Proof Theoretical Studies on Semilattice Relevant Logics

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Abstract

The semilattice relevant logics ${}^{\cup}\mathbf{R}$, ${}^{\cup}\mathbf{T}$, ${}^{\cup}\mathbf{RW}$, and ${}^{\cup}\mathbf{TW}$ (slightly different from the orthodox relevant logics $\mathbf{R}$, $\mathbf{T}$, $\mathbf{RW}$, and $\mathbf{TW}$) are defined by "semilattice models" in which conjunction and disjunction are interpreted in a natural way. In this paper, we prove the equivalence between "LK-style" and "LJ-style" labelled sequent calculi for these logics. (LK-style sequents have plural succedents, while they are singletons in LJ-style.) Moreover, using this equivalence, we give the following. (1) New Hilbert-style axiomatizations for ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$. (2) Equivalence between two semantics (commutative monoid model and distributive semilattice model) for the "contractionless" logics ${}^{\cup}\mathbf{RW}$ and ${}^{\cup}\mathbf{TW}$.

1 Introduction

Relevant logics which are defined by "semilattice models" are called *semilattice relevant logics*; they have been studied in the literature, e.g., [3, 4, 5, 8, 9]. A typical semilattice relevant logic ${}^{\cup}\mathbf{R}$ is defined to be *the set of formulas that are valid in any* ${}^{\cup}\mathbf{R}$*-models*, where a ${}^{\cup}\mathbf{R}$-model $\langle I, \cdot, \mathbf{e}\rangle$ is a semilattice with identity $\mathbf{e}$ (i.e., $\langle I, \cdot, \mathbf{e}\rangle$ is an idempotent commutative monoid). This structure can be considered to be a "structure of information"—I is a set of pieces of information, $\cdot$ is a binary operator which combines two pieces of information, and $\mathbf{e}$ is an empty piece of information. The notion "$\alpha \models A$" (a formula A holds according to a piece α of information) is inductively defined in a natural way:

$$\alpha \models A{\to}B \iff \forall\beta \in I\ [(\beta \models A)\ \Rightarrow\ (\alpha{\cdot}\beta \models B)].$$

$$\alpha \models A{\wedge}B \iff (\alpha \models A) \text{ and } (\alpha \models B).$$

$$\alpha \models A{\vee}B \iff (\alpha \models A) \text{ or } (\alpha \models B).$$

(In this paper, formulas are constructed by the connectives $\to$ (implication), $\wedge$ (conjunction) and $\vee$ (disjunction).) We say that a formula A is *valid in* the model if and only if $\mathbf{e} \models A$.

In addition to ${}^{\cup}\mathbf{R}$, the semilattice relevant logics ${}^{\cup}\mathbf{T}$, ${}^{\cup}\mathbf{RW}$, and ${}^{\cup}\mathbf{TW}$ appear in this paper, where they are obtained by modifying the definition of ${}^{\cup}\mathbf{R}$-model and/or the definition of "$\models$" as follows. The definition of "$\alpha \models A{\to}B$" in ${}^{\cup}\mathbf{T}/{}^{\cup}\mathbf{TW}$-models is described as

$$\forall\beta \succeq \alpha\ [(\beta \models A)\ \Rightarrow\ (\alpha{\cdot}\beta \models B)]$$

where $\succeq$ is a binary relation on I, and ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$-models are commutative monoids without the idempotence postulate: $\alpha{\cdot}\alpha = \alpha$.

A point of semilattice models is that *"$\alpha \models A \Rightarrow \alpha{\cdot}\beta \models A$" is not always true*, which reflects the principle of relevant logics—adding an irrelevant assumption (β) makes a fall of truth value.

This excludes "paradoxical" formulas, e.g., $A{\to}(B{\to}A)$, while acceptable formulas such as $A{\to}A$, $(A{\wedge}B){\to}A$, $A{\to}(A{\vee}B)$ and $(A{\wedge}(B{\vee}C)){\to}((A{\wedge}B){\vee}C)$ are valid in any semilattice models. Moreover, there are some marginal formulas that correspond to some postulates on models. For example, the formula $(A{\to}(A{\to}B)){\to}(A{\to}B)$ corresponds to the idempotence postulate: this formula is not in ${}^{\cup}\mathbf{RW}$ or ${}^{\cup}\mathbf{TW}$ while it is in ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$. (Since this formula is called *contraction*, the logics ${}^{\cup}\mathbf{RW}$ and ${}^{\cup}\mathbf{TW}$ are also called *contractionless logics*.) These examples show an advantage of semilattice models: various features of relevant logics are represented by simple devices of models.

The "orthodox" relevant logics (see, e.g., [1, 2]) and the semilattice relevant logics coincide in the $\{\to,\wedge\}$-fragments, but they have a slight difference if the connective $\vee$ exists. For example, it is known ([9]) that $\mathbf{R}_{\to\wedge\vee} \subsetneq {}^{\cup}\mathbf{R}$ where $\mathbf{R}_{\to\wedge\vee}$ is the set of provable formulas in the $\{\to,\wedge,\vee\}$-fragment of the well-known relevant logic $\mathbf{R}$. The orthodox relevant logics are usually defined by their proof systems, and models of them are relatively complicated. On the other hand semilattice relevant logics are defined by simple models; then development of proof theoretical studies is a natural requirement for semilattice relevant logics. This paper is a study in such a direction.

The results of this paper are summarized as follows.

1. Equivalence between the LK-style and LJ-style labelled sequent calculi. For the logic ${}^{\cup}\mathbf{R}$, there is a cut-free labelled sequent calculus, called $\mathbf{LK}^{\cup}\mathbf{R}$, where the labels $\alpha, \beta, \ldots$ are finite sets, and the both sides of a sequent are multisets of labelled formulas.

Axioms of $\mathbf{LK}^{\cup}\mathbf{R}$: $\quad \alpha{:}A \Rightarrow \alpha{:}A$

Inference rules of $\mathbf{LK}^{\cup}\mathbf{R}$:

- The weakening/contraction rules for both side. For example:

$$\frac{\Gamma \Rightarrow \Delta}{\alpha{:}A, \Gamma \Rightarrow \Delta}\ (\text{weakening left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha{:}A, \alpha{:}A}{\Gamma \Rightarrow \Delta, \alpha{:}A}\ (\text{contraction right})$$

- Rules for $\to$:

$$\frac{\Gamma \Rightarrow \Delta, \beta{:}A \qquad \alpha{\cup}\beta{:}B, \Pi \Rightarrow \Sigma}{\alpha{:}A{\to}B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\to \text{left}) \qquad \frac{\{a\}{:}A, \Gamma \Rightarrow \Delta, \alpha{\cup}\{a\}{:}B}{\Gamma \Rightarrow \Delta, \alpha{:}A{\to}B}\ (\to \text{right})$$

  where a does not appear in the lower sequent of "$\to$ right".

