A Proof of the Standardization Theorem in λ-Calculus

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Abstract

We present a new proof of the standardization theorem in λ-calculus, which is performed by inductions based on an inductive definition of β-reducibility with a standard sequence.

1 Introduction

The standardization theorem is a fundamental theorem in reduction theory of λ-calculus, which states that if a λ-term M β-reduces to a λ-term N, then there is a "standard" β-reduction sequence from M to N. This paper gives a new simple proof of this and some related theorems.

In literature (e.g., [1], [2], [3]), there have been some proofs of the standardization theorem. Compared with these, a feature of the presented proof is that we use neither the notion of "residuals" nor the separation of the "head" and "internal" reductions. The key to our proof is an inductive definition of β-reducibility with a standard sequence (Definition 3.2). In virtue of this definition, all the proof can be performed by easy inductions.

In Section 2, we give basic definitions. In Section 3, we prove the standardization theorem. In Section 4, we prove the quasi-leftmost reduction theorem, and we mention a result concerning the length of the standard β-reduction sequences.

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2 Preliminaries

We follow the notations and terminology of [1] unless otherwise stated. Capital letters $A, B, \ldots$ denote arbitrary (type-free) λ-terms, and small letters $x, y, \ldots$ denote arbitrary variables. Terms of the form $\lambda x.M$ are called *abstractions*. The symbol $\equiv$ means syntactic equality modulo α-congruence. $M[x := N]$ denotes the result of substituting N for all the free occurrences of x in M with adequate change of bound variables. By $\mathrm{r}(M)$, we mean the number of all the occurrences of β-redexes in M.

Definition 2.1 *For λ-terms M, N and a natural number $n \geq 1$, we define a relation $M \xrightarrow{n} N$ inductively as follows.*

(1) $(\lambda x.A)B \xrightarrow{1} A[x := B]$.

(2) *If $A \xrightarrow{n} B$ and A is not an abstraction, then $AC \xrightarrow{n} BC$.*

(3) *If $A \xrightarrow{n} B$ and A is an abstraction, then $AC \xrightarrow{n+1} BC$.*

(4) *If $A \xrightarrow{n} B$ and C is not an abstraction, then $CA \xrightarrow{n+\mathrm{r}(C)} CB$.*

(5) *If $A \xrightarrow{n} B$ and C is an abstraction, then $CA \xrightarrow{n+\mathrm{r}(C)+1} CB$.*

(6) *If $A \xrightarrow{n} B$, then $\lambda x.A \xrightarrow{n} \lambda x.B$.*

$M \xrightarrow{n} N$ represents that N is obtained from M by contracting the n-th β-redex in M. The usual notions $M \to_\beta N$ (i.e., N is obtained from M by one step β-reduction) and $M \to_\ell N$ (i.e., N is obtained from M by one step leftmost reduction) and their sequences $\twoheadrightarrow_\beta$ and $\twoheadrightarrow_\ell$ are defined as follows.

Definition 2.2

- *$A \to_\beta B$ if $A \xrightarrow{n} B$ for some n.*
- *$A \to_\ell B$ if $A \xrightarrow{1} B$.*
- *$\twoheadrightarrow_\beta$ and $\twoheadrightarrow_\ell$ are the reflexive transitive closure of $\to_\beta$ and $\to_\ell$ respectively.*

The notions of *standard* and *quasi-leftmost* β-reduction sequences are defined as follows.

Definition 2.3

- *A β-reduction sequence $A_0 \xrightarrow{n_1} A_1 \xrightarrow{n_2} \cdots \xrightarrow{n_k} A_k$ is called standard if $n_1 \leq n_2 \leq \cdots \leq n_k$.*
- *An infinite β-reduction sequence is called quasi-leftmost if it contains infinitely many leftmost reduction steps $\to_\ell$.*

Now the theorems are precisely stated as follows. While they have some proofs in literature, we present a simpler proof in the succeeding sections.

Theorem 2.4 (Standardization Theorem) *If $M \twoheadrightarrow_\beta N$, then there is a standard β-reduction sequence from M to N.*

Theorem 2.5 (Quasi-Leftmost Reduction Theorem) *If M has a β-normal form, then there is no infinite quasi-leftmost β-reduction sequence from M.*

Note that *if $N' \xrightarrow{n} N$ and N is a β-normal form, then $n = 1$.* Therefore the following is a special case of the standardization theorem.

