathrm{E3}\}$ and any $L_\mathrm{b} \in \{L_\mathrm{C}, L_\mathrm{Z}\}$, $\mathrm{P}(\Gamma^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$ is equivalent to QOP (2). By letting $\mathbb{K}^{1+n} = \mathbb{S}^{1+n}_+$, $\mathbb{D}^{1+n}_\mathrm{NN}$ and $\mathbb{C}^{1+n}_\mathrm{PP}$, SDP, DNN and CPP relaxations of QOP (2) are obtained, respectively. The following theorem, obtained as a corollary of Lemmas 3.1 and 3.2, shows their relations.

Theorem 3.1. *For any* $L_\mathrm{a} \in \{L_\mathrm{E1}, L_\mathrm{E2}, L_\mathrm{E3}\}$*, we have*

$$\begin{aligned} \mathrm{Feas}(\mathbb{S}^{1+n}_+, L_\mathrm{a}, L_\mathrm{Z}) &\supset \mathrm{Feas}(\mathbb{S}^{1+n}_+, L_\mathrm{a}, L_\mathrm{C}) \\ &\supset \mathrm{Feas}(\mathbb{D}^{1+n}_\mathrm{NN}, L_\mathrm{a}, L_\mathrm{Z}) = \mathrm{Feas}(\mathbb{D}^{1+n}_\mathrm{NN}, L_\mathrm{a}, L_\mathrm{C}) \\ &\supset \mathrm{Feas}(\mathbb{C}^{1+n}_\mathrm{PP}, L_\mathrm{a}, L_\mathrm{Z}) = \mathrm{Feas}(\mathbb{C}^{1+n}_\mathrm{PP}, L_\mathrm{a}, L_\mathrm{C}). \end{aligned}$$

Moreover, if $|J_p| = 2 \ (1 \leq p \leq m)$*, then* $\mathrm{Feas}(\mathbb{S}^{1+n}_+, L_\mathrm{a}, L_\mathrm{Z}) = \mathrm{Feas}(\mathbb{S}^{1+n}_+, L_\mathrm{a}, L_\mathrm{C})$.

3.4 Eliminating $\boldsymbol{x}$ from $\mathrm{P}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$

The elimination is based on [9]. We assume that $\mathbb{K}^{1+n} \in \left\{\mathbb{S}^{1+n}_+, \mathbb{D}^{1+n}_\mathrm{NN}, \mathbb{C}^{1+n}_\mathrm{PP}\right\}$ throughout this section. Let $\boldsymbol{h}$ be an arbitrary convex combination of $\boldsymbol{e}_{J_p}$ $(1 \leq p \leq m)$, *i.e.*, $\boldsymbol{h} = \sum_{p=1}^m \lambda_p \boldsymbol{e}_{J_p}$, $1 = \sum_{p=1}^m \lambda_p$ and $\lambda_i \geq 0 \ (1 \leq p \leq m)$. Then, $\boldsymbol{x} = \boldsymbol{x}\boldsymbol{x}^T\boldsymbol{h}$ and $\boldsymbol{h}^T\boldsymbol{x}\boldsymbol{x}^T\boldsymbol{h} = 1$ are valid constraints for QOP (2), and its lifted equalities $\boldsymbol{x} = \boldsymbol{Y}\boldsymbol{h}$ and $\boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} = 1$ can be added to the constraints of $\mathrm{P}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$ for each $L_\mathrm{a} \in \{L_\mathrm{E1}, L_\mathrm{E2}, L_\mathrm{E3}\}$ and $L_\mathrm{b} \in \{L_\mathrm{C}, L_\mathrm{Z}\}$. Define L_S by (3). The following lemma shows that adding the lifted equalities is redundant.

Lemma 3.3.

(i) $L_\mathrm{E2} \subset L_\mathrm{S}$.

(ii) $L_\mathrm{S} \cap L_\mathrm{a} \cap \mathbb{S}^{1+n}_+ = L_\mathrm{a} \cap \mathbb{S}^{1+n}_+$ *for any* $L_\mathrm{a} \in \{L_\mathrm{E1}, L_\mathrm{E2}, L_\mathrm{E3}\}$.

Proof. Let $\boldsymbol{Z} \in L_\mathrm{E2}$. Then, $\boldsymbol{h}^T\boldsymbol{x} = \left(\sum_{p=1}^m \lambda_p \boldsymbol{e}_{J_p}\right)^T \boldsymbol{x} = \sum_{p=1}^m \lambda_p \boldsymbol{e}_{J_p}^T \boldsymbol{x} = x_0$ and $\boldsymbol{x} = \sum_{p=1}^m \lambda_p \boldsymbol{x} = \sum_{p=1}^m \lambda_p \boldsymbol{Y}\boldsymbol{e}_{J_p} = \boldsymbol{Y}\boldsymbol{h}$. As a result, $\boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} = \boldsymbol{h}^T\boldsymbol{x} = x_0$. Thus we have shown Assertion (i). Assertion (ii) follows from (i) and (ii) of Lemma 3.1. □

By Lemma 3.3, the simplification technique described in Section 2.3 can be applied to $\mathrm{P}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$ without adding the lifted equalities $\boldsymbol{x} = \boldsymbol{Y}\boldsymbol{h}$ and $\boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} = x_0$. Note that each of (E1), (E2), (E3), (Z) and (C) consists of $\boldsymbol{Z} \in M$ and $\boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1$ for some linear subspace M of $\mathbb{S}^{1+n}$. By Lemma 2.1, the corresponding simplified condition is obtained by replacing them with $M' = \boldsymbol{\Phi}^{-1}(M) = \{\boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{\Phi}(\boldsymbol{Y}) \in M\}$ and $\boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = 1$. If M is represented as $M = \{\boldsymbol{Z} \in \mathbb{S}^{1+n} : \boldsymbol{H}_p \bullet \boldsymbol{Z} = 0 \ (1 \leq p \leq k)\}$ for some k and $\boldsymbol{H}_p \in \mathbb{S}^n$ $(1 \leq p \leq k)$, then M' turns out to be $\{\boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{\Psi}(\boldsymbol{H}_p) \bullet \boldsymbol{Y} = 0 \ (1 \leq p \leq k)\}$. M' is also obtained by substituting $\boldsymbol{x} = \boldsymbol{Y}\boldsymbol{h}$ and $x_0 = \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h}$ into the homogeneous linear equalities in $\boldsymbol{Z} \in \mathbb{S}^{1+n}$ describing M. As a result, (E1), (E2), (E3), (Z) and (C) are simplified as follows:

(E1)' $\boldsymbol{Y} \in L'_{\mathrm{E1}} \equiv \left\{\boldsymbol{Y} \in \mathbb{K}^n : \boldsymbol{\Psi}(\boldsymbol{E}^+_{J_p}) \bullet \boldsymbol{Y} = 0 \ (1 \leq p \leq m)\right\}$ and $\boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = 1$.

(E2)' $\boldsymbol{Y} \in L'_{\mathrm{E2}} \equiv \left\{\boldsymbol{Y} \in \mathbb{K}^n : \begin{array}{l} \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{e}_{J_p} = \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h}, \\ \boldsymbol{Y}\boldsymbol{h} = \boldsymbol{Y}\boldsymbol{e}_{J_p} \ (1 \leq p \leq m) \end{array}\right\}$ and $\boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = 1$.

(E3)' $\boldsymbol{Y} \in L'_{\mathrm{E3}} \equiv \left\{\boldsymbol{Y} \in \mathbb{K}^n : \begin{array}{l} \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{e}_{J_p} = \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h}, \\ \boldsymbol{e}^T_{J_p}\boldsymbol{Y}\boldsymbol{e}_{J_p} = \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} \ (1 \leq p \leq m) \end{array}\right\}$ and $\boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = 1$.

(Z)' $\boldsymbol{Y} \in L'_{\mathrm{Z}} \equiv \{\boldsymbol{Y} \in \mathbb{K}^n : [\boldsymbol{Y}\boldsymbol{h}]_j = Y_{jj} \ (j \in J_p, \ 1 \leq p \leq m)\}$.

(C)' $\boldsymbol{Y} \in L'_{\mathrm{C}} \equiv \{\boldsymbol{Y} \in \mathbb{K}^n : Y_{ij} = 0 \ (i, j \in J_p, \ i \neq j, \ 1 \leq p \leq m)\}$.

Define

$$L'_0 \ = \ \boldsymbol{\Phi}^{-1}(L_0) = \left\{\boldsymbol{Y} \in \mathbb{S}^n : \boldsymbol{F}'_j \bullet \boldsymbol{Y} = 0 \quad (1 \leq j \leq \ell_0)\right\},$$

where $\boldsymbol{F}'_j = \boldsymbol{\Psi}(\boldsymbol{F}_j)$ $(1 \leq j \leq \ell_0)$. Then, for every $L'_\mathrm{a} \in \{L'_{\mathrm{E1}}, L'_{\mathrm{E2}}, L'_{\mathrm{E1}}\}$ and every $L'_\mathrm{b} \in \{L'_\mathrm{Z}, L'_\mathrm{C}\}$, we obtain the following COP, denoted as $\mathrm{P}'(\mathbb{K}^n, L'_0 \cap L'_\mathrm{a} \cap L'_\mathrm{b})$, for the simplified reformulation of $\mathrm{P}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$:

$$\zeta'_\mathrm{p}(\mathbb{K}^n, L'_0 \cap L'_\mathrm{a} \cap L'_\mathrm{b}) = \min\left\{\boldsymbol{F}'_0 \bullet \boldsymbol{Y} : \boldsymbol{Y} \in \mathrm{Feas}'(\mathbb{K}^n, L'_\mathrm{a}, L'_\mathrm{b})\right\},$$

where $\mathrm{Feas}'(\mathbb{K}^n, L'_\mathrm{a}, L'_\mathrm{b}) = \left\{\boldsymbol{Y} \in \mathbb{K}^n : \ \boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = 1, \ \boldsymbol{Y} \in L'_0 \cap L'_\mathrm{a} \cap L'_\mathrm{b} \ \right\}$.

Lemma 3.1, Lemma 3.2 and Theorem 3.1 still remain valid when changing L_{E1} through L_C to L'_{E1} through L'_C, $\mathrm{P}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$ to $\mathrm{P}'(\mathbb{K}^n, L'_0 \cap L'_\mathrm{a} \cap L'_\mathrm{b})$, and Feas and ζ_p to Feas′ and ζ'_p. The key difference is that these changes reduce the cones of the dimension $1 + n$ to n. The details are omitted.

In the remainder of this section, we impose an additional restriction that $\boldsymbol{h}$ is the barycenter of $\boldsymbol{e}_{J_p}$ $(1 \leq p \leq m)$, *i.e.*, $\boldsymbol{h} = \frac{1}{m}\sum_{p=1}^m \boldsymbol{e}_{J_p}$, and further simplify (E3)' to

(E3)" $\boldsymbol{Y} \in L''_{\mathrm{E3}} \equiv \left\{\boldsymbol{Y} \in \mathbb{K}^n : \ \boldsymbol{e}^T_{J_p}\boldsymbol{Y}\boldsymbol{e}_{J_p} = \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} \ (1 \leq p \leq m) \ \right\}$ and $\boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = 1$.

We need the following lemmas to show the equivalence between (E3)' and (E3)".

Lemma 3.4. *[22, Lemma 2] Let* $\boldsymbol{A} \in \mathbb{S}^m_+$ *and* $\sum_{p=1}^m \sum_{q=1}^m A_{pq} = \left(\sum_{p=1}^m \sqrt{A_{pp}}\right)^2$. *Then,* $A_{pq} = \sqrt{A_{pp}A_{qq}}$ $(1 \leq p, q \leq m)$.

Lemma 3.5. *Suppose that* $\boldsymbol{Y} \in \mathbb{S}^n_+$ *and* $\boldsymbol{e}_{J_p}^T \boldsymbol{Y} \boldsymbol{e}_{J_p} = 1$ $(1 \leq p \leq m)$. *Then,* $\boldsymbol{h}^T \boldsymbol{Y} \boldsymbol{h} = 1$ *if and only if* $\boldsymbol{e}_{J_p}^T \boldsymbol{Y} \boldsymbol{e}_{J_q} = 1$ $(1 \leq p, q \leq m)$.

Proof. Since we know that

$$\boldsymbol{h}^T \boldsymbol{Y} \boldsymbol{h} \ = \ \left(\frac{1}{m}\sum_{p=1}^m \boldsymbol{e}_{J_p}\right)^T \boldsymbol{Y} \left(\frac{1}{m}\sum_{q=1}^m \boldsymbol{e}_{J_p}\right) = \frac{1}{m^2}\sum_{p=1}^m \sum_{q=1}^m \boldsymbol{e}_{J_p}^T \boldsymbol{Y} \boldsymbol{e}_{J_q}, \tag{9}$$

the 'if part' follows. Assume that $\boldsymbol{h}^T \boldsymbol{Y} \boldsymbol{h} = 1$ holds. Consider the $m \times m$ matrix

$$\boldsymbol{A} \ = \ \begin{pmatrix} \boldsymbol{e}_{J_1}^T \boldsymbol{Y} \boldsymbol{e}_{J_1} & \dots & \boldsymbol{e}_{J_1}^T \boldsymbol{Y} \boldsymbol{e}_{J_m} \\ \dots & \dots & \dots \\ \boldsymbol{e}_{J_m}^T \boldsymbol{Y} \boldsymbol{e}_{J_1} & \dots & \boldsymbol{e}_{J_m}^T \boldsymbol{Y} \boldsymbol{e}_{J_m} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_{J_1}^T \\ \vdots \\ \boldsymbol{e}_{J_m}^T \end{pmatrix} \boldsymbol{Y} \begin{pmatrix} \boldsymbol{e}_{J_1} & \dots & \boldsymbol{e}_{J_m} \end{pmatrix} \in \mathbb{S}^m_+.$$

By the assumption, $A_{pp} = \boldsymbol{e}_{J_p}^T \boldsymbol{Y} \boldsymbol{e}_{J_p} = 1$ $(1 \leq p \leq m)$. Thus, we see from (9) that $\sum_{p=1}^m \sum_{q=1}^m A_{pq} = \sum_{p=1}^m \sum_{q=1}^m \boldsymbol{e}_{J_p}^T \boldsymbol{Y} \boldsymbol{e}_{J_q} = m^2 = \left(\sum_{p=1}^m \sqrt{A_{pp}}\right)^2$. Therefore, by Lemma 3.4, $\boldsymbol{e}_{J_p}^T \boldsymbol{Y} \boldsymbol{e}_{J_q} = A_{pq} = \sqrt{A_{pp} A_{qq}} = 1$ $(1 \leq p, \ q \leq m)$. □

Theorem 3.2. *Suppose that* $\boldsymbol{Y} \in \mathbb{S}^n_+$. *Then, (E3)' and (E3)" are equivalent.*

Proof. Since $L'_{\mathrm{E3}} \subset L''_{\mathrm{E3}}$, (E3)' implies (E3)". Now assume that (E3)" holds. By Lemma 3.5, we see that $\boldsymbol{e}_{J_p}^T \boldsymbol{Y} \boldsymbol{e}_{J_q} = A_{pq} = 1$ $(1 \leq p, \ q \leq m)$. Thus,

$$\boldsymbol{h}^T \boldsymbol{Y} \boldsymbol{e}_{J_p} \ = \ \left(\frac{1}{m}\sum_{q=1}^m \boldsymbol{e}_{J_q}\right)^T \boldsymbol{Y} \boldsymbol{e}_{J_p} = \frac{1}{m}\sum_{q=1}^m \boldsymbol{e}_{J_q}^T \boldsymbol{Y} \boldsymbol{e}_{J_p} = 1 \ (1 \leq p \leq m).$$

Therefore, (E3)' holds. □

4 Issues on computational efficiency and stability

DNN relaxations have been studied as an effective computational method for obtaining good lower bounds for the QOP (2). They are considered as a good compromise between the numerically intractable CPP relaxations and the less effective SDP relaxations. When we consider DNN relaxations for computation, we realize that various DNN relaxations are available. For instance, $\mathrm{P}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L_0 \cap L_{\mathrm{a}} \cap L_{\mathrm{b}})$ with each $L_{\mathrm{a}} \in \{L_{\mathrm{E1}}, L_{\mathrm{E2}}, L_{\mathrm{E3}}\}$ and each $L_{\mathrm{b}} \in \{L_{\mathrm{Z}}, L_{\mathrm{C}}\}$ as well as $\mathrm{P}'(\mathbb{D}^n_{\mathrm{NN}}, L'_0 \cap L'_{\mathrm{a}} \cap L'_{\mathrm{b}})$ with each $L'_{\mathrm{a}} \in \{L'_{\mathrm{E1}}, L'_{\mathrm{E2}}, L'_{\mathrm{E3}}\}$ and each $L'_{\mathrm{b}} \in \{L'_{\mathrm{Z}}, L'_{\mathrm{C}}\}$, which are equivalent to each other from the theoretical point of view.