- Rules for $\wedge$ and $\vee$. For example,

$$\frac{\alpha{:}A, \Gamma \Rightarrow \Delta \qquad \alpha{:}B, \Gamma \Rightarrow \Delta}{\alpha{:}A{\vee}B, \Gamma \Rightarrow \Delta}\ (\vee \text{ left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha{:}A}{\Gamma \Rightarrow \Delta, \alpha{:}A{\vee}B}\ (\vee \text{ right})$$

  (Rules for $\wedge$ are the dual forms of the ones for $\vee$.)

This system is known to be complete in the following sense: $A \in {}^{\cup}\mathbf{R}$ if and only if $\mathbf{LK}^{\cup}\mathbf{R} \vdash\ \Rightarrow \emptyset{:}A$ (i.e., the sequent $\Rightarrow \emptyset{:}A$ is provable in $\mathbf{LK}^{\cup}\mathbf{R}$). On the other hand, there is another labelled sequent calculus, called $\mathbf{LJ}^{\cup}\mathbf{R}$, that is defined from $\mathbf{LK}^{\cup}\mathbf{R}$ by imposing the restriction that *the right-hand side of each sequent is a singleton.* (The names $\mathbf{LK}^{\cup}\mathbf{R}$ and $\mathbf{LJ}^{\cup}\mathbf{R}$ come from the well-known sequent calculi **LK** for the classical logic and **LJ** for the intuitionistic logic, where **LK** and **LJ** are equivalent to the label-free parts of $\mathbf{LK}^{\cup}\mathbf{R}$ and $\mathbf{LJ}^{\cup}\mathbf{R}$, respectively.) In this paper, we prove that $\mathbf{LK}^{\cup}\mathbf{R}$ *is a conservative extension of* $\mathbf{LJ}^{\cup}\mathbf{R}$ (although **LK** is *not* a conservative extension of **LJ**); therefore the following three conditions are shown to be equivalent: (1) $A \in {}^{\cup}\mathbf{R}$. (2) $\mathbf{LK}^{\cup}\mathbf{R} \vdash\ \Rightarrow \emptyset{:}A$. (3) $\mathbf{LJ}^{\cup}\mathbf{R} \vdash\ \Rightarrow \emptyset{:}A$. Moreover, similar results for ${}^{\cup}\mathbf{T}$, ${}^{\cup}\mathbf{RW}$, and ${}^{\cup}\mathbf{TW}$ are proved. These are shown with a delicate transformation of sequent-proofs. Note that these results were already stated in [5], but the proof there was wrong (see the beginning of Section 3).

2. Hilbert-style axiomatization of ${}^{\cup}\mathbf{R}/{}^{\cup}\mathbf{T}$. We prove that Hilbert-style systems for ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$ are obtained from the orthodox systems $\mathbf{R}$ and $\mathbf{T}$ by adding an extra inference rule. In

[3], such a result for $^\cup\mathbf{R}$ has been proved where the extra rule is slightly different from this paper. Our extra rule corresponds to the "$\vee$ left" rule of $\mathbf{LJ}^\cup\mathbf{R}$, and our proof method is "translation of $\mathbf{LJ}^\cup\mathbf{R}(^\cup\mathbf{T})$-sequents into formulas", which is clearer than the method of [3]. For $^\cup\mathbf{T}$, no other completeness proof for Hilbert-style systems has been known. Unfortunately we do not yet get such a result for $^\cup\mathbf{RW}/^\cup\mathbf{TW}$.

3. Equivalence between the commutative monoid semantics and the distributive semilattice semantics for $^\cup\mathbf{RW}/^\cup\mathbf{TW}$. In [4, 5, 8], two kinds of models are introduced for $^\cup\mathbf{RW}/^\cup\mathbf{TW}$: one is "commutative monoid" and the other is "distributive semilattice". In this paper we introduce labelled sequent calculi for both semantics. Then, by the analysis of these systems, we show the equivalence of the two semantics in the sense that *a formula A is valid in any commutative monoid models if and only if A is valid in any distributive semilattice models.* This equivalence for $^\cup\mathbf{RW}$ and for the $\{\to,\wedge\}$-fragment of $^\cup\mathbf{TW}$ have been proved in [4, 8] (where the proof methods are different from this paper), but the result for $^\cup\mathbf{TW}$ with the connective $\vee$ has not been proved so far.

Thus our results are classified into three, where the first one is a fundamental result on the proof theory of the semilattice relevant logics and the second and third ones solve some open problems, making good use of the first result. We hope that some more important property on the semilattice relevant logics will be shown by virtue of the fundamental result; for example, a decision procedure for $^\cup\mathbf{R}(^\cup\mathbf{T})$ with the cut-free system $\mathbf{LJ}^\cup\mathbf{R}(^\cup\mathbf{T})$ is an attractive goal. (It is open whether they are decidable or not.) We also hope to extend these results to other relevant logics, for example, $\mathbf{E}$ and the neighbors appearing in [6].

The structure of this paper is as follows. In Section 2, we present the definitions of the semilattice relevant logics $^\cup\mathbf{R}$ and $^\cup\mathbf{T}$ with the semilattice models. Then we introduce the labelled sequent calculi $\mathbf{LK}^\cup\mathbf{R}$ and $\mathbf{LK}^\cup\mathbf{T}$, and we prove the completeness of them. In Section 3, we introduce the labelled sequent calculi $\mathbf{LJ}^\cup\mathbf{R}$ and $\mathbf{LJ}^\cup\mathbf{T}$, and we prove that $\mathbf{LK}^\cup\mathbf{R}$ (or $\mathbf{LK}^\cup\mathbf{T}$) is a conservative extension of $\mathbf{LJ}^\cup\mathbf{R}$ (or $\mathbf{LJ}^\cup\mathbf{T}$, respectively). In Section 4, we show that $\mathbf{LJ}^\cup\mathbf{R}$ and $\mathbf{LJ}^\cup\mathbf{T}$ can be changed into systems without the "weakening" rule. These systems are called $\mathbf{LI}^\cup\mathbf{R}$ and $\mathbf{LI}^\cup\mathbf{T}$, which are proved, in Section 5, to be sound with respect to the semilattice models. The results of Sections 2–5 imply the Main Theorem 5.4, which claims the equivalence between the conditions: $A\in X$, $\mathbf{LK}X\vdash\Rightarrow\emptyset\!:\!A$, $\mathbf{LJ}X\vdash\Rightarrow\emptyset\!:\!A$, and $\mathbf{LI}X\vdash\Rightarrow\emptyset\!:\!A$, for $X\in\{^\cup\mathbf{R},{}^\cup\mathbf{T}\}$. In Section 6, we present Hilbert-style systems for $^\cup\mathbf{R}$ and $^\cup\mathbf{T}$, and we show the soundness and completeness of them. In Section 7, we investigate the logics $^\cup\mathbf{RW}$ and $^\cup\mathbf{TW}$. We introduce two kinds of labelled sequent calculi which reflect the two kinds of models (commutative monoid and distributive semilattice). All the results of Sections 2–5 are translated for $^\cup\mathbf{RW}/^\cup\mathbf{TW}$, and then we obtain the equivalence between the two semantics.