Theorem 2.6 (Leftmost Reduction Theorem) *If $M \twoheadrightarrow_\beta N$ and N is a β-normal form, then $M \twoheadrightarrow_\ell N$.*

3 Proof of the standardization theorem

Two binary relations $\twoheadrightarrow_{\mathrm{hap}}$ and $\twoheadrightarrow_{\mathrm{st}}$ on the set of λ-terms are inductively defined as follows, which are the keys to our proof. ("hap" and "st" stand for "head reduction in application" and "standard" respectively.)

Definition 3.1

(1) $A \twoheadrightarrow_{\mathrm{hap}} A$.

(2) $(\lambda x.A_0)A_1A_2\cdots A_n \twoheadrightarrow_{\mathrm{hap}} A_0[x := A_1]A_2\cdots A_n$, *where* $n \geq 1$.

(3) *If* $A \twoheadrightarrow_{\mathrm{hap}} B$ *and* $B \twoheadrightarrow_{\mathrm{hap}} C$, *then* $A \twoheadrightarrow_{\mathrm{hap}} C$.

Definition 3.2

(1) *If* $L \twoheadrightarrow_{\mathrm{hap}} x$, *then* $L \twoheadrightarrow_{\mathrm{st}} x$.

(2) *If* $L \twoheadrightarrow_{\mathrm{hap}} AB$, $A \twoheadrightarrow_{\mathrm{st}} C$, *and* $B \twoheadrightarrow_{\mathrm{st}} D$, *then* $L \twoheadrightarrow_{\mathrm{st}} CD$.

(3) *If* $L \twoheadrightarrow_{\mathrm{hap}} \lambda x.A$ *and* $A \twoheadrightarrow_{\mathrm{st}} B$, *then* $L \twoheadrightarrow_{\mathrm{st}} \lambda x.B$.

Lemma 3.3

(1) *If* $M \twoheadrightarrow_{\mathrm{hap}} N$, *then* $M \twoheadrightarrow_\ell N$.

(2) *If* $M \twoheadrightarrow_{\mathrm{st}} N$, *then there is a standard β-reduction sequence from M to N.*

Proof (1) By induction on the definition of $M \twoheadrightarrow_{\mathrm{hap}} N$. (2) By induction on the definition of $M \twoheadrightarrow_{\mathrm{st}} N$, using (1). ∎

Lemma 3.4

(1) $M \twoheadrightarrow_{\mathrm{st}} M$.

(2) *If* $M \twoheadrightarrow_{\mathrm{hap}} N$, *then* $MP \twoheadrightarrow_{\mathrm{hap}} NP$.

(3) *If* $L \twoheadrightarrow_{\mathrm{hap}} M \twoheadrightarrow_{\mathrm{st}} N$, *then* $L \twoheadrightarrow_{\mathrm{st}} N$.

(4) *If* $M \twoheadrightarrow_{\mathrm{hap}} N$, *then* $M[z := P] \twoheadrightarrow_{\mathrm{hap}} N[z := P]$.

(5) *If* $M \twoheadrightarrow_{\rm st} N$ *and* $P \twoheadrightarrow_{\rm st} Q$, *then* $M[z := P] \twoheadrightarrow_{\rm st} N[z := Q]$.

Proof (1) By induction on the structure of M. (2) By induction on the definition of $M \twoheadrightarrow_{\rm hap} N$. (3) By the definition of $M \twoheadrightarrow_{\rm st} N$ and the transitivity of $\twoheadrightarrow_{\rm hap}$. (4) By induction on the definition of $M \twoheadrightarrow_{\rm hap} N$. (5) By induction on the definition of $M \twoheadrightarrow_{\rm st} N$, using (3) and (4). ∎

Lemma 3.5 *If* $L \twoheadrightarrow_{\rm st} (\lambda x.M)N$, *then* $L \twoheadrightarrow_{\rm st} M[x := N]$.