The size of conic relaxations, the existence of primal/dual interior feasible solutions and the primal/dual nondegeneracy are three most important factors for efficient and stable implementation of conic relaxations. The size of a conic relaxation means the dimension of the variable matrix and the number of linear equality constraints. The size $(1+n)$ of the variable matrix $\boldsymbol{Z}$ has been reduced to n by the simplification technique in Section 3.4. In Section 4.1, we compare the numbers of linear equalities involved in L_{E1} through L_{C}, and discuss how to reduce the number of linear equalities in $\mathrm{P}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L_0 \cap L_{\mathrm{E1}} \cap L_{\mathrm{C}})$. The technique to reduce the number of linear equalities can also be applied to L'_{E1} through L'_{C}.

The existence of interior feasible solutions in the conic relaxations of QOP (2) is crucial to the successful computation of a primal-dual interior-point method and SDPNAL+, which were designed by assuming the existence of primal and dual interior feasible solutions. When the assumption is not satisfied, they may give inaccurate approximate optimal solutions and/or converge very slowly.

The primal and dual nondegeneracy, which plays an important role in the local convergence analysis of primal-dual interior-point methods and SDPNAL+, is discussed in Section 4.3.

4.1 The number of linear equalities

Consider the DNN relaxation $\mathrm{P}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L_0 \cap L_{\mathrm{a}} \cap L_{\mathrm{b}})$ with some $L_{\mathrm{a}} \in \{L_{\mathrm{E1}}, L_{\mathrm{E2}}, L_{\mathrm{E3}}\}$ and some $L_{\mathrm{b}} \in \{L_{\mathrm{Z}}, L_{\mathrm{C}}\}$ for computing a lower bound for QOP (2). As L_{E1}, L_{E2} and L_{E3} involve m, $m + mn$ and $2m$ linear equalities, respectively, L_{E2} may not be the best choice for computational efficiency. From the positive semidefiniteness of the coefficient matrix $\boldsymbol{E}^+_{J_p}$ of each linear equality in L_{E1} $(1 \leq p \leq m)$, we see that $\boldsymbol{E}^+_{J_p} \bullet \boldsymbol{Z} \geq 0$ for every $\boldsymbol{Z} \in \mathbb{S}^{1+n}_+$. Thus, by defining

$$L_{\mathrm{E1sum}} \equiv \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \left(\textstyle\sum_{p=1}^m \boldsymbol{E}^+_{J_p} \right) \bullet \boldsymbol{Z} = 0 \right\}, \tag{10}$$

we obtain $L_{\mathrm{E1}} \cap \mathbb{S}^{1+n}_+ = L_{\mathrm{E1sum}} \cap \mathbb{S}^{1+n}_+$. We note that L_{E1sum} contains only one linear equality. This can be regarded as a clear advantage of Condition (E1).

Now, we compare Conditions (Z) and (C). As L_{Z} involves $\sum_{p=1}^m |J_p|$ linear equalities and L_{C} involves $\sum_{p=1}^m (|J_p| - 1)|J_p|/2$ linear equalities ($Y_{ij} = Y_{ji}$ $(1 \leq i, j \leq n)$ for $\boldsymbol{Y} \in \mathbb{S}^n$), (Z) may seem more efficient than (C). Since $Y_{ij} \geq 0$ $(1 \leq i, j \leq n)$ for every $\boldsymbol{Z} \in \mathbb{N}^{1+n}$, however, we see that $L_{\mathrm{C}} \cap \mathbb{N}^{1+n} = L_{\mathrm{Csum}} \cap \mathbb{N}^{1+n}$, where

$$L_{\mathrm{Csum}} = \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \sum_{p=1}^m \sum_{i \in J_p} \sum_{j \in J_p, j > i} Y_{ij} = 0 \right\}.$$

As L_{Csum} involves only one equality, it is more efficient to employ L_{Csum} than L_Z or L_C for the DNN relaxation.

We note that the constraints $\boldsymbol{Z} \in L_{\mathrm{Z}} \cap \mathbb{N}^{1+n}$ and $\boldsymbol{Z} \in L_{\mathrm{C}} \cap \mathbb{N}^{1+n}$ can be treated as cone constraints. In fact, both $L_{\mathrm{Z}} \cap \mathbb{N}^{1+n}$ and $L_{\mathrm{C}} \cap \mathbb{N}^{1+n}$ form closed convex cones of $\mathbb{S}^{1+n}$. The metric projection of $\boldsymbol{Z} \in \mathbb{S}^{1+n}$ onto $L_{\mathrm{Z}} \cap \mathbb{N}^{1+n}$ was presented for the bisection and projection algorithm in [15]. Computing the metric projection of $\boldsymbol{Z} \in \mathbb{S}^{1+n}$ onto $L_{\mathrm{C}} \cap \mathbb{N}^{1+n}$ is trivial as the elements Y_{ij} are replaced by 0 if $i \in J_p$, $j \in J_p$, $i \neq j$ and $1 \leq p \leq m$, and $\max\{0, Y_{ij}\}$ otherwise. Both computation have been used for the bisection and projection algorithm [16], and they can also be incorporated in SDPNAL+ [27].

4.2 The existence of interior feasible solutions of COP and its dual

Let $\mathbb{K}^{1+n}$ be a closed convex cone. The dual of $\mathrm{P}(\mathbb{K}^{1+n}, L)$ is written as $\mathrm{D}(\mathbb{K}^{1+n}, L)$:

$$\zeta_{\mathrm{d}}(\mathbb{K}^{1+n}, L) = \max \left\{ y_0 : \boldsymbol{F}_0 - \boldsymbol{H}_0 y_0 - \boldsymbol{W} = \boldsymbol{S} \in (\mathbb{K}^{1+n})^*, \ \boldsymbol{W} \in L^\perp \right\}.$$

Suppose that two distinct linear subspaces $L = L_1$ and $L = L_2 \subset L_1$ provide an equivalent feasible region for $\mathrm{P}(\mathbb{K}^{1+n}, L)$. Obviously, $\boldsymbol{Z}$ is an interior feasible solution of

$\mathrm{P}(\mathbb{K}^{1+n}, L_1)$ iff it is an interior feasible solution of $\mathrm{P}(\mathbb{K}^{1+n}, L_2)$. However, this is not true in their dual problems $\mathrm{D}(\mathbb{K}^{1+n}, L_1)$ and $\mathrm{D}(\mathbb{K}^{1+n}, L_2)$, since the feasible region of $\mathrm{D}(\mathbb{K}^{1+n}, L_1)$ can be a proper subset of the feasible region of $\mathrm{D}(\mathbb{K}^{1+n}, L_2)$. Consequently, $\mathrm{D}(\mathbb{K}^{1+n}, L_1)$ may not have an interior feasible solution even when $\mathrm{D}(\mathbb{K}^{1+n}, L_2)$ has one. As the dimension of the subspace L becomes smaller, the possibility of the existence of a dual interior-feasible solution increases. From the relation $L_{\mathrm{E2}} \subset L_{\mathrm{E3}} \subset L_{\mathrm{E1}}$ and $L_{\mathrm{E2}} \cap L_{\mathrm{C}} \subset L_{\mathrm{E2}} \cap L_{\mathrm{Z}}$ shown in Lemmas 3.1, 3.2 and their proofs, the pair (L_{E2},L_{C}) seems to be the best choice for the numerical stability of computational methods that assumes the existence of interior feasible solution of $\mathrm{D}(\mathbb{K}^{1+n}, L)$. The pair ($L_{\mathrm{E1}}$,$L_{\mathrm{C}}$) or the pair ($L_{\mathrm{E1}}$,$L_{\mathrm{Z}}$) seems to be the worst choice for the numerical stability of such methods. Interestingly, the pair (L_{E1},L_{Z}) has been the best choice with respect to the number of linear equality constraints as discussed in the previous section.

4.3 Nondegeneracy

The numerical stability and the local convergence of many numerical methods, for instance, primal-dual interior-point methods and SDPNAL+ [27] for COPs, are affected by the primal and dual nondegeneracy of the optimal solutions of the problem being solved. For SDPs, the primal and dual nondegeneracy was studied in [1]. (See [17] for the local convergence of the primal-dual interior point method under the nondegeneracy assumption.) The definition in [1] can be extended to the primal-dual pair, $\mathrm{P}(\mathbb{K}^{1+n}, L)$ and $\mathrm{D}(\mathbb{K}^{1+n}, L)$, in a straightforward manner.

Let $\mathbb{J}$ be a closed convex cone in $\mathbb{S}^k$. For every $\boldsymbol{U} \in \mathbb{J}$, let $\mathcal{F}_{\boldsymbol{U}}(\mathbb{J})$ denote the minimal face of $\mathbb{J}$ containing $\boldsymbol{U}$, and $\mathcal{T}_{\boldsymbol{U}}(\mathbb{J})$ the subspace of $\mathbb{S}^k$ spanned by $\mathcal{F}_{\boldsymbol{U}}(\mathbb{J})$ (the tangent subspace of $\mathbb{J}$ at $\boldsymbol{U} \in \mathbb{J}$). Let $\bar{L} = \{\boldsymbol{Z} \in L : \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 0\}$. Then a feasible (or optimal) solution $\boldsymbol{Z}$ of $\mathrm{P}(\mathbb{K}^{1+n}, L)$ is called *(primal) nondegenerate* if $\mathbb{S}^{1+n} = \bar{L} + \mathcal{T}_{\boldsymbol{Z}}(\mathbb{K}^{1+n})$, and a feasible (or optimal) solution $(y_0, \boldsymbol{W}, \boldsymbol{S})$ of $\mathrm{D}(\mathbb{K}^{1+n}, L)$ *(dual) nondegenerate* if $\mathbb{S}^{1+n} = \bar{L}^{\perp} + \mathcal{T}_{\boldsymbol{S}}((\mathbb{K}^{1+n})^*)$. This definition may be regarded as a special case of the nondegeneracy for a feasible solution of a general nonlinear optimization problem in [24].

For solving a DNN problem, $\mathrm{P}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L)$, a primal-dual interior-point method and SDPNAL+ would in fact solve the following SDP:

$$\min\left\{\boldsymbol{F}_0 \bullet \boldsymbol{Z} : (\boldsymbol{Z}, \boldsymbol{U}) \in \mathbb{S}^{1+n}_{+} \times \mathbb{N}^{1+n},\ \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1,\ (\boldsymbol{Z}, \boldsymbol{U}) \in M\right\}, \tag{11}$$

which is equivalent $\mathrm{P}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L)$. Here $M = \{(\boldsymbol{Z}, \boldsymbol{U}) \in \mathbb{S}^{1+n} \times \mathbb{S}^{1+n} : Z \in L,\ \boldsymbol{Z} - \boldsymbol{U} = \boldsymbol{O}\}$. $\boldsymbol{Z}$ is a feasible (or optimal) solution of $\mathrm{P}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L)$ iff $(\boldsymbol{Z}, \boldsymbol{Z})$ is a feasible (or optimal) solution of SDP (11). The definition of the primal and dual nondegeneracy can be applied to SDP (11). More precisely, a feasible solution $(\boldsymbol{Z}, \boldsymbol{Z})$ of SDP (11) is *(primal) nondegenerate* if $\mathbb{S}^{1+n} \times \mathbb{S}^{1+n} = \bar{M} + \mathcal{T}_{\boldsymbol{Z}}(\mathbb{S}^{1+n}_{+}) \times \mathcal{T}_{\boldsymbol{Z}}(\mathbb{N}^{1+n})$, where $\bar{M} = \{(\boldsymbol{Z}, \boldsymbol{U}) \in M : \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 0\} = \{(\boldsymbol{Z}, \boldsymbol{Z}) : \boldsymbol{Z} \in L, \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 0\}$. It can be proved that if $(\boldsymbol{Z}, \boldsymbol{Z})$ is a nondegenerate feasible solution of SDP (11) then $\boldsymbol{Z}$ is a nondegenerate feasible solution of $\mathrm{P}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L)$. The dual problem of (11) can be shown to be given by:

$$\begin{aligned} &\max\left\{y_0 : (\boldsymbol{F}_0, \boldsymbol{O}) - (\boldsymbol{H}_0, \boldsymbol{O})y_0 - (\boldsymbol{V}, \boldsymbol{W}) = (\boldsymbol{S}, \boldsymbol{T}) \in \mathbb{S}^{1+n}_{+} \times \mathbb{N}^{1+n},\ (\boldsymbol{V}, \boldsymbol{W}) \in M^{\perp}\right\} \quad (12) \\ &= \max\left\{y_0 :\ \boldsymbol{F}_0 - \boldsymbol{H}_0 y_0 - \boldsymbol{S} - \boldsymbol{T} \in L^{\perp}, (\boldsymbol{S}, \boldsymbol{T}) \in \mathbb{S}^{1+n}_{+} \times \mathbb{N}^{1+n}\ \right\}, \end{aligned}$$

where we used the fact that $M^\perp = \{(\boldsymbol{V}, \boldsymbol{W}) \in \mathbb{S}^{1+n} \times \mathbb{S}^{1+n} : \boldsymbol{V} + \boldsymbol{W} \in L^\perp\}$. A feasible solution $(\boldsymbol{S}, \boldsymbol{T})$ is said to (dual) nondegenerate for the dual problem (12) if $\mathbb{S}^{1+n} \times \mathbb{S}^{1+n} = \bar{M}^\perp + \mathcal{T}_{\boldsymbol{S}}(\mathbb{S}^{1+n}_+) \times \mathcal{T}_{\boldsymbol{T}}(\mathbb{N}^{1+n})$. It can be shown that the above condition is equivalent to

$$\mathbb{S}^{1+n} = \bar{L}^\perp + \mathcal{T}_{\boldsymbol{S}}(\mathbb{S}^{1+n}_+) + \mathcal{T}_{\boldsymbol{T}}(\mathbb{N}^{1+n}). \tag{13}$$

With larger (or smaller) L, the possibility of the primal nondegeneracy (or the dual nondegeneracy) can be expected to increase by both definitions of nondegeneracy for $\mathrm{P}(\mathbb{K}^{1+n}, L)$ and SDP (11).