2 Labelled sequent calculi and completeness

In this section we present the definitions of semilattice models and labelled sequent calculi for $^\cup\mathbf{R}$ and $^\cup\mathbf{T}$, and we prove the completeness of these calculi. (This completeness result appeared in [5].)

Formulas are constructed from propositional variables and the binary connectives $\to$, $\wedge$ and $\vee$. We assume that the set of propositional variables is countable. We use the metavariables $p,q,\ldots$ for propositional variables, and $A,B,\ldots$ for formulas.

A pair $M=\langle\langle I,\cdot,\mathbf{e}\rangle,V\rangle$ is said to be a $^\cup\mathbf{R}$-*model* if it satisfies the following conditions.

- $\langle I,\cdot,\mathbf{e}\rangle$ is an idempotent commutative monoid (a semilattice with identity $\mathbf{e}$); that is, I is a non-empty set, $\cdot$ is a binary operator on I, and $\mathbf{e}\in I$ such that $(\alpha\cdot\beta)\cdot\gamma=\alpha\cdot(\beta\cdot\gamma)$, $\alpha\cdot\beta=\beta\cdot\alpha$, $\alpha\cdot\alpha=\alpha$, and $\alpha\cdot\mathbf{e}=\alpha$ hold for any $\alpha,\beta,\gamma\in I$.

- V is a subset of $I \times \mathbf{Var}$, where $\mathbf{Var}$ is the set of propositional variables.

For an element α in I and a formula A, we define a notion

$$\alpha \models_M A$$

inductively as follows.

$$\alpha \models_M p \iff (\alpha, p) \in V.$$

$$\alpha \models_M A \to B \iff \forall \beta \in I \ \big[(\beta \models_M A) \ \Rightarrow \ (\alpha \cdot \beta \models_M B)\big].$$

$$\alpha \models_M A \wedge B \iff (\alpha \models_M A) \text{ and } (\alpha \models_M B).$$

$$\alpha \models_M A \vee B \iff (\alpha \models_M A) \text{ or } (\alpha \models_M B).$$

We say that a formula A is *valid in* the model M if and only if $\mathbf{e} \models_M A$.

A ${}^{\cup}\mathbf{T}$-*model* is obtained from a ${}^{\cup}\mathbf{R}$-model by adding a binary relation on the base set. A pair $M = \langle\langle I, \cdot, \mathbf{e}, \preceq\rangle, V\rangle$ is said to be a ${}^{\cup}\mathbf{T}$-model if the $\preceq$-free part is a ${}^{\cup}\mathbf{R}$-model and $\preceq$ is a transitive binary relation on I such that $\mathbf{e} \preceq \alpha$ and $(\alpha \preceq \beta \ \Rightarrow \ \alpha \cdot \gamma \preceq \beta \cdot \gamma)$. The notion $\alpha \models_M A$ is defined like ${}^{\cup}\mathbf{R}$-models where the clause for implication is changed as

$$\alpha \models_M A \to B \iff \forall \beta \succeq \alpha \ \big[(\beta \models_M A) \ \Rightarrow \ (\alpha \cdot \beta \models_M B)\big].$$

The logics ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$ are defined to be the sets of formulas:

$${}^{\cup}\mathbf{R} = \{A \mid A \text{ is valid in any } {}^{\cup}\mathbf{R}\text{-model.}\}$$

$${}^{\cup}\mathbf{T} = \{A \mid A \text{ is valid in any } {}^{\cup}\mathbf{T}\text{-model.}\}$$

We introduce labelled sequent calculi for ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$. A *label* is a finite set of positive integers. If α is a label and A is a formula, then the expression $\alpha : A$ is called a *labelled formula*. We use the letters $\alpha, \beta, \ldots$ to represent labels, and $\Gamma, \Delta, \ldots$ for multisets of labelled formulas. As usual, an expression, e.g. $(\Gamma, \Delta, \alpha : A, \beta : B)$ denotes the multiset $\Gamma \cup \Delta \cup \{\alpha : A\} \cup \{\beta : B\}$ where $\cup$ is the "multiset union". The set of positive integers appearing in the labels Γ is denoted by $\mathcal{L}(\Gamma)$; that is to say, $\mathcal{L}(\Gamma)$ is the union of labels in Γ.

An expression $\Gamma \Rightarrow \Delta$ is called a *labelled sequent* where Γ and Δ are finite multiset of labelled formulas (Γ and Δ are called the *antecedent* and *succedent*, respectively). Intuitively, a label α corresponds to an element of the base set of a model M, and the meaning of a labelled sequent

$$\alpha_1 : A_1, \ldots, \alpha_m : A_m \Rightarrow \beta_1 : B_1, \ldots, \beta_n : B_n$$

is, like the sequent calculus $\mathbf{LK}$ for classical logics, the following:

$$(\alpha_1 \not\models_M A_1) \text{ or } \cdots \text{ or } (\alpha_m \not\models_M A_m) \text{ or } (\beta_1 \models_M B_1) \text{ or } \cdots \text{ or } (\beta_n \models_M B_n).$$

A (cut-free) labelled sequent calculus $\mathbf{LK}^{\cup}\mathbf{R}$ is defined as follows. Axioms of $\mathbf{LK}^{\cup}\mathbf{R}$ are

$$\alpha : A \Rightarrow \alpha : A.$$

Inference rules of $\mathbf{LK}^{\cup}\mathbf{R}$ are "weakening left/right", "contraction left/right", "$\to$ left/right", "$\wedge$ left/right", and "$\vee$ left/right" as follows:

$$\frac{\Gamma \Rightarrow \Delta}{\alpha : A,\ \Gamma \Rightarrow \Delta} \ (\text{weakening left}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\ \alpha : A} \ (\text{weakening right})$$

$$\frac{\alpha : A,\ \alpha : A,\ \Gamma \Rightarrow \Delta}{\alpha : A,\ \Gamma \Rightarrow \Delta} \ (\text{contraction left}) \qquad \frac{\Gamma \Rightarrow \Delta,\ \alpha : A,\ \alpha : A}{\Gamma \Rightarrow \Delta,\ \alpha : A} \ (\text{contraction right})$$

$$\frac{\Gamma \Rightarrow \Delta,\ \beta{:}A \qquad \alpha{\cup}\beta{:}B,\ \Pi \Rightarrow \Sigma}{\alpha{:}A{\to}B,\ \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\to \text{left})$$

$$\frac{\{a\}{:}A,\ \Gamma \Rightarrow \Delta,\ \alpha{\cup}\{a\}{:}B}{\Gamma \Rightarrow \Delta,\ \alpha{:}A{\to}B}\ (\to \text{right}) \text{ with the proviso:}$$

(Label Condition): $a \notin \mathcal{L}(\Gamma, \Delta) \cup \alpha$ (i.e., a does not appear in the lower sequent).