Proof By the definition of $L \twoheadrightarrow_{\rm st} (\lambda x.M)N$, we have

(i) $L \twoheadrightarrow_{\rm hap} PN'$,

(ii) $P \twoheadrightarrow_{\rm st} \lambda x.M$,

(iii) $N' \twoheadrightarrow_{\rm st} N$,

for some P and N'; and similarly by (ii), we have

(iv) $P \twoheadrightarrow_{\rm hap} \lambda x.M'$,

(v) $M' \twoheadrightarrow_{\rm st} M$,

for some M'. Then we have

$$\begin{array}{llll} L & \twoheadrightarrow_{\rm hap} & PN' & \text{(by (i))} \\ & \twoheadrightarrow_{\rm hap} & (\lambda x.M')N' & \text{(by (iv) and Lemma 3.4(2))} \\ & \twoheadrightarrow_{\rm hap} & M'[x := N'] & \text{(by Definition 3.1(2))} \\ & \twoheadrightarrow_{\rm st} & M[x := N] & \text{(by (v), (iii) and Lemma 3.4(5))} \end{array}$$

and the transitivity of $\twoheadrightarrow_{\rm hap}$ and Lemma 3.4(3) imply $L \twoheadrightarrow_{\rm st} M[x := N]$. ∎

Lemma 3.6 *If* $L \twoheadrightarrow_{\rm st} M \rightarrow_\beta N$, *then* $L \twoheadrightarrow_{\rm st} N$.

Proof By induction on the definition of $M \rightarrow_\beta N$ (i.e., $M \xrightarrow{n} N$ for some n). Here we show two cases (the other cases are similar).

(Case 1): $M \rightarrow_\beta N$ is obtained by Definition 2.1(1). This is just the previous Lemma 3.5.

(Case 2): $M \rightarrow_\beta N$ is obtained by Definition 2.1(2); that is, $M \equiv AC$, $N \equiv BC$, and

(i) $A \rightarrow_\beta B$.

In this case, we have

(ii) $L \twoheadrightarrow_{\rm hap} A'C'$,

(iii) $A' \twoheadrightarrow_{\rm st} A$,

(iv) $C' \twoheadrightarrow_{\rm st} C$, for some A' and C' because of the definition of $L \twoheadrightarrow_{\mathrm{st}} M(\equiv AC)$. Then (iii), (i) and the induction hypothesis imply the fact $A' \twoheadrightarrow_{\mathrm{st}} B$, which shows $L \twoheadrightarrow_{\mathrm{st}} BC(\equiv N)$ using (ii), (iv) and the definition of $\twoheadrightarrow_{\mathrm{st}}$. ∎

Lemma 3.7 *If $M \twoheadrightarrow_\beta N$, then $M \twoheadrightarrow_{\mathrm{st}} N$.*

Proof Suppose $M \equiv M_0 \rightarrow_\beta M_1 \rightarrow_\beta \cdots \rightarrow_\beta M_k \equiv N$. We can show $M \twoheadrightarrow_{\mathrm{st}} M_i$ for $i = 0, 1, \ldots, k$, by Lemmas 3.4(1) and 3.6. ∎

Now the Standardization Theorem 2.4 is obvious by Lemmas 3.7 and 3.3(2).

4 Other results

Lemma 4.1 *If $L \twoheadrightarrow_\beta M \rightarrow_\ell N$, then $L \rightarrow_\ell\twoheadrightarrow_\beta N$, that is, $L \rightarrow_\ell L' \twoheadrightarrow_\beta N$ for some L'.*

Proof By virture of Lemma 3.7, it is sufficient to show that *if $L \twoheadrightarrow_{\mathrm{st}} M \rightarrow_\ell N$, then $L \rightarrow_\ell\twoheadrightarrow_\beta N$.* This claim is proved by induction on the definition of $M \rightarrow_\ell N$ (i.e., $M \xrightarrow{1} N$). Here we show two cases (the other cases are similar).

(Case 1): $M \xrightarrow{1} N$ is obtained by Definition 2.1(1). In this case, the proof of Lemma 3.5 shows $L \rightarrow_\ell\twoheadrightarrow_\beta N$. (In the proof, $L \twoheadrightarrow_{\mathrm{hap}} M'[x := N']$ contains at least one step $\rightarrow_\ell$.)