5 Nondegeneracy of the simplified COP

It is suggested by Burer in [9] that the simplification technique can increase the possibility of the existence of interior feasible solutions. His discussion can be applied to the simplified COP, $\mathrm{P}'(\mathbb{K}^n, L')$, derived from $\mathrm{P}(\mathbb{K}^{1+n}, L)$. Recall that the simplification depends on a nonzero $\boldsymbol{h} \in \mathbb{R}^n_+$ satisfying $\boldsymbol{x} = \boldsymbol{Y}\boldsymbol{h}$ and $\boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} = 1$ for every feasible solution $\boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix}$ of $\mathrm{P}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$. For further simplification of (E3)' to (E3)" in Section 3.4, $\boldsymbol{h} = \left(\sum_{p=1}^m \boldsymbol{e}_{J_p}\right)/m$ is used. We use this $\boldsymbol{h}$ and assume that $\mathbb{K}^k \in \{\mathbb{S}^k_+, \mathbb{D}^k_\mathrm{NN}, \mathbb{C}^k_\mathrm{PP}\}$ ($k \in \{n, 1+n\}$) throughout this section.

The primal and dual nondegeneracy of the simplified COP is discussed in Sections 5.1 and 5.3, respectively. Specifically, we show that the simplification is effective for increasing the possibility of the nondegeneracy. In Section 5.2, the dual of $\mathrm{P}'(\mathbb{K}^n, L'_0 \cap L'_\mathrm{a} \cap L'_\mathrm{b})$ is derived, and its relations to the dual of the original $\mathrm{P}(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$ are investigated. We prove that the possibility of the existence of interior feasible solutions becomes greater. In particular, if the union of J_p ($1 \le p \le m$) coincides with $\{1, \ldots, n\}$ or $\boldsymbol{h} > \boldsymbol{0}$, then the simplified dual COP always has an interior feasible solution which can be easily computed for any choice of $L'_\mathrm{a} \in \{L'_\mathrm{E1}, L'_\mathrm{E2}, L'_\mathrm{E3}, L''_\mathrm{E3}\}$ and $L'_\mathrm{b} \in \{L'_\mathrm{Z}, L'_\mathrm{C}\}$. The BIQP and the QAP in Sections 6.1 and 6.2 are such cases.

Throughout this section, we consider $\mathrm{P}(\mathbb{K}^{1+n}, L)$ with

$$\begin{aligned} L &= L_\mathrm{S} \cap L_0 \cap L_\mathrm{a} \cap L_\mathrm{b} = \boldsymbol{\Phi}(\mathbb{S}^n) \cap L_0 \cap L_\mathrm{a} \cap L_\mathrm{b} \\ &= \left\{ \boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \in \mathbb{S}^{1+n} : \boldsymbol{h}^T\boldsymbol{Y}\boldsymbol{h} = x_0,\ \boldsymbol{x} = \boldsymbol{Y}\boldsymbol{h},\ \boldsymbol{Z} \in L_0 \cap L_\mathrm{a} \cap L_\mathrm{b} \right\} \subset \boldsymbol{\Phi}(\mathbb{S}^n) \end{aligned}$$

for a fixed $(L_\mathrm{a}, L_\mathrm{b}) \in \{L_\mathrm{E1}, L_\mathrm{E2}, L_\mathrm{E3}\} \times \{L_\mathrm{Z}, L_\mathrm{C}\}$. We recall that the linear maps $\boldsymbol{\Phi} : \mathbb{S}^n \to \mathbb{S}^{1+n}$, $\boldsymbol{\Psi} : \mathbb{S}^{1+n} \to \mathbb{S}^n$ and $\boldsymbol{\Pi} : \mathbb{S}^{1+n} \to \mathbb{S}^n$ have been defined in Section 2.3, and that the linear subspace L_S has been given in (3). If $L_\mathrm{a} = L_\mathrm{E2}$, then $L = L_0 \cap L_\mathrm{a} \cap L_\mathrm{b}$ by (i) of Lemma 3.3. In other cases, $L \subset L_0 \cap L_\mathrm{a} \cap L_\mathrm{b}$ but $\zeta(\mathbb{K}^{1+n}, L) = \zeta(\mathbb{K}^{1+n}, L_0 \cap L_\mathrm{a} \cap L_\mathrm{b})$ holds by (ii) of Lemma 3.3. We should note that the existence of an interior feasible solution of the dual of $\mathrm{P}(\mathbb{K}^{1+n}, L)$ and the primal and dual nondegeneracy depend on specific choices of L_a and L_b. In the following discussions, however, all assertions are valid for any choices of L_a and L_b. Thus, we omit describing the dependency.

5.1 The primal nondegeneracy in the simplified COP $\mathbf{P}'(\mathbb{K}^n, L')$

By Lemma 2.1, the simplified COP of $\mathrm{P}(\mathbb{K}^{1+n}, L)$ is described as $\mathrm{P}'(\mathbb{K}^n, L')$, where

$$L' = \boldsymbol{\Phi}^{-1}(L) = \boldsymbol{\Pi}(L) = \left\{ \boldsymbol{Y} : \boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \in L \right\}.$$

(To see the identity $\boldsymbol{\Phi}^{-1}(L) = \boldsymbol{\Pi}(L)$, we recall that $L \subset \boldsymbol{\Phi}(\mathbb{S}^n)$ and (4).)

Let $\bar{\boldsymbol{Z}} = \begin{pmatrix} 1 & \bar{\boldsymbol{x}}^T \\ \bar{\boldsymbol{x}} & \bar{\boldsymbol{Y}} \end{pmatrix}$ be a feasible solution of $\mathrm{P}(\mathbb{K}^{1+n}, L)$. By Lemma 2.1, $\bar{\boldsymbol{Y}} = \boldsymbol{\Pi}(\bar{\boldsymbol{Z}})$ is a feasible solution of $\mathrm{P}'(\mathbb{K}^n, L')$. Let $\bar{L} = \{\boldsymbol{Z} \in L : \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 0\}$, and

$$\bar{L}' \;=\; \{\boldsymbol{Y} \in L' : \boldsymbol{h}\boldsymbol{h}^T \bullet \boldsymbol{Y} = 0\} = \boldsymbol{\Phi}^{-1}(\bar{L}) = \boldsymbol{\Pi}(\bar{L}). \tag{14}$$

By definition, $\bar{\boldsymbol{Z}}$ is nondegenerate in $\mathrm{P}(\mathbb{K}^{1+n}, L)$ if $\mathbb{S}^{1+n} = \bar{L} + \mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})$, and $\bar{\boldsymbol{Y}}$ nondegenerate in $\mathrm{P}'(\mathbb{K}^n, L')$ if $\mathbb{S}^n = \bar{L}' + \mathcal{T}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$.

Theorem 5.1. *Assume that $\bar{\boldsymbol{Z}}$ is nondegenerate. Then $\bar{\boldsymbol{Y}} = \boldsymbol{\Pi}(\bar{\boldsymbol{Z}})$ is nondegenerate.*

Proof. It suffices to prove that $\mathbb{S}^n \subset \bar{L}' + \mathcal{T}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$ because the converse inclusion is obvious. In the following we will show that $\mathbb{S}^n \subset \bar{L}' + \boldsymbol{\Pi}(\mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n}))$ in (i), and $\boldsymbol{\Pi}(\mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})) \subset \mathcal{T}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$ in (iii).

(i) Take an arbitrary $\boldsymbol{Y} \in \mathbb{S}^n$. Let $\boldsymbol{Z} = \begin{pmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{Y} \end{pmatrix} \in \mathbb{S}^{1+n}$. By the nondegeneracy assumption on the $\bar{\boldsymbol{Z}}$, there exist $\boldsymbol{Z}^1 \in \bar{L}$ and $\boldsymbol{Z}^2 \in \mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})$ such that $\boldsymbol{Z} = \boldsymbol{Z}^1 + \boldsymbol{Z}^2$. Since $\boldsymbol{\Pi} : \mathbb{S}^{1+n} \to \mathbb{S}^n$ is linear, we see that $\boldsymbol{Y} = \boldsymbol{\Pi}(\boldsymbol{Z}) = \boldsymbol{\Pi}(\boldsymbol{Z}^1) + \boldsymbol{\Pi}(\boldsymbol{Z}^2)$. Thus $\mathbb{S}^n \subset \boldsymbol{\Pi}(\bar{L}) + \boldsymbol{\Pi}(\mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n}))$. Using $\boldsymbol{\Pi}(\bar{L}) = \bar{L}'$ by (14), we obtain that $\mathbb{S}^n \subset \bar{L}' + \boldsymbol{\Pi}(\mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n}))$.

(ii) Next we show that $\boldsymbol{\Pi}(\mathcal{F}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})) \subset \mathcal{F}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$. Obviously, $\mathcal{F}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n) \ni \bar{\boldsymbol{Y}}$. Let $\boldsymbol{\Pi}(\mathcal{F}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})) \ni \boldsymbol{Y} \neq \bar{\boldsymbol{Y}}$. Then, there is a $\boldsymbol{Z} \in \mathcal{F}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})$ such that $\boldsymbol{Z} = \begin{pmatrix} x_0 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{Y} \end{pmatrix} \neq \bar{\boldsymbol{Z}}$. Since $\bar{\boldsymbol{Z}}$ lies in the relative interior of $\mathcal{F}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})$, $\bar{\boldsymbol{Z}}$ can be represented as a convex combination of $\boldsymbol{Z}$ and some $\boldsymbol{U} \in \mathcal{F}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})$ such that $\boldsymbol{U} \neq \bar{\boldsymbol{Z}}$; $\bar{\boldsymbol{Z}} = \lambda \boldsymbol{Z} + (1-\lambda)\boldsymbol{U}$ for some $\lambda \in (0,1)$. It follows that $\mathcal{F}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n) \ni \bar{\boldsymbol{Y}} = \boldsymbol{\Pi}(\bar{\boldsymbol{Z}}) = \lambda \boldsymbol{Y} + (1-\lambda)\boldsymbol{\Pi}(\boldsymbol{U})$, $\bar{\boldsymbol{Y}} \neq \boldsymbol{Y} \in \mathbb{K}^n$ and $\boldsymbol{\Pi}(\boldsymbol{U}) \in \mathbb{K}^n$. Since $\mathcal{F}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$ is a face of $\mathbb{K}^n$, we obtain that $\boldsymbol{Y} \in \mathcal{F}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$. Thus, we have shown $\boldsymbol{\Pi}(\mathcal{F}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})) \subset \mathcal{F}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$.

(iii) Let $\boldsymbol{Z}^1, \ldots, \boldsymbol{Z}^q$ be a basis of $\mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})$. They can be taken from $\mathcal{F}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})$. Then we know by (ii) that $\boldsymbol{\Pi}(\boldsymbol{Z}^1), \ldots, \boldsymbol{\Pi}(\boldsymbol{Z}^q) \in \mathcal{F}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$. Therefore we see that

$$\boldsymbol{\Pi}(\mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})) = \boldsymbol{\Pi}\left(\mathrm{lin}\{\boldsymbol{Z}^1, \ldots, \boldsymbol{Z}^q\}\right) = \mathrm{lin}\{\boldsymbol{\Pi}(\boldsymbol{Z}^1), \ldots, \boldsymbol{\Pi}(\boldsymbol{Z}^q)\} \subset \mathcal{T}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n),$$

where $\mathrm{lin}(A)$ denotes the smallest linear subspace of $\mathbb{S}^k$ that contains $A \subset \mathbb{S}^k$. Consequently, we have shown that $\boldsymbol{\Pi}(\mathcal{T}_{\bar{\boldsymbol{Z}}}(\mathbb{K}^{1+n})) \subset \mathcal{T}_{\bar{\boldsymbol{Y}}}(\mathbb{K}^n)$. □

5.2 The dual of $\mathbf{P}'(\mathbb{K}^n, L')$

The dual of $\mathrm{P}(\mathbb{K}^{1+n}, L)$ can be written as $\mathrm{D}(\mathbb{K}^{1+n}, L)$. We consider the dual of the simplified $\mathrm{P}'(\mathbb{K}^n, L')$, denoted by $\mathrm{D}'(\mathbb{K}^n, L')$:

$$\zeta_{\mathrm{d}}'(\mathbb{K}^n, L') = \max\left\{ y_0 : \boldsymbol{F}_0' - \boldsymbol{h}\boldsymbol{h}^T y_0 - \boldsymbol{W}' = \boldsymbol{S}' \in (\mathbb{K}^n)^*,\ \boldsymbol{W}' \in (L')^{\perp} \right\}.$$

Lemma 5.1.

(i) $(L')^\perp = \boldsymbol{\Psi}(L^\perp)$.