$$\frac{\alpha{:}A,\ \Gamma \Rightarrow \Delta}{\alpha{:}A{\wedge}B,\ \Gamma \Rightarrow \Delta}\ (\wedge \text{left}) \qquad \frac{\alpha{:}B,\ \Gamma \Rightarrow \Delta}{\alpha{:}A{\wedge}B,\ \Gamma \Rightarrow \Delta}\ (\wedge \text{left})$$

$$\frac{\Gamma \Rightarrow \Delta,\ \alpha{:}A \qquad \Gamma \Rightarrow \Delta,\ \alpha{:}B}{\Gamma \Rightarrow \Delta,\ \alpha{:}A{\wedge}B}\ (\wedge \text{right})$$

$$\frac{\alpha{:}A,\ \Gamma \Rightarrow \Delta \qquad \alpha{:}B,\ \Gamma \Rightarrow \Delta}{\alpha{:}A{\vee}B,\ \Gamma \Rightarrow \Delta}\ (\vee \text{left})$$

$$\frac{\Gamma \Rightarrow \Delta,\ \alpha{:}A}{\Gamma \Rightarrow \Delta,\ \alpha{:}A{\vee}B}\ (\vee \text{right}) \qquad \frac{\Gamma \Rightarrow \Delta,\ \alpha{:}B}{\Gamma \Rightarrow \Delta,\ \alpha{:}A{\vee}B}\ (\vee \text{right})$$

Note that the rules

$$\frac{\Gamma,\ \alpha{:}A,\ \beta{:}B,\ \Delta \Rightarrow \Pi}{\Gamma,\ \beta{:}B,\ \alpha{:}A,\ \Delta \Rightarrow \Pi}\ (\text{exchange left}) \qquad \frac{\Gamma \Rightarrow \Delta,\ \alpha{:}A,\ \beta{:}B,\ \Pi}{\Gamma \Rightarrow \Delta,\ \beta{:}B,\ \alpha{:}A,\ \Pi}\ (\text{exchange right})$$

are implicitly available because the antecedent and succedent are multisets.

A labelled sequent calculus $\mathbf{LK}^{\cup}\mathbf{T}$ is obtained from $\mathbf{LK}^{\cup}\mathbf{R}$ by replacing "$\to$ left/right" rules as follows, where $\max(\alpha)$ *is the numerically largest element in* α *if* α *is not empty, and* $\max(\emptyset) = 0$.

$$\frac{\Gamma \Rightarrow \Delta,\ \beta{:}A \qquad \alpha{\cup}\beta{:}B,\ \Pi \Rightarrow \Sigma}{\alpha{:}A{\to}B,\ \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\to \text{left}) \text{ with the proviso:}$$

(Label Condition): $\max(\alpha) \leq \max(\beta)$.

$$\frac{\{a\}{:}A,\ \Gamma \Rightarrow \Delta,\ \alpha{\cup}\{a\}{:}B}{\Gamma \Rightarrow \Delta,\ \alpha{:}A{\to}B}\ (\to \text{right}) \text{ with the proviso:}$$

(Label Condition): $a \notin \mathcal{L}(\Gamma, \Delta) \cup \alpha$, and $\max(\alpha) < a$.

The symbol "$\vdash$" represents provability; for example, "$\mathbf{LK}^{\cup}\mathbf{R} \vdash \Gamma \Rightarrow \Delta$" means "$\Gamma \Rightarrow \Delta$ is provable in $\mathbf{LK}^{\cup}\mathbf{R}$."

We show the completeness of these calculi in the sense that $\mathbf{LK}^{\cup}\mathbf{R}(\mathbf{LK}^{\cup}\mathbf{T}) \vdash\ \Rightarrow \emptyset{:}A$ if $A \in {}^{\cup}\mathbf{R}({}^{\cup}\mathbf{T})$.

The set of all the finite set of positive integers (i.e., the set of all the labels) will be denoted by **Label**. If V is a subset of $\mathbf{Label} \times \mathbf{Var}$, then the pairs $\langle\langle \mathbf{Label}, \cup, \emptyset\rangle, V\rangle$ and $\langle\langle \mathbf{Label}, \cup, \emptyset, \preceq\rangle, V\rangle$ are said to be a *label* ${}^{\cup}\mathbf{R}$*-model* and a *label* ${}^{\cup}\mathbf{T}$*-model* where the binary relation $\preceq$ is defined by

$$\alpha \preceq \beta \Leftrightarrow \max(\alpha) \leq \max(\beta).$$

(It is easy to verify that label models satisfy the definitions of ${}^{\cup}\mathbf{R}/{}^{\cup}\mathbf{T}$-models.) We say that a labelled formula $\alpha\!:\!A$ is *true* (or *false*) in a label model M if $\alpha \models_M A$ (or $\alpha \not\models_M A$, respectively).

A pair $\langle \Gamma, \Delta \rangle$ of multisets of labelled formulas is said to be ${}^{\cup}\mathbf{R}$-*saturated* if the following six conditions hold for any formulas A, B and any labels α, β.

(1) If $\alpha\!:\!A{\to}B \in \Gamma$, then $[\beta\!:\!A \in \Delta$ or $\alpha{\cup}\beta\!:\!B \in \Gamma]$.

(2) If $\alpha\!:\!A{\to}B \in \Delta$, then $[\{a\}\!:\!A \in \Gamma$ and $\alpha{\cup}\{a\}\!:\!B \in \Delta$ for some $a > \max(\alpha)]$.

(3) If $\alpha\!:\!A{\wedge}B \in \Gamma$, then $[\alpha\!:\!A \in \Gamma$ and $\alpha\!:\!B \in \Gamma]$.

(4) If $\alpha\!:\!A{\wedge}B \in \Delta$, then $[\alpha\!:\!A \in \Delta$ or $\alpha\!:\!B \in \Delta]$.

(5) If $\alpha\!:\!A{\vee}B \in \Gamma$, then $[\alpha\!:\!A \in \Gamma$ or $\alpha\!:\!B \in \Gamma]$.

(6) If $\alpha\!:\!A{\vee}B \in \Delta$, then $[\alpha\!:\!A \in \Delta$ and $\alpha\!:\!B \in \Delta]$.

Similarly "${}^{\cup}\mathbf{T}$-*saturated*" is defined by these clauses where (1) is changed into the following.

(1) If $[\alpha\!:\!A{\to}B \in \Gamma$ and $\max(\alpha) \leq \max(\beta)]$, then $[\beta\!:\!A \in \Delta$ or $\alpha{\cup}\beta\!:\!B \in \Gamma]$.

Note that the condition "$a > \max(\alpha)$" in the clause (2) is not essential for "${}^{\cup}\mathbf{R}$-saturated"; we adopt this clause only for the unity of the arguments for ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$.