(Case 2): $M \xrightarrow{1} N$ is obtained by Definition 2.1(4); that is, $M \equiv CA$, $N \equiv CB$, and

(i) $A \xrightarrow{1} B$,

where C is a β-normal form and is not an abstraction. In this case, we have

(ii) $L \twoheadrightarrow_{\mathrm{hap}} C'A'$,

(iii) $C' \twoheadrightarrow_{\mathrm{st}} C$,

(iv) $A' \twoheadrightarrow_{\mathrm{st}} A$,

for some C' and A' because of the definition of $L \twoheadrightarrow_{\mathrm{st}} M(\equiv CA)$; and each of them implies the following.

(ii)$^+$ $L \equiv C'A'$ or $L \rightarrow_\ell\twoheadrightarrow_\beta C'A'$. (By Lemma 3.3(1).)

(iii)$^+$ $C' \equiv C$ or $C' \rightarrow_\ell\twoheadrightarrow_\beta C$. (By the Leftmost Reduction Theorem 2.6.)

(iv)$^+$ $A' \rightarrow_\ell\twoheadrightarrow_\beta B$. (By (i) and the induction hypothesis.)

Moreover, we can show that C' is not an abstraction (otherwise C becomes an abstraction). Then, (ii)$^+$, (iii)$^+$, and (iv)$^+$ imply $L \to_\ell \twoheadrightarrow_\beta CB (\equiv N)$. ∎

Now the Quasi-Leftmost Reduction Theorem 2.5 is easily proved as follows. Suppose there is an infinite quasi-leftmost β-reduction sequence

$$M \twoheadrightarrow_\beta \to_\ell \twoheadrightarrow_\beta \to_\ell \cdots \twoheadrightarrow_\beta \to_\ell \cdots$$

By Lemma 4.1, each $\to_\ell$ step in this sequence can be moved to the left; and we can construct an infinite sequence of $\to_\ell$ starting from M. Thus, by the Leftmost Reduction Theorem 2.6, M cannot have a β-normal form.

Finally, we make a remark on a result in [3], where Xi proved the standardization theorem involving evaluation of the length of the standard β-reduction sequence. We can prove this result (Lemma 3.3 of [3]) by our method if we add "evaluation of the number of reduction steps" to all the argument in Section 3, as follows. **(Definition 3.1) (1)** $A \twoheadrightarrow_{\text{hap}}^{\{0\}} A$. **(2)** $(\lambda x.A_0)A_1A_2\cdots A_n \twoheadrightarrow_{\text{hap}}^{\{1\}} A_0[x := A_1]A_2\cdots A_n$. **(3)** If $A \twoheadrightarrow_{\text{hap}}^{\{n\}} B$ and $B \twoheadrightarrow_{\text{hap}}^{\{m\}} C$, then $A \twoheadrightarrow_{\text{hap}}^{\{n+m\}} C$. **(Definition 3.2) (1)** If $L \twoheadrightarrow_{\text{hap}}^{\{n\}} x$, then $L \twoheadrightarrow_{\text{st}}^{\{n\}} x$. **(2)** If $L \twoheadrightarrow_{\text{hap}}^{\{n\}} AB$, $A \twoheadrightarrow_{\text{st}}^{\{m\}} C$, and $B \twoheadrightarrow_{\text{st}}^{\{k\}} D$, then $L \twoheadrightarrow_{\text{st}}^{\{n+m+k\}} CD$. **(3)** If $L \twoheadrightarrow_{\text{hap}}^{\{n\}} \lambda x.A$ and $A \twoheadrightarrow_{\text{st}}^{\{m\}} B$, then $L \twoheadrightarrow_{\text{st}}^{\{n+m\}} \lambda x.B$. **(Lemma 3.4(5))** If $M \twoheadrightarrow_{\text{st}}^{\{m\}} N$ and $P \twoheadrightarrow_{\text{st}}^{\{p\}} Q$, then $M[z := P] \twoheadrightarrow_{\text{st}}^{\{m+\alpha p\}} N[z := Q]$, where $\alpha = |N|_z =$ the number of free occurrences of the variable z in N. **(Lemma 3.5)** If $L \twoheadrightarrow_{\text{st}}^{\{n\}} (\lambda x.M)N$, then $L \twoheadrightarrow_{\text{st}}^{\{m\}} M[x := N]$ for some $m \leq 1 + \max\{|M|_x, 1\} \cdot n$. The other lemmas are similarly altered.

References

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