(ii) $\boldsymbol{\Psi}((\mathbb{K}^{1+n})^*) \subset (\mathbb{K}^n)^*$ *and* $\boldsymbol{\Psi}(\mathrm{int}((\mathbb{K}^{1+n})^*)) \subset \mathrm{int}((\mathbb{K}^n)^*)$. *Here* $\mathrm{int}((\mathbb{K}^k)^*)$ *denotes the interior of* $(\mathbb{K}^k)^*$ $(k \in \{n, 1+n\})$*.*

Proof. To prove Assertion (i), we observe that

$$\begin{aligned} \boldsymbol{Y} \in L' \quad & \text{iff} \quad \boldsymbol{\Phi}(\boldsymbol{Y}) \in L \\ & \text{iff} \quad 0 = \boldsymbol{\Phi}(\boldsymbol{Y}) \bullet \boldsymbol{W} = \boldsymbol{Y} \bullet \boldsymbol{\Psi}(\boldsymbol{W}) \text{ for all } \boldsymbol{W} \in L^\perp \\ & \text{iff} \quad \boldsymbol{Y} \in \boldsymbol{\Psi}(L^\perp)^\perp. \end{aligned}$$

Hence $\boldsymbol{W}' \in (L')^\perp$ iff $\boldsymbol{W}' \in \boldsymbol{\Psi}(L^\perp)$, which implies (i). For Assertion (ii), suppose that $\boldsymbol{S} \in (\mathbb{K}^{1+n})^*$ and $\boldsymbol{S}' = \boldsymbol{\Psi}(\boldsymbol{S})$. We note that $\boldsymbol{S}' \in (\mathbb{K}^n)^*$ iff $\boldsymbol{S}' \bullet \boldsymbol{Y} \geq 0$ for every nonzero $\boldsymbol{Y} \in \mathbb{K}^n$ and that $\boldsymbol{S}' \in \mathrm{int}((\mathbb{K}^n)^*)$ iff $\boldsymbol{S}' \bullet \boldsymbol{Y} > 0$ for every nonzero $\boldsymbol{Y} \in \mathbb{K}^n$. Choose a nonzero $\boldsymbol{Y} \in \mathbb{K}^n$ arbitrarily. Then $\boldsymbol{O} \neq \boldsymbol{\Phi}(\boldsymbol{Y}) \in \mathbb{K}^{1+n}$ by the definition of $\boldsymbol{\Phi}$ and (6). Hence $\boldsymbol{S}' \bullet \boldsymbol{Y} = \boldsymbol{\Psi}(\boldsymbol{S}) \bullet \boldsymbol{Y} = \boldsymbol{S} \bullet \boldsymbol{\Phi}(\boldsymbol{Y}) \geq 0$. If $\boldsymbol{S} \in \mathrm{int}((\mathbb{K}^{1+n})^*)$, then the inequality is strict. Thus we have shown (ii). □

Theorem 5.2. *Assume that* $(y_0, \boldsymbol{W}, \boldsymbol{S}) \in \mathbb{R} \times \mathbb{S}^{1+n} \times \mathbb{S}^{1+n}$ *is a feasible solution of* $\mathrm{D}(\mathbb{K}^{1+n}, L)$*. Let* $\boldsymbol{W}' = \boldsymbol{\Psi}(\boldsymbol{W})$ *and* $\boldsymbol{S}' = \boldsymbol{\Psi}(\boldsymbol{S})$*. Then* $(y_0, \boldsymbol{W}', \boldsymbol{S}')$ *is a feasible solution of* $\mathrm{D}'(\mathbb{K}^n, L')$*. If* $\boldsymbol{S}$ *lies in the interior of* $(\mathbb{K}^{1+n})^*$*, then* $\boldsymbol{S}'$ *lies in the interior of* $(\mathbb{K}^n)^*$*.*

Proof. From the assumption,

$$\boldsymbol{F}_0 - \boldsymbol{H}_0 y_0 - \boldsymbol{W} = \boldsymbol{S}, \ \boldsymbol{W} \in L^\perp, \ \boldsymbol{S} \in (\mathbb{K}^{1+n})^*.$$

By applying the linear map $\boldsymbol{\Psi} : \mathbb{S}^{1+n} \to \mathbb{S}^n$ to these relations, we obtain

$$\boldsymbol{F}_0' - \boldsymbol{h}\boldsymbol{h}^T y_0 - \boldsymbol{W}' = \boldsymbol{S}', \ \boldsymbol{W}' \in \boldsymbol{\Psi}(L^\perp), \ \boldsymbol{S}' \in \boldsymbol{\Psi}((\mathbb{K}^{1+n})^*).$$

By Lemma 5.1, we know that $\boldsymbol{W}' \in (L')^\perp$ and $\boldsymbol{S}' \in (\mathbb{K}^n)^*$. Hence $(y_0, \boldsymbol{W}', \boldsymbol{S}')$ is a feasible solution of $\mathrm{D}'(\mathbb{K}^n, L')$. The second assertion also follows from Lemma 5.1. □

By the weak duality, $\zeta_{\mathrm{d}}(\mathbb{K}^{1+n}, L) \leq \zeta_{\mathrm{p}}(\mathbb{K}^{1+n}, L)$ and $\zeta_{\mathrm{d}}'(\mathbb{K}^n, L') \leq \zeta_{\mathrm{p}}'(\mathbb{K}^n, L')$. Theorem 5.2 ensures that $\zeta_{\mathrm{d}}(\mathbb{K}^{1+n}, L) \leq \zeta_{\mathrm{d}}'(\mathbb{K}^n, L')$. By Lemma 2.1, we know that $\zeta_{\mathrm{p}}(\mathbb{K}^{1+n}, L) = \zeta_{\mathrm{p}}'(\mathbb{K}^n, L')$. Therefore, if the strong duality relation $\zeta_{\mathrm{d}}(\mathbb{K}^{1+n}, L) = \zeta_{\mathrm{p}}(\mathbb{K}^{1+n}, L)$ holds between $\mathrm{P}(\mathbb{K}^{1+n}, L)$ and $\mathrm{D}(\mathbb{K}^{1+n}, L)$, then it also holds between $\mathrm{P}'(\mathbb{K}^n, L')$ and $\mathrm{D}'(\mathbb{K}^n, L')$, and all the optimal values ζ_{d}, ζ_{p}, ζ_{d}' and ζ_{p}' coincide with each other.

Theorem 5.3. *Assume that* $\mathbb{K}^k \in \{\mathbb{D}^k_{\mathrm{NN}}, \mathbb{C}^k_{\mathrm{PP}}\}$ $(k \in \{n, 1+n\})$ *and that* $\boldsymbol{h} > \boldsymbol{0}$*. Let* $\boldsymbol{W}' \in (\mathbb{K}^n)^*$ *(for example, let* $\boldsymbol{W}' = \boldsymbol{O}$*). Choose* $y_0 \in \mathbb{R}$ *such that* $\boldsymbol{S}' = \boldsymbol{F}_0' - \boldsymbol{h}\boldsymbol{h}^T y_0 - \boldsymbol{W}' > \boldsymbol{O}$ *(such a choice of* y_0 *is possible since* $\boldsymbol{h}\boldsymbol{h}^T > \boldsymbol{O}$*). Then* $(y_0, \boldsymbol{W}', \boldsymbol{S}')$ *is an interior feasible solution of* $\mathrm{D}'(\mathbb{K}^n, L')$*.*

Proof. By construction, $(y_0, \boldsymbol{W}', \boldsymbol{S}')$ satisfies the equality constraint of $\mathrm{D}'(\mathbb{K}^n, L')$ and $\boldsymbol{W}' \in (\mathbb{K}^n)^*$, and $\boldsymbol{S}' > \boldsymbol{O}$ lies in the interior of $\mathbb{N}^n$. Since $\mathbb{N}^n \subset (\mathbb{K}^n)^*$, $\boldsymbol{S}'$ lies in the interior of $(\mathbb{K}^n)^*$. Therefore $(y_0, \boldsymbol{W}', \boldsymbol{S}')$ is an interior feasible solution of $\mathrm{D}'(\mathbb{K}^n, L')$. □

Note that if the union of J_p $(1 \leq p \leq m)$ coincides with $\{1, \ldots, n\}$, the choice $\boldsymbol{h} = \frac{1}{m}\sum_{p=1}^m \boldsymbol{e}_{J_p}$ taken in Section 3.4 satisfies $\boldsymbol{h} > \boldsymbol{0}$ and hence the simplified dual COP has an interior feasible solution.

5.3 The dual nondegeneracy in the simplified COP

Suppose that $(\bar{y}_0, \bar{\boldsymbol{W}}, \bar{\boldsymbol{S}})$ is a feasible solution of $\mathrm{D}(\mathbb{K}^{1+n}, L)$. Let $\bar{\boldsymbol{W}}' = \boldsymbol{\Psi}(\bar{\boldsymbol{W}})$ and $\bar{\boldsymbol{S}}' = \boldsymbol{\Psi}(\boldsymbol{S})$. By Theorem 5.2, $(\bar{y}_0, \bar{\boldsymbol{W}}', \bar{\boldsymbol{S}}')$ is a feasible solution of $\mathrm{D}'(\mathbb{K}^n, L')$. By definition, $(\bar{y}_0, \bar{\boldsymbol{W}}, \bar{\boldsymbol{S}})$ is nondegenerate in $\mathrm{D}(\mathbb{K}^{1+n}, L)$ if $\mathbb{S}^{1+n} = \bar{L}^{\perp} + \mathcal{T}_{\bar{\boldsymbol{S}}}((\mathbb{K}^{1+n})^*)$, and $(\bar{y}_0, \bar{\boldsymbol{W}}', \bar{\boldsymbol{S}}')$ nondegenerate in $\mathrm{D}'(\mathbb{K}^n, L')$ if $\mathbb{S}^n = \bar{L}'^{\perp} + \mathcal{T}_{\bar{\boldsymbol{S}}'}((\mathbb{K}^n)^*)$. We can prove the following theorem similarly to Theorem 5.1, and the proof is omitted.

Theorem 5.4. *Assume that $(\bar{y}_0, \bar{\boldsymbol{W}}, \bar{\boldsymbol{S}})$ is nondegenerate. Then $(\bar{y}_0, \bar{\boldsymbol{W}}', \bar{\boldsymbol{S}}')$ is nondegenerate.*

6 Applications

We consider two well-known combinatorial QOPs, the simple BIQP in Section 6.1, and the QAP, which is widely known as one of the most difficult combinatorial QOPs, in Section 6.2. For their importance in applications, many conic relaxations of both QOPs have been extensively studied. We investigate some of them in terms of $\mathrm{P}(\mathbb{K}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{b}})$ with $L_{\mathrm{a}} \in \{L_{\mathrm{E1}}, L_{\mathrm{E2}}, L_{\mathrm{E3}}\}$ and $L_{\mathrm{b}} \in \{L_{\mathrm{C}}, L_{\mathrm{Z}}\}$, and $\mathrm{P}'(\mathbb{K}^n, L'_{\mathrm{a}} \cap L'_{\mathrm{b}})$ with $L'_{\mathrm{a}} \in \{L'_{\mathrm{E1}}, L'_{\mathrm{E2}}, L'_{\mathrm{E3}}, L''_{\mathrm{E3}}\}$ and $L'_{\mathrm{b}} \in \{L'_{\mathrm{C}}, L'_{\mathrm{Z}}\}$.

6.1 The binary integer quadratic problem

Let $\boldsymbol{Q} \in \mathbb{S}^r$. The binary quadratic optimization problem (BIQP) is given by

$$\begin{aligned} \zeta^*_{\mathrm{BIQP}} &= \min \left\{ \boldsymbol{Q} \bullet \boldsymbol{u}\boldsymbol{u}^T : u_j \in \{0,1\} \ (1 \leq j \leq r) \right\} \\ &= \min \left\{ \boldsymbol{Q} \bullet \boldsymbol{u}\boldsymbol{u}^T : \boldsymbol{u} = \mathrm{diag}(\boldsymbol{u}\boldsymbol{u}^T) \right\}. \end{aligned} \tag{15}$$

By the definition of Γ^{1+r}, $\boldsymbol{U} = \boldsymbol{u}\boldsymbol{u}^T$ with $\boldsymbol{u} \in \mathbb{R}^r_+$ iff $\begin{pmatrix} 1 & \boldsymbol{u}^T \\ \boldsymbol{u} & \boldsymbol{U} \end{pmatrix} \in \Gamma^{1+r}$. Hence we can rewrite BIQP (15) as

$$\eta(\mathbb{K}^{1+r}) = \min \left\{ \boldsymbol{Q} \bullet \boldsymbol{U} : \begin{pmatrix} 1 & \boldsymbol{u}^T \\ \boldsymbol{u} & \boldsymbol{U} \end{pmatrix} \in \mathbb{K}^{1+r}, \ \boldsymbol{u} = \mathrm{diag}(\boldsymbol{U}) \right\} \tag{16}$$

with $\mathbb{K}^{1+r} = \Gamma^{1+r}$. Then, the standard SDP, DNN and CPP relaxations are obtained if $\mathbb{K}^{1+r} = \Gamma^{1+r}$ is replaced by $\mathbb{S}^{1+r}_+$, $\mathbb{D}^{1+r}_{\mathrm{NN}}$ and $\mathbb{C}^{1+r}_{\mathrm{PP}}$, respectively. Since $\mathbb{S}^{1+r}_+ \supset \mathbb{D}^{1+r}_{\mathrm{NN}} \supset \mathbb{C}^{1+r}_{\mathrm{PP}} \supset \Gamma^{1+r}$, it follows that $\eta(\mathbb{S}^{1+r}_+) \leq \eta(\mathbb{D}^{1+r}_{\mathrm{NN}}) \leq \eta(\mathbb{C}^{1+r}_{\mathrm{PP}}) \leq \eta(\Gamma^{1+r}) = \zeta^*_{\mathrm{BIQP}}$.

To strengthen the DNN relaxation and derive the CPP reformulation [4, 9] of BIQP (15), we transform BIQP (15) to the following problem by introducing slack variables $v_j = 1 - u_j \geq 0$ $(1 \leq i \leq r)$.

$$\zeta^*_{\mathrm{BIQP}} = \min_{\boldsymbol{x} = [\boldsymbol{u}; \boldsymbol{v}] \in \mathbb{R}^{2r}} \left\{ f_0(\boldsymbol{x}) : 1 = u_j + v_j, \ x_j \in \{0,1\} \ (1 \leq j \leq r) \right\}, \tag{17}$$

where $f_0(\boldsymbol{x}) = \boldsymbol{Q} \bullet \boldsymbol{u}\boldsymbol{u}^T$ for every n-dimensional column vector $\boldsymbol{x} = [\boldsymbol{u}; \boldsymbol{v}]$ formed by concatenating $\boldsymbol{u}$ and $\boldsymbol{v}$ vertically. Let $n = 2r$, $m = r$ and $J_p = \{p, p + r\}$ $(1 \leq p \leq m)$. Then, the constraint of the problem (17) is stated as (G), and (17) is written as $\zeta^*_{\mathrm{BIQP}} =$

$\min_{\boldsymbol{x}\in\mathbb{R}^n}\{f_0(\boldsymbol{x}) : \text{(G)}\}$, which is a special case of QOP (2) with $\ell_0 = 0$. Therefore, the discussions in Section 3 can be applied for deriving various conic relaxations of BIQP (17) as special cases of $\mathrm{P}(\mathbb{K}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{b}})$ and $\mathrm{P}'(\mathbb{K}^n, L'_{\mathrm{a}} \cap L'_{\mathrm{b}})$. Note that $L_0 = \mathbb{S}^{1+n}$ and $L'_0 = \mathbb{S}^n$ since $\ell_0 = 0$. In particular, each J_p consists of two elements $(1 \leq p \leq m)$. Applying (iii) of Lemma 3.2, we may replace Condition (Z) with

(Zu) $\boldsymbol{Z} \in L_{\mathrm{Zu}} \equiv \{\boldsymbol{Z} \in \mathbb{S}^{1+n} : x_i = Y_{ii} \ (1 \leq i \leq r)\}$.