Lemma 2.1 *If* $\mathbf{LK}X \not\vdash \Gamma \Rightarrow \Delta$, *then there exists a label X-model M such that any labelled formula in Γ is true in M and any labelled formula in Δ is false in M, for* $X = {}^{\cup}\mathbf{R}$ *and* ${}^{\cup}\mathbf{T}$.

Proof We call any pair $\langle \varphi\!:\!F, \psi \rangle$ a *seed for saturation* where $\varphi\!:\!F$ is a labelled formula and ψ is a label. Since both the set of finite sets of positive integers and the set of formulas are countable, we can enumerate all seeds for saturations as

$$\langle \varphi_1\!:\!F_1, \psi_1 \rangle,\ \langle \varphi_2\!:\!F_2, \psi_2 \rangle,\ \ldots$$

so that every seed occurs infinitely often in the enumeration. Using this enumeration, we define a sequence $\Gamma_i \Rightarrow \Delta_i$ $(i = 0, 1, 2, \ldots)$ of unprovable labelled sequents as follows.

[Step 0] $(\Gamma_0 \Rightarrow \Delta_0) = (\Gamma \Rightarrow \Delta)$

[Step k] Suppose that $\Gamma_{k-1} \Rightarrow \Delta_{k-1}$ is already defined and is not provable in $\mathbf{LK}X$. Then we define $\Gamma_k \Rightarrow \Delta_k$ according to the seed $\langle \varphi_k\!:\!F_k, \psi_k \rangle$.

(Case 1): F_k is of the form $A{\to}B$ and $\varphi_k\!:\!A{\to}B \in \Gamma_{k-1}$; moreover, $\max(\varphi_k) \leq \max(\psi_k)$ if $X = {}^{\cup}\mathbf{T}$. In this case,

$$(\Gamma_k \Rightarrow \Delta_k) = \begin{cases} (\Gamma_{k-1} \Rightarrow \Delta_{k-1},\ \psi_k\!:\!A) & \text{(if this labelled sequent is not provable in } \mathbf{LK}X\text{)}, \\ (\varphi_k{\cup}\psi_k\!:\!B,\ \Gamma_{k-1} \Rightarrow \Delta_{k-1}) & \text{(otherwise).} \end{cases}$$

The fact $\mathbf{LK}X \not\vdash \Gamma_k \Rightarrow \Delta_k$ is guaranteed by the rules "$\to$ left" and contraction.

(Case 2): F_k is of the form $A{\to}B$ and $\varphi_k : A{\to}B \in \Delta_{k-1}$. In this case, we take an integer a such that $a > \max(\varphi_k)$ and $a \notin \mathcal{L}(\Gamma_{k-1}, \Delta_{k-1})$. (Since the elements of $\mathcal{L}(\Gamma_{k-1}, \Delta_{k-1})$ are finite, we can take such a.) Then we define

$$(\Gamma_k \Rightarrow \Delta_k) = (\{a\}\!:\!A,\ \Gamma_{k-1} \Rightarrow \Delta_{k-1},\ \varphi_k{\cup}\{a\}\!:\!B).$$

The fact $\mathbf{LK}X \not\vdash \Gamma_k \Rightarrow \Delta_k$ is guaranteed by the rules "$\to$ right" and contraction.

(Case 3): F_k is of the form $A{\wedge}B$ and $\varphi_k\!:\!A{\wedge}B \in \Gamma_{k-1}$. In this case,

$$(\Gamma_k \Rightarrow \Delta_k) = (\varphi_k\!:\!A,\ \varphi_k\!:\!B,\ \Gamma_{k-1} \Rightarrow \Delta_{k-1}).$$

The fact $\mathbf{LK}X \not\vdash \Gamma_k \Rightarrow \Delta_k$ is guaranteed by the rules "$\wedge$ left" and contraction.

(Case 4): F_k is of the form $A \wedge B$ and $\varphi_k : A \wedge B \in \Delta_{k-1}$. In this case,

$$(\Gamma_k \Rightarrow \Delta_k) = \begin{cases} (\Gamma_{k-1} \Rightarrow \Delta_{k-1},\ \varphi_k : A) & \text{(if this labelled sequent is not provable in } \mathbf{LK}X\text{)}, \\ (\Gamma_{k-1} \Rightarrow \Delta_{k-1},\ \varphi_k : B) & \text{(otherwise)}. \end{cases}$$

The fact $\mathbf{LK}X \not\vdash \Gamma_k \Rightarrow \Delta_k$ is guaranteed by the rules "$\wedge$ right" and contraction.

(Case 5): F_k is of the form $A \vee B$ and $\varphi_k : A \vee B \in \Gamma_{k-1}$. This case is similar to the case 4.

(Case 6): F_k is of the form $A \vee B$ and $\varphi_k : A \vee B \in \Delta_{k-1}$. This case is similar to the case 3.

(Case 7): None of the above conditions for (1)–(6) hold. Then

$$(\Gamma_k \Rightarrow \Delta_k) = (\Gamma_{k-1} \Rightarrow \Delta_{k-1}).$$

This completes the construction of the infinite sequence $\Gamma_i \Rightarrow \Delta_i$ $(i = 0, 1, 2, \ldots)$. Then we define $\Gamma_\infty = \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta_\infty = \bigcup_{i=0}^{\infty} \Delta_i$, and we show the following: (1) $\langle \Gamma_\infty, \Delta_\infty \rangle$ is X-saturated. (2) $\Gamma_\infty \cap \Delta_\infty = \emptyset$.

[Proof of (1)] We verify the clause (1) of the definition of X-saturatedness (the other clauses are similar). Suppose $\alpha : A \to B \in \Gamma_\infty$; that is, $\alpha : A \to B \in \Gamma_p$ for some p. Since the seed $\langle \alpha : A \to B, \beta \rangle$ occurs infinitely often in the enumeration, there is a natural number $k \geq p$ that "hits" the above construction of $\Gamma_k \Rightarrow \Delta_k$; that is, there is a natural number k such that $\beta : A \in \Delta_k$ or $\alpha \cup \beta : B \in \Gamma_k$.

[Proof of (2)] If there is a labelled formula $\alpha : A$ in $\Gamma_\infty \cap \Delta_\infty$, then there is a natural number k such that $\alpha : A \in \Gamma_k \cap \Delta_k$, and this contradicts the fact $\mathbf{LK}X \not\vdash \Gamma_k \Rightarrow \Delta_k$.

Now we define a label X-model $M = \langle \langle \mathbf{Label}, \cup, \emptyset(, \preceq) \rangle, V \rangle$ by

$$V = \{(\alpha, p) \mid \alpha : p \in \Gamma_\infty\}.$$

Then we have the following:

- If $\alpha : A \in \Gamma_\infty$, then it is true in M.
- If $\alpha : A \in \Delta_\infty$, then it is false in M.