We note that (Zu) is different from (Z) in that $x_i = Y_{ii}$ $(r+1 \leq i \leq n)$ are not imposed in (Zu).

If Burer's CPP reformulation [9] of a class of linearly constrained QOPs in binary and continuous variables is applied to BIQP (17), then $\mathrm{P}(\mathbb{C}^{1+n}_{\mathrm{PP}}, L_{\mathrm{E3}} \cap L_{\mathrm{Zu}})$ is obtained. On the other hand, the application of the CPP reformulation proposed by Arima, Kim and Kojima in [3] (see also [5, 16]) for a class of quadratically constrained QOPs to BIQP (17) would result in $\mathrm{P}(\mathbb{C}^{1+n}_{\mathrm{PP}}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$. The 0-1 condition (Zu) is utilized in Burer's reformulation while the complementarity condition (C) is employed in the reformulation of Arima, Kim and Kojima. As both of them are equivalent to (17), the equivalence of their reformulations follows. Lemmas 3.1 and 3.2 ensure not only their equivalence but also the equivalence of their DNN and SDP relaxations, i.e., $\zeta_p(\mathbb{D}^{1+n}_{\mathrm{NN}}, L_{\mathrm{E3}} \cap L_{\mathrm{Zu}}) = \zeta_p(\mathbb{D}^{1+n}_{\mathrm{NN}}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$, $\zeta_p(\mathbb{S}^{1+n}_+, L_{\mathrm{E3}} \cap L_{\mathrm{Zu}}) = \zeta_p(\mathbb{S}^{1+n}_+, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$.

The DNN relaxation (with optimal value $\eta(\mathbb{D}^{1+r}_{\mathrm{NN}})$) of BIQP (15) and the DNN relaxation $\mathrm{P}(\mathbb{D}^{1+2r}_{\mathrm{NN}}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$ (with the optimal value $\zeta_p(\mathbb{D}^{1+2r}_{\mathrm{NN}}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$) of BIQP (17) were compared in detail in [15, Section 6] by Kim, Kojima and Toh. They showed that the second relaxation is at least as effective as the first, *i.e.*, $\eta(\mathbb{D}^{1+r}_{\mathrm{NN}}) \leq \zeta_p(\mathbb{D}^{1+2r}_{\mathrm{NN}}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$ in theory, and provided randomly generated numerical instances for which strict inequalities hold. Moreover, Kim and Kojima [14] presented a numerical instance of BIQP (15) with $r = 3$ such that its DNN relaxation (16) with $\mathbb{K}^{1+r} = \mathbb{D}^{1+3}_{\mathrm{NN}}$ attained only a strict lower bound $\hat{\zeta}$ for the optimal value ζ^*_{BIQP}. Since it is known in [11] that $\mathbb{D}^n_{\mathrm{NN}} = \mathbb{C}^n_{\mathrm{PP}}$ if $n \leq 4$, we also have $\eta(\mathbb{C}^{1+3}_{\mathrm{PP}}) = \eta(\mathbb{D}^{1+3}_{\mathrm{NN}}) = \hat{\zeta} < \zeta^*_{\mathrm{BIQP}}$ in that instance. In other words, the CPP relaxation (16) with $\mathbb{K}^{1+r} = \mathbb{C}^{1+3}_{\mathrm{PP}}$ attains only the strict lower bound $\hat{\zeta} < \zeta^*_{\mathrm{BIQP}}$. In that instance, they also illustrated that $\zeta_p(\mathbb{D}^{1+6}_{\mathrm{NN}}, L_{\mathrm{E1}} \cap L_{\mathrm{C}}) = \hat{\zeta} < \zeta^*_{\mathrm{BIQP}}$. In theory, however, $\zeta_p(\mathbb{C}^{1+6}_{\mathrm{PP}}, L_{\mathrm{E1}} \cap L_{\mathrm{C}}) = \zeta^*_{\mathrm{BIQP}}$ is guaranteed.

6.2 The quadratic assignment problem

Based on the results of Section 3, we investigate the relations among several existing conic relaxations [2, 9, 22, 28, 23] of the QAP, which are summarized in Table 1. In addition to the existing conic relaxations in Table 1, there can be other conic relaxations of the QAP, which are obtained by combining the conditions in the rows and columns of Table 1.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be given $r \times r$ matrices. Then the QAP is described as

$$\zeta^*_{\mathrm{QAP}} = \min\left\{\boldsymbol{X} \bullet (\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^T) : \boldsymbol{X} \text{ is a permutation matrix}\right\}. \tag{18}$$

QAP can be regarded as a QOP in an $r \times r$ matrix variable $\boldsymbol{X}$ subject to the following combinatorial conditions:

(GC) All elements of each pth column of $\boldsymbol{X}$ are 0 except one with the value 1.

Table 1: A sketch of the correspondence of some existing relaxations of the QAP to $\mathrm{P}(\mathbb{K}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{b}})$ with $\mathbb{K}^{1+n} = \mathbb{S}^{1+n}_{+}$, $\mathbb{D}^{1+n}_{\mathrm{NN}}$ or $\mathbb{C}^{1+n}_{\mathrm{PP}}$, and $\mathrm{P}'(\mathbb{K}^{n}, L'_{\mathrm{a}} \cap L'_{\mathrm{b}})$ with $\mathbb{K}^{n} = \mathbb{S}^{n}_{+}$, $\mathbb{D}^{n}_{\mathrm{NN}}$ or $\mathbb{C}^{n}_{\mathrm{PP}}$. AKKT means the conic relaxation presented in [3, 5], $\mathrm{QAP}_{\mathrm{R}_0}$, $\mathrm{QAP}_{\mathrm{R}_2}$ and $\mathrm{QAP}_{\mathrm{R}_3}$ the ones in [23], and Burer the one in [9]. $\mathrm{QAP}^{-}_{\mathrm{R}_2}$ and $\mathrm{QAP}^{-}_{\mathrm{R}_3}$ mean conic relaxations obtained from $\mathrm{QAP}_{\mathrm{R}_2}$ and $\mathrm{QAP}_{\mathrm{R}_3}$ by eliminating the redundant constraint (Z), respectively. sBurer corresponds to Burer's simplified conic relaxation applied to the QAP. $\mathrm{QAP}_{\mathrm{CP}}$, $\mathrm{QAP}_{\mathcal{K}^{0*}_{n}}$, $\mathrm{QAP}_{\mathrm{AW+}}$ and $\mathrm{QAP}_{\mathrm{ZKRW1}}$ are from [22], but their correspondence to $\mathrm{P}'(\mathbb{K}^{n}, L''_{\mathrm{E3}} \cap L'_{\mathrm{C}})$ is not exact. More precisely, their optimal values are the same, but the descriptions of the feasible regions are different. The details are shown in Theorem 6.1. No existing conic relaxations, to the authors' best knowledge, correspond to other COPs. Detailed explanations are included in Sections 6.2.1 and 6.2.2.

		L_{b}		
		L_{Z}	L_{C}	$L_{\mathrm{Z}} \cap L_{\mathrm{C}}$
L_{a}	L_{E1}		AKKT	
	L_{E2}	$\mathrm{QAP}_{\mathrm{R}_0}$	$\mathrm{QAP}^{-}_{\mathrm{R}_2}$, $\mathrm{QAP}^{-}_{\mathrm{R}_3}$	$\mathrm{QAP}_{\mathrm{R}_2}$, $\mathrm{QAP}_{\mathrm{R}_3}$
	L_{E3}	Burer		

↓ via the simplification in [9]

		L'_{b}		
		L'_{Z}	L'_{C}	$L'_{\mathrm{Z}} \cap L'_{\mathrm{C}}$
L'_{a}	L'_{E1}			
	L'_{E2}			
	L'_{E3}	sBurer		
	L''_{E3}		$\mathrm{QAP}_{\mathrm{CP}}$, $\mathrm{QAP}_{\mathcal{K}^{0*}_{n}}$, $\mathrm{QAP}_{\mathrm{AW+}}$, $\mathrm{QAP}_{\mathrm{ZKRW1}}$	

(GR) All elements of each ith row of $\boldsymbol{X}$ are 0 except one with the value 1.

To derive the conic relaxation using the systematic method presented in Section 3, we need to transform the problem (18) in the matrix variable $\boldsymbol{X} \in \mathbb{R}^{r \times r}$ to a QOP in an n-dimensional column vector variable $\boldsymbol{x}$, where $n = r^2$. For every $\boldsymbol{X} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^r) \in \mathbb{R}^{r \times r}$, where $\boldsymbol{x}^p$ denotes the p-th column vector of $\boldsymbol{X}$, we consider the n-dimensional vector $\boldsymbol{x} = [\boldsymbol{x}^1; \ldots; \boldsymbol{x}^r]$ by arranging $\boldsymbol{x}^p$ $(1 \leq p \leq r)$ vertically. Then the objective function can be rewritten as $\boldsymbol{X} \bullet (\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^T) = \boldsymbol{x}^T(\boldsymbol{B} \otimes \boldsymbol{A})\boldsymbol{x}$. The conditions (GC) and (GR) on $\boldsymbol{X}$ can also be restated as the condition (G) on $\boldsymbol{x}$ with $m = 2r$ and

$$\begin{aligned} J_p &= \{(p-1)r+1, (p-1)r+2, \ldots, (p-1)+r\} \ (1 \leq p \leq r), \\ J_{r+i} &= \{i, i+r, \ldots, i+(r-1)r\} \ (1 \leq i \leq r), \end{aligned}$$

where J_p and J_{r+i} correspond to (GC) and (GR), respectively. Hence, QAP (18) is equivalently formulated as follows:

$$\zeta^*_{\mathrm{QAP}} = \min \left\{ \boldsymbol{x}^T(\boldsymbol{B} \otimes \boldsymbol{A})\boldsymbol{x} : (\mathrm{G}) \right\}, \tag{19}$$

which is a special case of QOP (2) with $\ell_0 = 0$. Thus, various conic relaxations of QAP (19) can be derived from $\mathrm{P}(\mathbb{K}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{b}})$ and $\mathrm{P}'(\mathbb{K}^{n}, L'_{\mathrm{a}} \cap L'_{\mathrm{b}})$ as stated in Section 3. Note that $L_0 = \mathbb{S}^{1+n}$ and $L'_0 = \mathbb{S}^{n}$ since $\ell_0 = 0$.

Let

$$\begin{aligned} \boldsymbol{Y}^{pq} &= \text{the } (p,q)\text{th } r \times r \text{ block submatrix of } \boldsymbol{Y} \text{ corresponding to } \boldsymbol{x}^p(\boldsymbol{x}^q)^T, \\ C_p &= J_p \ (1 \le p \le r), \ R_i = J_{r+i} \ (1 \le i \le r), \ \boldsymbol{E}_J = \boldsymbol{e}_J \boldsymbol{e}_J^T \in \Gamma^n. \end{aligned}$$

We note that both $\{C_p : 1 \le p \le r\}$ and $\{R_i : 1 \le i \le r\}$ form partitions of the index set $\{1, \ldots, n\}$. This is an important feature of QOP (19) derived from QAP (18), which is utilized in Section 6.2.2. Some of Conditions (E1) through (C), which are mainly considered in Section 6.2.2, are rewritten as follows:

$$\textbf{(E3)} \ \boldsymbol{Z} \in L_{\mathrm{E3}} = \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \begin{array}{l} \boldsymbol{e}_{C_p}^T \boldsymbol{x} = x_0, \ \boldsymbol{E}_{C_p} \bullet \boldsymbol{Y} = x_0, \\ \boldsymbol{e}_{R_p}^T \boldsymbol{x} = x_0, \ \boldsymbol{E}_{R_p} \bullet \boldsymbol{Y} = x_0 \\ (1 \le p \le r) \end{array} \right\} \text{ and } \boldsymbol{H}_0 \bullet \boldsymbol{Z} = 1.$$

$$\textbf{(C)} \ \boldsymbol{Z} \in L_{\mathrm{C}} = \left\{ \boldsymbol{Z} \in \mathbb{S}^{1+n} : \ Y_{ij}^{pp} = 0 \ (i \ne j, \ 1 \le p \le r), \ Y_{ii}^{pq} = 0 \ (p \ne q, \ 1 \le i \le r) \ \right\}.$$

Other conditions appear in Sections 6.2.1 and 6.2.2 but their precise forms are not relevant to the discussions there.

6.2.1 Conic relaxations in the space $\mathbb{S}^{1+n}$

We now compare the CPP, DNN, and SDP relaxations of QAP (19) in $\mathbb{S}^{1+n}$. The CPP reformulation of Arima-Kim-Kojima-Toh [3, 5] applied to QAP (19) is $\mathrm{P}(\mathbb{C}_{\mathrm{PP}}^{1+n}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$, while Burer's CPP reformulation [9] applied to QAP (19) is $\mathrm{P}(\mathbb{C}_{\mathrm{PP}}^{1+n}, L_{\mathrm{E3}} \cap L_{\mathrm{Z}})$. Replacing the CPP cone $\mathbb{C}_{\mathrm{PP}}^{1+n}$ with the DNN cone $\mathbb{D}_{\mathrm{NN}}^{1+n}$, DNN relaxations $\mathrm{P}(\mathbb{D}_{\mathrm{NN}}^{1+n}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$ and $\mathrm{P}(\mathbb{D}_{\mathrm{NN}}^{1+n}, L_{\mathrm{E3}} \cap L_{\mathrm{Z}})$ are obtained respectively, and they are equivalent by Theorem 3.1. Replacing $\mathbb{D}_{\mathrm{NN}}^{1+n}$ with $\mathbb{S}_{+}^{1+n}$ leads to SDP relaxations $\mathrm{P}(\mathbb{S}_{+}^{1+n}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$ and $\mathrm{P}(\mathbb{S}_{+}^{1+n}, L_{\mathrm{E3}} \cap L_{\mathrm{Z}})$, respectively. By Theorem 3.1, the lower bound $\zeta_p(\mathbb{S}_{+}^{1+n}, L_{\mathrm{E3}} \cap L_{\mathrm{Z}})$ obtained by the second SDP relaxation for the optimal value of QAP (19) can not exceed the one $\zeta_p(\mathbb{S}_{+}^{1+n}, L_{\mathrm{E1}} \cap L_{\mathrm{C}})$ provided by the first SDP relaxation.

$\mathrm{P}(\mathbb{K}^{1+n}, L_{\mathrm{E2}} \cap L_{\mathrm{C}})$ with $\mathbb{K}^{1+n} = \mathbb{D}_{\mathrm{NN}}^{1+n}$ and $\mathbb{K}^{1+n} = \mathbb{S}_{+}^{1+n}$ also provides DNN and SDP relaxations of QAP (19). These two relaxations coincide with the relaxations $\mathrm{QAP}_{\mathrm{R}_3}$ and $\mathrm{QAP}_{\mathrm{R}_2}$ presented in [23] except that for the latter two relaxations, the 0-1 condition (Z) is imposed in addition to the complementarity condition (C). But (Z) is redundant by Lemma 3.2. On the other hand, $\mathrm{P}(\mathbb{S}_{+}^{1+n}, L_{\mathrm{E2}} \cap L_{\mathrm{Z}})$ coincides with the relaxations $\mathrm{QAP}_{\mathrm{R}_0}$ presented in [23]. See also Section 4 of [22].

In general, all CPP relaxations $\mathrm{P}(\mathbb{C}_{\mathrm{PP}}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{b}})$ with $L_{\mathrm{a}} \in \{L_{\mathrm{E1}}, L_{\mathrm{E2}}, L_{\mathrm{E3}}\}$ and $L_{\mathrm{b}} \in \{L_{\mathrm{Z}}, L_{\mathrm{C}}\}$ are equivalent by Theorem 3.1. Thus, they all provide CPP reformulations of QAP (19). By Theorem 3.1, all DNN relaxations $\mathrm{P}(\mathbb{D}_{\mathrm{NN}}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{b}})$ with $L_{\mathrm{a}} \in \{L_{\mathrm{E1}}, L_{\mathrm{E2}}, L_{\mathrm{E3}}\}$ and $L_{\mathrm{b}} \in \{L_{\mathrm{Z}}, L_{\mathrm{C}}\}$ are equivalent. But only the inequality $\zeta_p(\mathbb{S}_{+}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{Z}}) \le \zeta_p(\mathbb{S}_{+}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{C}})$ is guaranteed for any choice of $L_{\mathrm{a}} \in \{L_{\mathrm{E1}}, L_{\mathrm{E2}}, L_{\mathrm{E3}}\}$.

6.2.2 Conic relaxations in the space $\mathbb{S}^n$

We examine and compare the CPP, DNN and SDP relaxations proposed by [2, 22, 28, 23] in view of the discussion in Sections 3.2 and 3.3. Let $\boldsymbol{h} = \left(\sum_{p=1}^{2r} \boldsymbol{e}_{J_p} \right) / 2r$. As both $\{C_p : 1 \le p \le r\}$ and $\{R_i : 1 \le i \le r\}$ form partitions of the index set $\{1, \ldots, n\}$, we see that $\boldsymbol{h} = \left(\sum_{p=1}^r \boldsymbol{e}_{C_p}\right)/r = \left(\sum_{i=1}^r \boldsymbol{e}_{R_i}\right)/r = \boldsymbol{e}/r$. Thus, $\boldsymbol{h}\boldsymbol{h}^T = \boldsymbol{E}/r^2$, where $\boldsymbol{E}$ denotes the $n \times n$ matrix of 1's. Let $\mathbb{K}^n \in \{\mathbb{S}^n_+, \mathbb{D}^n_{\rm NN}, \mathbb{C}^n_{\rm PP}\}$. Then, $\mathrm{P}'(\mathbb{K}^n, L'_{\rm a} \cap L'_{\rm b})$ with each pair of $L'_{\rm a} \in \{L'_{\rm E1}, L'_{\rm E2}, L'_{\rm E3}, L''_{\rm E3}\}$ and $L'_{\rm b} \in \{L'_{\rm Z}, L'_{\rm C}\}$ entails an SDP, DNN or CPP relaxation of QAP (19) in the space $\mathbb{S}^n$, respectively. Our main focus is on $\mathrm{P}'(\mathbb{K}^n, L''_{\rm E3} \cap L'_{\rm C})$ among the relaxations, which will be related to some existing conic relaxations [2, 22, 28, 23] of QAPs. First, we rewrite Conditions (E3)" and (C)' as follows:

$$\textbf{(E3)"} \ \ \boldsymbol{Y} \in L''_{\rm E3} \equiv \left\{ \boldsymbol{Y} \in \mathbb{S}^n : \begin{array}{l} \boldsymbol{E}_{C_p} \bullet \boldsymbol{Y} = (\boldsymbol{E}/r^2) \bullet \boldsymbol{Y} \ (1 \leq p \leq r), \\ \boldsymbol{E}_{R_i} \bullet \boldsymbol{Y} = (\boldsymbol{E}/r^2) \bullet \boldsymbol{Y} \ (1 \leq i \leq r) \end{array} \right\} \text{ and } \boldsymbol{E} \bullet \boldsymbol{Y} = r^2.$$

$$\textbf{(C)'} \ \ \boldsymbol{Y} \in L'_{\rm C} \equiv \left\{ \boldsymbol{Y} \in \mathbb{S}^n : \ Y^{pp}_{ij} = 0 \ (i \neq j, \ 1 \leq p \leq r), \ Y^{pq}_{ii} = 0 \ (p \neq q, \ 1 \leq i \leq r) \ \right\}.$$

The permutation matrix $\boldsymbol{X}$ is characterized by the conditions $\boldsymbol{X} \geq \boldsymbol{O}$ and $\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}$. Based on this characterization, Povh and Rendl [22] reformulated QAP (18) by adding the redundant constraints $\boldsymbol{X}\boldsymbol{X}^T = \boldsymbol{I}$ and $\left(\sum_{i=1}^r \sum_{p=1}^r X_{ip}\right)^2 = r^2$ as follows:

$$\zeta^*_{\rm QAP} = \min_X \left\{ \boldsymbol{X} \bullet (\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^T) : \ \boldsymbol{X} \geq \boldsymbol{O}, \ \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}, \ \boldsymbol{X}\boldsymbol{X}^T = \boldsymbol{I}, \ \left(\sum_{i=1}^r \sum_{p=1}^r X_{ip} \right)^2 = r^2 \ \right\},$$

and derived its conic relaxations:

$$\zeta_{\rm PR}(\mathbb{K}^n) = \min \left\{ (\boldsymbol{B} \otimes \boldsymbol{A}) \bullet \boldsymbol{Y} : \begin{array}{ll} \boldsymbol{Y} \in \mathbb{K}^n, & \boldsymbol{I} \bullet \boldsymbol{Y}^{pq} = \delta_{pq} \ (1 \leq p, q \leq r), \\ \sum_{p=1}^r \boldsymbol{Y}^{pp} = \boldsymbol{I}, & \boldsymbol{E} \bullet \boldsymbol{Y} = r^2 \end{array} \right\}. \tag{20}$$

Here $\mathbb{K}^n = \mathbb{C}^n_{\rm PP}$, $\mathbb{D}^n_{\rm NN}$ or $\mathbb{S}^n_+$ with $n = r^2$, δ_{pq} denotes the Kronecker delta such that $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ otherwise. We note that $\boldsymbol{I} \bullet \boldsymbol{Y}^{pq} = \delta_{pq}$ $(1 \leq p, q \leq r)$ and $\sum_{p=1}^r \boldsymbol{Y}^{pp} = \boldsymbol{I}$ are the liftings of the equalities $\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}$ and $\boldsymbol{X}\boldsymbol{X}^T = \boldsymbol{I}$ in the variable matrix $\boldsymbol{X} \in \mathbb{R}^{r \times r}$ to the equalities in the variable matrix $\boldsymbol{Y} \in \mathbb{S}^{n \times n}$, respectively. The CPP relaxation $\mathrm{QAP}_{\rm CP}$, DNN relaxation $\mathrm{QAP}_{\mathcal{K}^{0*}_n}$ and SDP relaxation $\mathrm{QAP}_{\rm AW+}$, which are all from Povh and Rendl [22], are obtained by setting $\mathbb{K}^n = \mathbb{C}^n_{\rm PP}$, $\mathbb{D}^n_{\rm NN}$ and $\mathbb{S}^n_+$, respectively. In particular, Povh and Rendl showed that $\mathrm{QAP}_{\rm CP}$ provided a CPP reformulation of QAP (18), *i.e.*, $\zeta_{\rm PR}(\mathbb{C}^n_{\rm PP}) = \zeta^*_{\rm QAP}$. The SDP relaxation $\mathrm{QAP}_{\rm AW+}$ corresponds to a slight variation of the Anstreicher-Wolkowicz relaxation [2].

As a stronger SDP relaxation than $\mathrm{QAP}_{\rm AW+}$, Povh and Rendl in [22] also presented the SDP relaxation $\mathrm{QAP}_{\rm ZRKW1}$, which was shown to be equivalent to the 'gangster-model' from [28]. The SDP relaxation $\mathrm{QAP}_{\rm ZRKW1}$ is naturally extended to a general conic relaxation of QAP by replacing the SDP cone $\mathbb{S}^n_+$ with a closed convex cone $\mathbb{K}^n \supset \Gamma^n$.

$$\zeta_{\rm ZRKW1}(\mathbb{K}^n) \ = \ \min_{\boldsymbol{Y}} \left\{ (\boldsymbol{B} \otimes \boldsymbol{A}) \bullet \boldsymbol{Y} : \begin{array}{ll} \boldsymbol{Y} \in \mathbb{K}^n, & \boldsymbol{I} \bullet \boldsymbol{Y}^{pp} = 1 \ (1 \leq p \leq n), \\ \sum_{q=1}^r Y^{qq}_{ii} = 1 \ (1 \leq i \leq n), & \\ \boldsymbol{Y} \in L'_{\rm C}, & \boldsymbol{E} \bullet \boldsymbol{Y} = r^2 \end{array} \right\} \tag{21}$$

We will relate $\mathrm{QAP}_{\rm CP}$, $\mathrm{QAP}_{\mathcal{K}^{0*}_n}$, $\mathrm{QAP}_{\rm AW+}$ and COP (21) to $\mathrm{P}'(\mathbb{K}^n, L''_{\rm E3} \cap L'_{\rm C})$.

Lemma 6.1. *Let $\mathbb{K}^n \subset \mathbb{S}^n_+$. Suppose that $\boldsymbol{Y} \in L'_{\mathrm{C}}$ holds. Then*

$$\begin{aligned}\sum_{i=1}^{r}\sum_{j\neq i} Y_{ij}^{pp} &= 0 \ (1 \leq p \leq r), \ \sum_{p=1}^{r}\sum_{q\neq p} Y_{ii}^{pq} = 0 \ (1 \leq i \leq r), \\ \boldsymbol{E}_{C_p} \bullet \boldsymbol{Y} &= \boldsymbol{I} \bullet \boldsymbol{Y}^{pp} \ (1 \leq p \leq r), \ \boldsymbol{E}_{R_i} \bullet \boldsymbol{Y} = \textstyle\sum_{p=1}^{r} Y_{ii}^{pp} \ (1 \leq i \leq r). \end{aligned} \tag{22}$$

Proof. $\boldsymbol{Z} \in L'_{\mathrm{C}}$ implies the first two equalities. The last two equalities follow from them. □

Lemma 6.2. *Let $\boldsymbol{Y} \in \mathbb{N}^n$. Suppose that $\boldsymbol{I} \bullet \boldsymbol{Y}^{pq} = \delta_{pq}$ $(1 \leq p, q \leq r)$ and $\sum_{p=1}^r \boldsymbol{Y}^{pp} = \boldsymbol{I}$. Then $\boldsymbol{Y} \in L'_{\mathrm{C}}$ holds.*

Proof. Let $p \neq q$. Then $\sum_{i=1}^r Y_{ii}^{pq} = \boldsymbol{I} \bullet \boldsymbol{Y}^{pq} = 0$. From $\sum_{p=1}^r \boldsymbol{Y}^{pp} = \boldsymbol{I}$, we see that $\sum_{p=1}^r Y_{ij}^{pp} = 0$ $(i \neq j)$. Since $\boldsymbol{Y} \in \mathbb{N}^n$, these identities imply $\boldsymbol{Z} \in L'_{\mathrm{C}}$. □

Theorem 6.1. *Let $\mathbb{K}^n \subset \mathbb{S}^n_+$.*

(i) $\boldsymbol{Y} \in \mathbb{K}^n$ is a feasible solution of COP (21) *iff it satisfies* (E3)" *and* (C)'.

(ii) If $\boldsymbol{Y} \in \mathbb{K}^n$ satisfies (E3)" *and* (C)', *then it is a feasible solution of COP* (20).

(iii) Assume that $\mathbb{K}^n \subset \mathbb{D}^n_{\mathrm{NN}}$. Then $\boldsymbol{Y} \in \mathbb{K}^n$ is a feasible solution of COP (20) *iff it satisfies* (E3)" *and* (C)'.

Proof. (i) By Lemma 6.1, (22) holds under (C)' which is assumed both in the 'if' and 'only if' parts. Hence assertion (i) follows.

(ii) Assume that $\boldsymbol{Y} \in \mathbb{K}^n$ satisfies (E3)" and (C)'. Then $\boldsymbol{E} \bullet \boldsymbol{Y} = r^2$. By Lemma 6.1 and (C)', we see that

$$\begin{aligned}\boldsymbol{I} \bullet \boldsymbol{Y}^{pq} &= \begin{cases} \boldsymbol{E}_{C_p} \bullet \boldsymbol{Y} = 1 & \text{if } p = q, \\ \sum_{i=1}^r Y_{ii}^{pq} = 0 & \text{otherwise,} \end{cases} \quad \textit{i.e.}, \ \boldsymbol{I} \bullet \boldsymbol{Y}^{pq} = \delta_{pq} \ (1 \leq p, q \leq r), \\ \textstyle\sum_{p=1}^r Y_{ij}^{pp} &= \begin{cases} \boldsymbol{E}_{R_i} \bullet \boldsymbol{Y} = 1 & \text{if } i = j, \\ \sum_{p=1}^r Y_{ij}^{pp} = 0 & \text{otherwise} \end{cases} \quad \textit{i.e.}, \ \textstyle\sum_{p=1}^r \boldsymbol{Y}^{pp} = \boldsymbol{I}.\end{aligned}$$

Therefore, $\boldsymbol{Y} \in \mathbb{K}^n$ is a feasible solution of COP (20).

(iii) Since the 'if' part is proved in (ii), it suffices to show the 'only if' part. Assume that $\boldsymbol{Y} \in \mathbb{K}^n$ is a feasible solution of COP (20) with $\mathbb{K}^n \subset \mathbb{D}^n_{\mathrm{NN}}$. By Lemma 6.2, (C)' holds. Hence, (22) holds by Lemma 6.1. The equality $\boldsymbol{E} \bullet \boldsymbol{Y} = r^2$ follows directly from the assumption. □

Corollary 6.1. *Suppose that $\Gamma^n \subset \mathbb{K}^n \subset \mathbb{S}^n_+$. Then*

$$\zeta_{\mathrm{PR}}(\mathbb{K}^n) \leq \zeta_{\mathrm{ZRKW1}}(\mathbb{K}^n) = \zeta'(\mathbb{K}^n, L''_{E3}, L'_C) \leq \zeta^*_{QAP}.$$

If, in addition, $\mathbb{K}^n \subset \mathbb{D}^n_{\mathrm{NN}}$, then $\zeta_{\mathrm{PR}}(\mathbb{K}^n) = \zeta_{\mathrm{ZRKW1}}(\mathbb{K}^n)$ holds. If $\mathbb{K}^n = \mathbb{C}^n_{\mathrm{PP}}$, then $\zeta_{\mathrm{PR}}(\mathbb{K}^n) = \zeta^*_{QAP}$; hence $\zeta_{\mathrm{PR}}(\mathbb{K}^n) = \zeta_{\mathrm{ZRKW1}}(\mathbb{K}^n) = \zeta'(\mathbb{K}^n, L''_{E3}, L'_C) = \zeta^*_{QAP}$.