This is proved by induction on the complexity of the formula A, using (1) and (2) above. Since $\Gamma \subseteq \Gamma_\infty$ and $\Delta \subseteq \Delta_\infty$, this implies that M is the required model. ∎

Theorem 2.2 (Completeness of $\mathbf{LK}^{\cup}\mathbf{R}$ and $\mathbf{LK}^{\cup}\mathbf{T}$) *If $A \in X$, then $\mathbf{LK}X \vdash \Rightarrow \emptyset : A$, for $X \in \{{}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$.*

Proof By the previous Lemma 2.1 where $\Gamma = \emptyset$ and $\Delta = \{\emptyset : A\}$. ∎

3 LJ-style systems

We define labelled sequent calculi $\mathbf{LJ}X$, for $X = {}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$, to be the systems obtained from $\mathbf{LK}X$ by imposing the restriction that the succedent of each labelled sequent is a singleton (recall that a succedent is a multiset of labelled formulas).

Axioms of $\mathbf{LJ}X$: $\alpha{:}A \Rightarrow \alpha{:}A$.

Inference rules of $\mathbf{LJ}X$:

$$\frac{\Gamma \Rightarrow \tau{:}F}{\alpha{:}A,\ \Gamma \Rightarrow \tau{:}F}\ (\text{weakening}) \qquad \frac{\alpha{:}A,\ \alpha{:}A,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A,\ \Gamma \Rightarrow \tau{:}F}\ (\text{contraction})$$

$$\frac{\Gamma \Rightarrow \beta{:}A \quad \alpha{\cup}\beta{:}B,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A{\to}B,\ \Gamma \Rightarrow \tau{:}F}\ (\to \text{left}) \text{ with the proviso:}$$

(Label Condition): $\max(\alpha) \le \max(\beta)$ if $X = {}^{\cup}\mathbf{T}$.

$$\frac{\{a\}{:}A,\ \Gamma \Rightarrow \alpha{\cup}\{a\}{:}B}{\Gamma \Rightarrow \alpha{:}A{\to}B}\ (\to \text{right}) \text{ with the proviso:}$$

(Label Condition): $a \notin \mathcal{L}(\Gamma) \cup \alpha$ if $X = {}^{\cup}\mathbf{R}$. $a \notin \mathcal{L}(\Gamma) \cup \alpha$ and $\max(\alpha) < a$ if $X = {}^{\cup}\mathbf{T}$.

$$\frac{\alpha{:}A,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A{\wedge}B,\ \Gamma \Rightarrow \tau{:}F}\ (\wedge \text{ left}) \qquad \frac{\alpha{:}B,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A{\wedge}B,\ \Gamma \Rightarrow \tau{:}F}\ (\wedge \text{ left})$$

$$\frac{\Gamma \Rightarrow \alpha{:}A \quad \Gamma \Rightarrow \alpha{:}B}{\Gamma \Rightarrow \alpha{:}A{\wedge}B}\ (\wedge \text{ right})$$

$$\frac{\alpha{:}A, \Gamma \Rightarrow \tau{:}F \quad \alpha{:}B,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A{\vee}B, \Gamma \Rightarrow \tau{:}F}\ (\vee \text{ left})$$

$$\frac{\Gamma \Rightarrow \alpha{:}A}{\Gamma \Rightarrow \alpha{:}A{\vee}B}\ (\vee \text{ right}) \qquad \frac{\Gamma \Rightarrow \alpha{:}B}{\Gamma \Rightarrow \alpha{:}A{\vee}B}\ (\vee \text{ right})$$

Note that the two forms of "$\to$ left"

$$\frac{\Gamma \Rightarrow \beta{:}A \quad \alpha{\cup}\beta{:}B,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A{\to}B,\ \Gamma \Rightarrow \tau{:}F} \quad \text{and} \quad \frac{\Gamma \Rightarrow \beta{:}A \quad \alpha{\cup}\beta{:}B,\ \Pi \Rightarrow \tau{:}F}{\alpha{:}A{\to}B,\ \Gamma, \Pi \Rightarrow \tau{:}F}$$

are mutually derivable with the help of the weakening and the contraction rules. We here adopt the former because of a technical reason—the upper sequents are uniquely determined if the lower sequent is given and if the "principal labelled formula" $\alpha{:}A{\to}B$ is identified.

In this section we show that *"$\mathbf{LK}X \vdash \Gamma \Rightarrow \alpha{:}A$" implies "$\mathbf{LJ}X \vdash \Gamma \Rightarrow \alpha{:}A$"* (Theorem 3.6); in other words, *$\mathbf{LK}X$ is a conservative extension of $\mathbf{LJ}X$.* (This fact is not surprising because the $\{\to, \wedge, \vee\}$-fragments of most relevant logics are weaker than the intuitionistic logic.) To show this claim, one may try to prove (one of) the following assertions by induction on the $\mathbf{LK}X$-proof of $\Gamma \Rightarrow \Delta$.

- If $\mathbf{LK}X \vdash \Gamma \Rightarrow \Delta$, then $\mathbf{LJ}X \vdash \Gamma \Rightarrow \alpha{:}A$ where A is the disjunction of all the formulas in Δ.

- If $\mathbf{LK}X \vdash \Gamma \Rightarrow \Delta$, then $\mathbf{LJ}X \vdash \Gamma \Rightarrow \alpha\!:\!A$ for some $\alpha\!:\!A \in \Delta$.

However, the former does not work because of the difference of Δ in labels (e.g., we cannot choose an adequate α such that $\alpha\!:\!F\vee G$ represents Δ when $\Delta = (\varphi\!:\!F, \psi\!:\!G)$ and $\varphi \neq \psi$); and the latter is wrong—there is a counterexample:

$\mathbf{LK}X \vdash \alpha\!:\!p_1\vee p_2 \Rightarrow \alpha\!:\!p_1, \alpha\!:\!p_2$.

$\mathbf{LJ}X \not\vdash \alpha\!:\!p_1\vee p_2 \Rightarrow \alpha\!:\!p_i$ for $i = 1, 2$.

(**Remark**: This is also a counterexample to Fact 6 of [5].) Then we will take another way.

We define labelled sequent calculi $\mathbf{LM}X$ ($X = {}^{\cup}\mathbf{R}$, ${}^{\cup}\mathbf{T}$), which are intermediate systems between $\mathbf{LK}X$ and $\mathbf{LJ}X$, as follows.

Axioms: $\alpha\!:\!A \Rightarrow \alpha\!:\!A$.