Proof. The first and second assertions follow from Theorem 6.1, and the last from [22, Corollary 4]. □

7 Numerical results on SDP and DNN relaxations of QAP

In Section 6.2, the QAP has been discussed as a special case of QOP (2), and some of existing conic relaxations have been related to $\mathrm{P}(\mathbb{K}^{1+n}, L_{\mathrm{a}} \cap L_{\mathrm{b}})$ with $(L_{\mathrm{a}}, L_{\mathrm{b}}) \in \{L_{\mathrm{E1}}, L_{\mathrm{E2}}, L_{\mathrm{E3}}\} \times \{L_{\mathrm{C}}, L_{\mathrm{Z}}\}$ and $\mathrm{P}'(\mathbb{K}^{1+n}, L'_{\mathrm{a}} \cap L'_{\mathrm{b}})$ with $(L'_{\mathrm{a}}, L'_{\mathrm{b}}) \in \{L'_{\mathrm{E1}}, L'_{\mathrm{E2}}, L''_{\mathrm{E3}}\} \times \{L'_{\mathrm{C}}, L'_{\mathrm{Z}}\}$. The purpose of the numerical experiments here is to compare the numerical results obtained by solving those COPs with SDPT3 [26], SDPNAL+ [27] and the bisection and projection method (BP) [6, 16, 15]. Specifically, we want to see whether the numerical results are in line with the theoretical results on the following issues:

(a) For every $L_{\mathrm{a}} \in \{L_{\mathrm{E1sum}}, L_{\mathrm{E1}}, L_{\mathrm{E2}}, L_{\mathrm{E3}}\}$ and $L'_{\mathrm{a}} \in \{L'_{\mathrm{E1sum}}, L'_{\mathrm{E1}}, L'_{\mathrm{E2}}, L''_{\mathrm{E3}}\}$, the equalities and the inequalities

$$\begin{aligned} \zeta_p(\mathbb{S}^{1+n}_+, L_{\mathrm{a}} \cap L_{\mathrm{Z}}) = \zeta_p(\mathbb{S}^n_+, L'_{\mathrm{a}} \cap L'_{\mathrm{Z}}) &\leq \zeta_p(\mathbb{S}^{1+n}_+, L_{\mathrm{a}} \cap L_{\mathrm{C}}) = \zeta_p(\mathbb{S}^n_+, L'_{\mathrm{a}} \cap L'_{\mathrm{C}}), \\ &\leq \zeta_p(\mathbb{D}^{1+n}_{\mathrm{NN}}, L_{\mathrm{a}} \cap L_{\mathrm{Z}}) = \zeta_p(\mathbb{D}^n_{\mathrm{NN}}, L'_{\mathrm{a}} \cap L'_{\mathrm{Z}}) \\ &= \zeta_p(\mathbb{D}^{1+n}_{\mathrm{NN}}, L_{\mathrm{a}} \cap L_{\mathrm{C}}) = \zeta_p(\mathbb{D}^n_{\mathrm{NN}}, L'_{\mathrm{a}} \cap L'_{\mathrm{C}}) \end{aligned}$$

hold. (Theorem 3.1 and the discussions in Section 3.3). Recall that L_{E1sum} has been defined by (10), and that its simplification L'_{E1sum} is given by $\mathbf{\Phi}^{-1}(L_{\mathrm{E1sum}})$.

(b) $\mathrm{P}(\mathbb{D}^{1+n}_{\mathrm{NN}}, L_{\mathrm{E2}} \cap L_{\mathrm{b}})$ with $L_{\mathrm{b}} \in \{L_{\mathrm{C}}, L_{\mathrm{Z}}\}$ and $\mathrm{P}'(\mathbb{D}^n_{\mathrm{NN}}, L'_{\mathrm{E2}} \cap L'_{\mathrm{b}})$ with $L'_{\mathrm{b}} \in \{L'_{\mathrm{C}}, L'_{\mathrm{Z}}\}$ are more time-consuming than the other COPs. (Section 4.1).

(c) The efficiency and effectiveness of solving a COP depend on numerical methods.

For (c), we experimented with SDPNAL+ and BP. The solver SDPNAL+ can be applied to solve all COPs in (a), although its fast local convergence and stable execution are considerably affected by the existence of interior feasible solutions and nondegeneracy. The BP method, on the other hand, is a first-order method which is designed to be robust for ill-conditioned problems having empty primal or dual interior feasible regions. However, it works on only $\mathrm{P}(\mathbb{K}^{1+n}, L_{\mathrm{E1sum}} \cap L_{\mathrm{b}})$ ($L_{\mathrm{b}} \in \{L_{\mathrm{C}}, L_{\mathrm{Z}}\}$) and $\mathrm{P}'(\mathbb{K}^n, L'_{\mathrm{E1sum}} \cap L'_{\mathrm{C}})$ (see [6, 16, 15] for more details). We should note that DNN relaxations of the tested QAP instances are too large for SDPT3 to solve within reasonable CPU time.

QAP instances for the experiments are obtained from QAPLIB [12]. The default parameters were used for SDPT3. For SDPNAL+, the parameter 'tol' was set to 10^{-6} and 'stopoptions' to 2 so that the solver continues to run even if it encounters some stagnations. For BP, the stopping criterion for the length of the target search interval was set to $\delta = 0.1$. SDPNAL+ and BP were terminated in 10000 seconds or 20000 iterations even if their stopping criteria were not satisfied. Linearly dependent equality constraints were removed from each COP before SDPT3 and SDPNAL+ were applied. All the computations were performed in MATLAB on a Mac Pro with Intel Xeon E5 CPU (2.7 GHZ) and 64 GB memory.

Table 2 shows the numerical results on two small-sized QAP instances from QAPLIB [12]. Note that we used the procedure in [6] to modify all computed bounds to be valid lower bounds. It is clear that the results in Table 2 are consistent with the inequalities in (a). For the equalities in (a), we observe that the lower bounds obtained from DNN relaxations

Table 2: Approximate optimal values of $\mathrm{P}(\mathbb{K}^{1+n}, L_{\rm a} \cap L_{\rm b})$ and $\mathrm{P}'(\mathbb{K}^n, L'_{\rm a} \cap L'_{\rm b})$ for small size QAP instances, tai10 and chr12a.

tai10a

SDP solved by SDPT3			DNN solved by SDPNAL+		
$\mathbb{K}^{1+n} = \mathbb{S}^{1+n}_+$	$L_{\rm b}$		$\mathbb{K}^{1+n} = \mathbb{D}^{1+n}_{\rm NN}$	$L_{\rm b}$	
$L_{\rm a}$	$L_{\rm Z}$	$L_{\rm C}$	$L_{\rm a}$	$L_{\rm Z}$	$L_{\rm C}$
$L_{\rm E1sum}$	32224.7	128738.6	$L_{\rm E1sum}$	135027.9	134972.8
$L_{\rm E1}$	32224.8	128701.6	$L_{\rm E1}$	135022.7	134955.1
$L_{\rm E2}$	32227.4	128739.9	$L_{\rm E2}$	135028.0	135024.8
$L_{\rm E3}$	32225.1	128734.5	$L_{\rm E3}$	135028.0	135028.0
$\mathbb{K}^n = \mathbb{S}^n_+$	$L'_{\rm b}$		$\mathbb{K}^n = \mathbb{D}^n_{\rm NN}$	$L'_{\rm b}$	
$L'_{\rm a}$	$L'_{\rm Z}$	$L'_{\rm C}$	$L'_{\rm a}$	$L'_{\rm Z}$	$L'_{\rm C}$
$L'_{\rm E1sum}$	32226.4	128739.3	$L'_{\rm E1sum}$	135028.0	134996.0
$L'_{\rm E1}$	32227.1	128738.9	$L'_{\rm E1}$	135027.7	134994.0
$L'_{\rm E2}$	32227.5	128739.9	$L'_{\rm E2}$	135023.4	135022.9
$L''_{\rm E3}$	32226.8	128722.4	$L''_{\rm E3}$	135028.0	135024.1

chr12a

$\mathbb{K}^{1+n} = \mathbb{S}^{1+n}_+$	$L_{\rm b}$		$\mathbb{K}^{1+n} = \mathbb{D}^{1+n}_{\rm NN}$	$L_{\rm b}$	
$L_{\rm a}$	$L_{\rm Z}$	$L_{\rm C}$	$L_{\rm a}$	$L_{\rm Z}$	$L_{\rm C}$
$L_{\rm E1sum}$	-103846.8	-4840.5	$L_{\rm E1sum}$	9551.4	9544.5
$L_{\rm E1}$	-103849.2	-4842.2	$L_{\rm E1}$	9551.6	9542.9
$L_{\rm E2}$	-103846.8	-4840.6	$L_{\rm E2}$	9552.0	9552.0
$L_{\rm E3}$	-103847.9	-4840.6	$L_{\rm E3}$	9552.0	9550.4
$\mathbb{K}^n = \mathbb{S}^n_+$	$L'_{\rm b}$		$\mathbb{K}^n = \mathbb{D}^n_{\rm NN}$	$L'_{\rm b}$	
$L'_{\rm a}$	$L'_{\rm Z}$	$L'_{\rm C}$	$L'_{\rm a}$	$L'_{\rm Z}$	$L'_{\rm C}$
$L'_{\rm E1sum}$	-103846.7	-4840.6	$L'_{\rm E1sum}$	9552.0	9546.7
$L'_{\rm E1}$	-103846.9	-4840.9	$L'_{\rm E1}$	9548.6	9542.3
$L'_{\rm E2}$	-103846.7	-4840.5	$L'_{\rm E2}$	9552.0	9552.0
$L''_{\rm E3}$	-103846.9	-4840.6	$L''_{\rm E3}$	9552.0	9548.5

with $(L_{\rm a}, L_{\rm b}) \in \{L_{\rm E1sum}, L_{\rm E1}\} \times \{L_{\rm C}\}$ are smaller than the ones from DNN relaxations with other $(L_{\rm a}, L_{\rm b})$, which suggests that the first pair of DNN relaxations do not work well with SDPNAL+ as they are ill-conditioned in the sense that we shall explain next. By Lemma 3.1, we know that $L^{\perp}_{\rm E1sum} \subset L^{\perp}_{\rm E1} \subset L^{\perp}_{\rm E3} \subset L^{\perp}_{\rm E2}$. In particular, $\dim(L^{\perp}_{\rm E1sum}) = 1$. Recall that $n = r^2$. The small value of $\dim\big((L_{\rm E1sum} \cap L_{\rm C})^{\perp}\big)$, which is at most $1 + r^2(r-1)$, relative to the dimension $(r^2+1)r^2/2$ of the linear space $\mathbb{S}^{1+n}$ implies that the dual nondegeneracy condition (13) is very likely to fail. In addition, the dual problems are also likely not to have interior feasible solutions due to the small dimension of $(L_{\rm E1sum} \cap L_{\rm C})^{\perp}$. We also know that the primal problems have no interior feasible solutions (see [16, Lemma 4]). All the issues just discussed has a negative impact on the numerical performance of SDPNAL+. The same observation can be made on the simplified DNN relaxations with $(L'_{\rm a}, L'_{\rm b}) \in \{L'_{\rm E1sum}, L'_{\rm E1}\} \times \{L'_{\rm C}\}$.

Table 2 displays the optimal values of 16 different DNN relaxations by SDPNAL+ for tai10 and chr12a. Although we solved the same set of 16 DNN relaxations for many small and large-scale QAP instances, detailed results on each DNN relaxation are omitted here to save space. We refer the reader to [13] for the results in detail. For a summary of the results, we introduce the following symbols and notation. Let $\hat{\zeta}(L_{\rm a} \cap L_{\rm b})$ and $\hat{\zeta}'(L'_{\rm a} \cap L'_{\rm b})$ denote the approximate optimal values of $\mathrm{P}(\mathbb{D}^{n+1}_{\rm NN}, L_{\rm a} \cap L_{\rm b})$ and $\mathrm{P}'(\mathbb{D}^n_{\rm NN}, L'_{\rm a} \cap L'_{\rm b})$ solved by SDPNAL+ and BP, and let $\tau(L_{\rm a} \cap L_{\rm b})$ and $\tau'(L'_{\rm a} \cap L'_{\rm b})$ be the corresponding execution time of SDPNAL+ and BP. These values are converted to differences in relative lower bounds with respect to the maximum lower bound, and the ratio of the execution time to the minimum execution time. More precisely, we computed

$$\hat{\zeta}_{\max} \equiv \max_{L_{\rm a}, L_{\rm b}, L'_{\rm a}, L'_{\rm b}, \text{SDPNAL+,BP}} \{\hat{\zeta}(L_{\rm a} \cap L_{\rm b}),\ \hat{\zeta}'(L'_{\rm a} \cap L'_{\rm b})\} \text{ (the maximum lower bound)},$$

$$\tau_{\min} \equiv \min_{L_{\rm a}, L_{\rm b}, L'_{\rm a}, L'_{\rm b}, \text{SDPNAL+,BP}} \{\tau(L_{\rm a} \cap L_{\rm b}),\ \tau'(L'_{\rm a} \cap L'_{\rm b})\} \text{ (the minimum execution time)},$$

$$\eta(L_{\rm a} \cap L_{\rm b}) \equiv (\hat{\zeta}_{\max} - \hat{\zeta}(L_{\rm a} \cap L_{\rm b}))/\hat{\zeta}_{\max}, \quad \eta'(L'_{\rm a} \cap L'_{\rm b}) \equiv (\hat{\zeta}_{\max} - \hat{\zeta}'(L'_{\rm a} \cap L'_{\rm b}))/\hat{\zeta}_{\max},$$

$$\sigma(L_{\rm a} \cap L_{\rm b}) \equiv \tau(L_{\rm a} \cap L_{\rm b})/\tau_{\min}, \quad \sigma'(L'_{\rm a} \cap, L'_{\rm b}) \equiv \tau'(L'_{\rm a} \cap L'_{\rm b})/\tau_{\min}.$$

Then, the computed means are denoted by

$$\eta_{\rm m}(L_{\rm a} \cap L_{\rm b}),\ \eta'_{\rm m}(L'_{\rm a} \cap L'_{\rm b}),\ \sigma_{\rm m}(L_{\rm a} \cap L_{\rm b}),\ \sigma'_{\rm m}(L'_{\rm a} \cap L'_{\rm b}),$$

and the worst-case values by

$$\eta_{\rm w}(L_{\rm a} \cap L_{\rm b}),\ \eta'_{\rm w}(L'_{\rm a} \cap L'_{\rm b}),\ \sigma_{\rm w}(L_{\rm a} \cap L_{\rm b}),\ \sigma'_{\rm w}(L'_{\rm a} \cap L'_{\rm b})$$

over the numerical results from a set of QAP instances for each $L_{\rm a}, L_{\rm b}, L'_{\rm a}, L'_{\rm b}$. We also compute their means and worst-case values over all of $L_{\rm a}, L_{\rm b}, L'_{\rm a}$, and $L'_{\rm b}$, respectively (the rows with 'all $L_{\rm a}$' and 'all $L'_{\rm a}$', the columns with 'all $L_{\rm b}$' and 'all $L'_{\rm b}$' in Table 3.)