Inference rules:

$$\frac{\Gamma \Rightarrow \Delta}{\alpha\!:\!A,\ \Gamma \Rightarrow \Delta}\ \text{(weakening left)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\ \alpha\!:\!A}\ \text{(weakening right)}$$

$$\frac{\alpha\!:\!A,\ \alpha\!:\!A,\ \Gamma \Rightarrow \Delta}{\alpha\!:\!A,\ \Gamma \Rightarrow \Delta}\ \text{(contraction left)} \qquad \frac{\Gamma \Rightarrow \Delta,\ \alpha\!:\!A,\ \alpha\!:\!A}{\Gamma \Rightarrow \Delta,\ \alpha\!:\!A}\ \text{(contraction right)}$$

$$\frac{\Gamma \Rightarrow \beta\!:\!A \qquad \alpha\cup\beta\!:\!B,\ \Gamma \Rightarrow \Delta}{\alpha\!:\!A\to B,\ \Gamma \Rightarrow \Delta}\ (\to \text{left})\ \text{with the proviso:}$$

(Label Condition): $\max(\alpha) \leq \max(\beta)$ if $X = {}^{\cup}\mathbf{T}$.

$$\frac{\{a\}\!:\!A,\ \Gamma \Rightarrow \alpha\cup\{a\}\!:\!B}{\Gamma \Rightarrow \alpha\!:\!A\to B}\ (\to \text{right})\ \text{with the proviso:}$$

(Label Condition): $a \notin \mathcal{L}(\Gamma) \cup \alpha$ if $X = {}^{\cup}\mathbf{R}$. $a \notin \mathcal{L}(\Gamma) \cup \alpha$ and $\max(\alpha) < a$ if $X = {}^{\cup}\mathbf{T}$.

$$\frac{\alpha\!:\!A,\ \Gamma \Rightarrow \Delta}{\alpha\!:\!A\wedge B,\ \Gamma \Rightarrow \Delta}\ (\wedge \text{ left}) \qquad \frac{\alpha\!:\!B,\ \Gamma \Rightarrow \Delta}{\alpha\!:\!A\wedge B,\ \Gamma \Rightarrow \Delta}\ (\wedge \text{ left})$$

$$\frac{\Gamma \Rightarrow \alpha\!:\!A \qquad \Gamma \Rightarrow \alpha\!:\!B}{\Gamma \Rightarrow \alpha\!:\!A\wedge B}\ (\wedge \text{ right})$$

$$\frac{\alpha\!:\!A,\ \Gamma \Rightarrow \Delta \qquad \alpha\!:\!B,\ \Gamma \Rightarrow \Delta}{\alpha\!:\!A\vee B,\ \Gamma \Rightarrow \Delta}\ (\vee \text{ left})$$

$$\frac{\Gamma \Rightarrow \alpha\!:\!A}{\Gamma \Rightarrow \alpha\!:\!A\vee B}\ (\vee \text{ right}) \qquad \frac{\Gamma \Rightarrow \alpha\!:\!B}{\Gamma \Rightarrow \alpha\!:\!A\vee B}\ (\vee \text{ right})$$

That is to say, $\mathbf{LM}X$ is obtained from $\mathbf{LK}X$ with the modification that "$\to/\wedge/\vee$ right" rules and the left upper sequent of "$\to$ left" rule are $\mathbf{LJ}X$-style. Then we will show

(A) if $\mathbf{LK}X \vdash \Gamma \Rightarrow \Delta$, then $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; and

(B) if $\mathbf{LM}X \vdash \Gamma \Rightarrow \alpha\!:\!A$, then $\mathbf{LJ}X \vdash \Gamma \Rightarrow \alpha\!:\!A$.

The claim (B) is easily shown by the following, which can be proved by induction on the $\mathbf{LM}X$-proofs.

Theorem 3.1 *If* $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$ *and* $\Delta = \{\alpha:A,\ldots,\alpha:A\}$ *(a non-empty multisets consisting only of* $\alpha\!:\!A$*), then* $\mathbf{LJ}X \vdash \Gamma \Rightarrow \alpha\!:\!A$*, where* $X \in \{{}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$.

The claim (A) is hard to prove. For this, we will show that *each inference rule of* $\mathbf{LK}X$ *is admissible in* $\mathbf{LM}X$. For example, suppose there are two $\mathbf{LM}X$-proofs P of $(\Gamma \Rightarrow \Delta, \beta\!:\!A)$ and Q of $(\alpha\cup\beta\!:\!B, \Pi \Rightarrow \Sigma)$. To show the admissibility of the $\mathbf{LK}X$-rule

$$\frac{\Gamma \Rightarrow \Delta,\ \beta\!:\!A \qquad \alpha\cup\beta\!:\!B,\ \Pi \Rightarrow \Sigma}{\alpha\!:\!A\to B,\ \Gamma,\Pi \Rightarrow \Delta,\Sigma,}\ (\to \text{left})$$

we should construct an $\mathbf{LM}X$-proof of this last sequent. The essence of this construction is explained as follows. First we make a proof P' by dropping some inferences from P so that the last sequent is of the form $\Gamma' \Rightarrow \beta\!:\!A$. Then the desired $\mathbf{LM}X$-proof is

$$\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\ P' \\ \Gamma' \Rightarrow \beta\!:\!A\end{array} \qquad \begin{array}{c}\vdots\ Q \\ \alpha\cup\beta\!:\!B,\ \Pi \Rightarrow \Sigma\end{array}}{\alpha\!:\!A\to B,\ \Gamma',\Pi \Rightarrow \Sigma}\ (\text{weakening and} \to \text{left}) \\
\vdots\ R \\
\alpha\!:\!A\to B,\ \Gamma,\Pi \Rightarrow \Delta,\Sigma
\end{array}$$

where R corresponds to the dropped inferences by which the sequent $\Gamma,\Phi \Rightarrow \Delta,\Psi$ is derived from the sequent $\Gamma',\Phi \Rightarrow \Psi$ for arbitrary Φ,Ψ. (A similar technique appears in a cut-elimination procedure for $\mathbf{E}_{\to}$ in [7].) A typical example is that P is

$$\frac{\beta\!:\!C \Rightarrow \beta\!:\!C \qquad \beta\!:\!A \Rightarrow \beta\!:\!A}{\beta\!:\!C\vee A \Rightarrow \beta\!:\!C,\ \beta\!:\!A}\ (\text{weakening and} \vee \text{left})$$

($\Gamma = \beta\!:\!C\vee A$, and $\Delta = \beta\!:\!C$), P' is an axiom $\beta\!:\!A \Rightarrow \beta\!:\!A$, and R consists of the axiom $\beta\!:\!C \Rightarrow \beta\!:\!C$ and the rules weakening and "$\vee$ left". The Main Lemma 3.3, which is a central technical result in this paper, and Lemma 3.4 will give the precise description of this construction.

Now we begin the proof in detail.

We define a relation $\precsim$ between two multisets of labelled formulas by

$$\Gamma \precsim \Delta \quad\Leftrightarrow\quad \exists(\alpha\!:\!A) \in \Gamma, \exists(\beta\!:\!B) \in \Delta\ [\alpha \subseteq \beta].$$

Lemma 3.2 *If* $\mathbf{LM}X \vdash \Gamma,\Pi \Rightarrow \Delta$ *and* $\Pi \not\precsim \Delta$*, then* $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$*, where* $X \in \{{}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$.

Proof By induction on the $\mathbf{LM}X$-proof of $\Gamma,\Pi \Rightarrow \Delta$. We divide cases according to the last inference of the proof, and here we show some nontrivial cases.