Table 3 shows a summary of the numerical results on three sets of QAP instances. More precisely, QAP instances of small to large size from QAPLIB [12] are categorized into Set 1 through Set 3. Set 1, for instance, includes QAPs such as had12, nug12, rou12, scr12, chr12a, chr12b, chr12c, tai10a, and tai10b. The mean and the worst-case value were computed over the numerical results of each set of QAPs obtained by SDPNAL+ and BP.

First, we focus on the numerical results by SDPNAL+. For Set 1 of small-sized QAPs, the values of $\eta_{\rm m}$ and $\eta_{\rm w}$ are smaller as shown in the rows $L_{\rm E2}$ and $L'_{\rm E2}$, which means that lower bounds of higher quality are obtained for these particular cases in the rows $L_{\rm E2}$ and $L'_{\rm E2}$. The large number $m + mn$ (which equals to $\dim(L_{\rm E2}^{\perp})$) of linear equalities in $L_{\rm E2}$ may have contributed to the positive result as dual interior feasible solutions are likely to exist and dual nondegeneracy is more likely to hold as discussed in Section 4. But $\sigma_m(\sigma_w)$ in the rows $L_{\rm E2}$ and $L'_{\rm E2}$ are much larger than the ones in the other rows, which indicates that solving DNN relaxations with $L_{\rm a} = L_{\rm E2}$ and $L_{\rm b} = L'_{\rm E2}$ by SDPNAL+ are much more expensive than the other cases as mentioned in (b). We omit these two time-consuming cases for the sets of larger QAPs.

The DNN relaxations with $L_{\rm a} \in \{L_{\rm E1sum}, L_{\rm E1}\}$ for Sets 1, 2 and 3 of QAPs, which have only 1 or m linear equalities, on the other hands, are inferior to the others with regard to lower bounds and execution time. The performance could have been affected by the lack of dual interior feasible solutions and/or the failure of dual nondegeneracy as a result of the small dimension of $L_{\rm a}^{\perp}$. The DNN relaxations with $L_{\rm a} = L_{\rm E3}$, which have intermediate numbers of linear equalities, show relatively good results for Sets 1, 2 and 3 of QAPs as far as lower bounds and execution time are concerned. The similar observation can be made for $L'_{\rm a} \in \{L'_{\rm E1sum}, L'_{\rm E1}\}$ and $L'_{\rm a} = L'_{\rm E3}$ cases.

When the DNN relaxation with $L_{\rm b} = L_{\rm C}$ and $L'_{\rm b} = L'_{\rm C}$ were solved by SDPNAL+, the mean and worst lower bounds (η_m and η_w) obtained were significantly inferior to the other cases of $L_{\rm b} = L_{\rm Z}$ and $L'_{\rm b} = L'_{\rm Z}$, especially when they are combined with $L_{\rm a} \in \{L_{\rm E1sum}, L_{\rm E1}\}$ and $L'_{\rm a} \in \{L'_{\rm E1sum}, L'_{\rm E1}\}$. This shows that SDPNAL+ works more effectively and stably on the DNN relaxations with $L_{\rm b} = L_{\rm Z}$ and $L'_{\rm b} = L'_{\rm Z}$.

In contrast to SDPNAL+, BP was able to stably solve the DNN relaxations with $L_{\rm b} = L_{\rm C}$ and $L'_{\rm b} = L'_{\rm C}$. In particular, BP has superior performance in solving the DNN relaxations with $L'_{\rm b} = L'_{\rm C}$ in each set of QAP instances. This demonstrates the robustness of BP for solving degenerate DNN problems, and the effectiveness of the simplification technique when it is combined with BP. However, BP solved the COPs with $L_{\rm b} = L_{\rm Z}$ less efficiently. Since the stability of BP depends on the stability of the numerical algorithm used to check the dual feasibility of a given matrix, simple Condition (C) which imposes constraints on each element independently would be numerically more preferable than Condition (Z).

In all cases, the simplification technique did not shorten the execution time much as expected in Section 3.3. This is because the simplification with $\boldsymbol{h} = \boldsymbol{e}/r$ destroyed the sparsity of the original DNN relaxations. We note that all the coefficient matrices involved in Conditions (E1) through (E3) are sparse while many of those in the simplified Conditions (E1)' through (E3)" (including $\boldsymbol{h}\boldsymbol{h}^T$) are fully dense. With regard to the issue (c) mentioned at the beginning of this section, we can conclude from the numerical results in Tables 2 and 3 that the DNN relaxation with $(L_{\rm a}, L_{\rm b}) = (L_{\rm E3}, L_{\rm Z})$ and the one $(L'_{\rm a}, L'_{\rm b}) = (L'_{\rm E3}, L'_{\rm Z})$ give better results when solved by SDPNAL+, while the DNN relaxation with $(L'_{\rm a}, L'_{\rm b}) = (L'_{\rm E1sum}, L'_{\rm C})$ is better solved with BP.

8 Concluding remarks

Many conic relaxations proposed for a combinatorial optimization problem are equivalent in the quality of the optimal values they provide. However, they differ in the size of the matrix variable, the number of linear equality and inequality constraints, the existence of an interior feasible solution, and the primal/dual degeneracy, which are crucial issues for the performance of a numerical method. We have proved the equivalences and differences in the SDP, DNN, and CPP relaxations of combinatorial optimization problems by examining several ways of representing the combinatorial condition. This approach has revealed the connections among the existing relaxations, and also provided new conic relaxations for the QAP as shown in Table 1. We have tested the theoretical results with QAP instances and obtained consistent numerical results using SDPT3, SDPNAL+ and BP.

For the combinatorial optimization problems that are not dealt with in this paper, for instance, the quadratic multiple knapsack problem, maximum stable set problem and graph partitioning problem, the same approach presented in this paper can be applied to show the equivalence and difference in their conic relaxations.

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Table 3: The mean and worst-case values of relative lower bound differences η and ratios of the execution time σ. The subscript 'm' stands for the mean and 'w' worst-case values.

Set 1 of small size QAP instances (had12, nug12, rou12, scr12, chr12a, chr12b, chr12c, tai10a, tai10b)

		L_Z		L_C		all L_b	
		$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$	$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$	$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$
BP	L_{E1sum}	3.99e-3 (1.37e-2)	3.33e0 (5.30e0)	3.50e-5 (1.34e-4)	2.06e0 (3.82e0)	2.01e-3 (1.37e-2)	2.70e0 (5.30e0)
SDPNAL+	L_{E1sum}	4.72e-5 (2.30e-4)	5.75e0 (1.17e1)	6.04e-4 (1.16e-3)	1.17e1 (2.09e1)	3.26e-4 (1.16e-3)	8.73e0 (2.09e1)
	L_{E1}	6.46e-5 (2.17e-4)	4.89e0 (8.43e0)	6.99e-4 (1.01e-3)	1.27e1 (2.62e1)	3.82e-4 (1.01e-3)	8.80e0 (2.62e1)
	L_{E2}	2.11e-5 (1.36e-4)	1.16e2 (2.30e2)	7.64e-5 (3.48e-4)	8.45e1 (1.72e2)	4.87e-5 (3.48e-4)	1.00e2 (2.30e2)
	L_{E3}	3.37e-5 (1.31e-4)	3.70e0 (7.02e0)	6.16e-5 (3.13e-4)	2.89e0 (7.06e0)	4.77e-5 (3.13e-4)	3.30e0 (7.06e0)
	all L_a	4.17e-5 (2.30e-4)	3.25e1 (2.30e2)	3.60e-4 (1.16e-3)	2.80e1 (1.72e2)	2.01e-4 (1.16e-3)	3.02e1 (2.30e2)
		L'_Z		L'_C		all L'_b	
		$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$	$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$	$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$
BP	L_{E1sum}	N/A	N/A	3.61e-5 (1.33e-4)	1.31e0 (3.08e0)	N/A	N/A
SDPNAL+	L_{E1sum}	2.78e-5 (1.25e-4)	4.44e0 (7.78e0)	2.93e-4 (6.62e-4)	7.46e0 (1.48e1)	1.60e-4 (6.62e-4)	5.95e0 (1.48e1)
	L'_{E1}	1.07e-4 (3.58e-4)	4.90e0 (1.04e1)	3.86e-4 (1.01e-3)	8.16e0 (1.66e1)	2.47e-4 (1.01e-3)	6.53e0 (1.66e1)
	L'_{E2}	2.10e-5 (1.31e-4)	1.35e2 (2.22e2)	1.08e-4 (3.36e-4)	1.63e2 (2.97e2)	6.46e-5 (3.36e-4)	1.49e2 (2.97e2)
	L''_{E3}	3.50e-5 (1.58e-4)	5.77e0 (1.22e1)	1.43e-4 (3.68e-4)	4.67e0 (1.12e1)	8.92e-5 (3.68e-4)	5.22e0 (1.22e1)
	all L'_a	4.78e-5 (3.58e-4)	3.75e1 (2.22e2)	2.33e-4 (1.01e-3)	4.59e1 (2.97e2)	1.40e-4 (1.01e-3)	4.17e1 (2.97e2)

Set 2 of medium size QAP instances (chr20a, chr20b, chr20c, had20, lipa20a, lipa20b, nug20, rou20, scr20, tai20a, tai20b)

		L_Z		L_C		all L_b	
		$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$	$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$	$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$
BP	L_{E1sum}	8.71e-3 (4.54e-2)	2.44e0 (3.82e0)	7.87e-5 (4.58e-4)	2.21e0 (5.37e0)	4.40e-3 (4.54e-2)	2.33e0 (5.37e0)
SDPNAL+	L_{E1sum}	1.42e-4 (6.64e-4)	6.04e0 (2.04e1)	9.67e-4 (3.97e-3)	1.90e1 (7.80e1)	5.54e-4 (3.97e-3)	1.25e1 (7.80e1)
	L_{E1}	1.12e-4 (4.67e-4)	4.47e0 (1.52e1)	8.08e-4 (1.24e-3)	1.46e1 (6.15e1)	4.60e-4 (1.24e-3)	9.55e0 (6.15e1)
	L_{E3}	1.06e-4 (4.05e-4)	3.28e0 (1.17e1)	2.21e-4 (7.87e-4)	3.39e0 (8.85e0)	1.63e-4 (7.87e-4)	3.34e0 (1.17e1)
	all L_a	1.20e-4 (6.64e-4)	4.60e0 (2.04e1)	6.65e-4 (3.97e-3)	1.23e1 (7.80e1)	3.93e-4 (3.97e-3)	8.46e0 (7.80e1)
		L'_Z		L'_C		all L'_b	
		$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$	$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$	$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$
BP	L_{E1sum}	N/A	N/A	6.83e-5 (4.56e-4)	1.73e0 (3.86e0)	N/A	N/A
SDPNAL+	L'_{E1sum}	1.28e-4 (7.02e-4)	6.48e0 (1.68e1)	2.94e-4 (1.24e-3)	9.45e0 (4.27e1)	2.11e-4 (1.24e-3)	7.97e0 (4.27e1)
	L'_{E1}	1.76e-4 (8.68e-4)	8.10e0 (2.20e1)	3.68e-4 (9.50e-4)	1.08e1 (3.30e1)	2.72e-4 (9.50e-4)	9.47e0 (3.30e1)
	L''_{E3}	1.13e-4 (6.89e-4)	7.63e0 (2.40e1)	1.54e-4 (7.31e-4)	7.92e0 (1.98e1)	1.34e-4 (7.31e-4)	7.77e0 (2.40e1)
	all L'_a	1.39e-4 (8.68e-4)	7.40e0 (2.40e1)	2.72e-4 (1.24e-3)	9.41e0 (4.27e1)	2.06e-4 (1.24e-3)	8.40e0 (4.27e1)

Set 3 of medium size QAP instances (bur26a, bur26b, bur26c, bur26d, bur26e, bur26f, bur26g, bur26h, nug25, chr25a)

		L_Z		L_C		all L_b	
		$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$	$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$	$\eta_m(\eta_w)$	$\sigma_m(\sigma_w)$
BP	L_{E1sum}	5.15e-3 (3.88e-2)	1.81e0 (4.13e0)	1.31e-4 (1.78e-4)	1.51e0 (2.56e0)	2.64e-3 (3.88e-2)	1.66e0 (4.13e0)
SDPNAL+	L_{E1sum}	1.30e-5 (5.30e-5)	4.88e0 (1.07e1)	4.15e-4 (1.04e-3)	4.59e0 (1.06e1)	2.14e-4 (1.04e-3)	4.74e0 (1.07e1)
	L_{E1}	5.26e-5 (1.01e-4)	5.61e0 (1.18e1)	4.44e-4 (1.38e-3)	5.34e0 (1.29e1)	2.48e-4 (1.38e-3)	5.48e0 (1.29e1)
	L_{E3}	4.97e-5 (9.98e-5)	3.57e0 (5.65e0)	1.31e-4 (7.46e-4)	2.37e0 (3.58e0)	9.02e-5 (7.46e-4)	2.97e0 (5.65e0)
	all L_a	3.84e-5 (1.01e-4)	4.69e0 (1.18e1)	3.30e-4 (1.38e-3)	4.10e0 (1.29e1)	1.84e-4 (1.38e-3)	4.40e0 (1.29e1)
		L'_Z		L'_C		all L'_b	
		$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$	$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$	$\eta'_m(\eta'_w)$	$\sigma'_m(\sigma'_w)$
BP	L_{E1sum}	N/A	N/A	1.15e-4 (1.69e-4)	1.26e0 (1.98e0)	N/A	N/A
SDPNAL+	L'_{E1sum}	5.56e-5 (3.87e-4)	5.70e0 (1.32e1)	2.20e-4 (8.99e-4)	4.18e0 (1.06e1)	1.38e-4 (8.99e-4)	4.94e0 (1.32e1)
	L'_{E1}	4.92e-5 (1.22e-4)	9.15e0 (2.17e1)	2.96e-4 (1.85e-3)	9.02e0 (2.49e1)	1.72e-4 (1.85e-3)	9.09e0 (2.49e1)
	L''_{E3}	2.50e-5 (7.58e-5)	1.01e1 (2.49e1)	1.50e-4 (1.03e-3)	8.60e0 (2.49e1)	8.73e-5 (1.03e-3)	9.33e0 (2.49e1)
	all L'_a	4.33e-5 (3.87e-4)	8.31e0 (2.49e1)	2.22e-4 (1.85e-3)	7.27e0 (2.49e1)	1.33e-4 (1.85e-3)	7.79e0 (2.49e1)

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