(Case 1): The last inference is

$$\frac{\Gamma',\Pi \Rightarrow \beta\!:\!A \qquad \alpha\cup\beta\!:\!B,\Gamma',\Pi \Rightarrow \Delta}{\alpha\!:\!A\to B,\Gamma',\Pi \Rightarrow \Delta}\ (\to \text{left})$$

and $\Gamma = (\alpha\!:\!A\to B, \Gamma')$.

(Subcase 1-1): $\Pi \not\precsim \beta\!:\!A$. In this case, we have

$$\frac{\begin{array}{c}\vdots\ \text{i.h.} \\ \Gamma' \Rightarrow \beta\!:\!A\end{array} \qquad \begin{array}{c}\vdots\ \text{i.h.} \\ \alpha\cup\beta\!:\!B,\Gamma' \Rightarrow \Delta\end{array}}{\alpha\!:\!A\to B,\Gamma' \Rightarrow \Delta}\ (\to \text{left})$$

where the Label Condition still holds when $X = {}^{\cup}\mathbf{T}$.

(Subcase 1-2): $\varPi \precsim \beta : A$. In this case, $\alpha\cup\beta : B \not\precsim \varDelta$ holds (and therefore $(\alpha\cup\beta : B, \varPi) \not\precsim \varDelta$ holds) because of the condition $\varPi \not\precsim \varDelta$. Then we have

$$\dfrac{\begin{matrix}\vdots\ \text{i.h.}\\ \varGamma' \Rightarrow \varDelta\end{matrix}}{\alpha : A {\to} B, \varGamma' \Rightarrow \varDelta.}\ (\text{weakening})$$

(Case 2): The last inference is

$$\dfrac{\varGamma, \varPi' \Rightarrow \beta : A \quad \varGamma, \alpha\cup\beta : B, \varPi' \Rightarrow \varDelta}{\varGamma, \alpha : A{\to}B, \varPi' \Rightarrow \varDelta}\ (\to \text{left})$$

and $\varPi = (\alpha : A{\to}B, \varPi')$. In this case, $(\alpha\cup\beta : B, \varPi') \not\precsim \varDelta$ holds, and $\mathbf{LM}X \vdash \varGamma \Rightarrow \varDelta$ is obtained by the induction hypothesis.

(Case 3): The last inference is

$$\dfrac{\{a\} : A, \varGamma, \varPi \Rightarrow \alpha\cup\{a\} : B}{\varGamma, \varPi \Rightarrow \alpha : A{\to}B}\ (\to \text{right})$$

and $\varDelta = \alpha : A{\to}B$. In this case, $\varPi \not\precsim \alpha\cup\{a\} : B$ holds because of the Label Condition and $\varPi \not\precsim \varDelta$. Then we have

$$\dfrac{\begin{matrix}\vdots\ \text{i.h.}\\ \{a\} : A, \varGamma \Rightarrow \alpha\cup\{a\} : B\end{matrix}}{\varGamma \Rightarrow \alpha : A{\to}B}\ (\to \text{right})$$

where the Label Condition still holds. ∎

Lemma 3.3 (Main Lemma) *Let $X \in \{{}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$. Suppose that $\mathbf{LM}X \vdash \varGamma, \varPi \Rightarrow \varDelta, \varSigma$ where $\varPi \not\precsim \varDelta$ and $\varSigma \neq \emptyset$. Then, at least one of the following two conditions holds:*

(1) $\mathbf{LM}X \vdash \varGamma \Rightarrow \varDelta$.

(2) There exist multisets $\varGamma_1, \varGamma_2, \ldots, \varGamma_n$ for some $n \geq 1$ such that

(2-1) $\mathcal{L}(\varGamma_1, \ldots, \varGamma_n) \subseteq \mathcal{L}(\varGamma, \varDelta)$;

(2-2) $\mathbf{LM}X \vdash \varGamma_i, \varPi \Rightarrow \varSigma$, *for* $i = 1, \ldots, n$; *and*

(2-3) the inference

$$\dfrac{\varGamma_1, \varPhi \Rightarrow \varPsi \quad \varGamma_2, \varPhi \Rightarrow \varPsi \quad \cdots \quad \varGamma_n, \varPhi \Rightarrow \varPsi}{\varGamma, \varPhi \Rightarrow \varPsi, \varDelta}\ \mathcal{R}^{\langle \varGamma_1; \cdots; \varGamma_n \rangle}_{\langle \varGamma \rangle \langle \varDelta \rangle}$$

is derivable in $\mathbf{LM}X$*; that is, for arbitrary $\varPhi$ and $\varPsi$, there is an* $\mathbf{LM}X$*-derivation from $\varGamma_i, \varPhi \Rightarrow \varPsi$ ($i = 1, \ldots, n$) to $\varGamma, \varPhi \Rightarrow \varPsi, \varDelta$.*

Proof The pair $\langle \varGamma; \varDelta \rangle$ will be called the *dropped part*, and the list $\langle \varGamma_1; \ldots; \varGamma_n \rangle$ will be called an *essential antecedents list*. Roughly speaking, the condition (2) expresses that the list $\langle \varGamma_1; \ldots; \varGamma_n \rangle$ is an essence of the dropped part $\langle \varGamma; \varDelta \rangle$.

The proof of this lemma is done by induction on the $\mathbf{LM}X$-proof of $\varGamma; \varPi \Rightarrow \varDelta; \varSigma$. (We will use semicolons in a sequent to indicate the boundary between the dropped part and the remaining part.) We divide cases according to the last inference of the proof.

(Case 1): $\varGamma; \varPi \Rightarrow \varDelta; \varSigma$ is an axiom $\alpha : A;\ \Rightarrow\ ; \alpha : A$ where $\varGamma = \varSigma = \{\alpha : A\}$ and $\varPi = \varDelta = \emptyset$. In this case, the condition (2) holds because $\langle \alpha : A \rangle$ is an essential antecedents list ($n = 1$).

(Case 2): $\varGamma; \varPi \Rightarrow \varDelta; \varSigma$ is an axiom $\ ; \alpha : A \Rightarrow\ ; \alpha : A$ where $\varPi = \varSigma = \{\alpha : A\}$ and $\varGamma = \varDelta = \emptyset$. In this case, the condition (2) holds because $\langle \emptyset \rangle$ is an essential antecedents list ($n = 1$).

(Case 3): The last inference is

$$\frac{\Gamma', \Pi \Rightarrow \Delta, \Sigma}{\alpha\!:\!A, \Gamma';\ \Pi \Rightarrow \Delta;\ \Sigma}\ \text{(weakening left)}$$

where $\Gamma = (\alpha\!:\!A, \Gamma')$. The induction hypothesis for the upper sequent $\Gamma'; \Pi \Rightarrow \Delta; \Sigma$ is available: (1) $\mathbf{LM}X \vdash \Gamma' \Rightarrow \Delta$; or (2) there is an essential antecedents list for the upper sequent. In (1), we have $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$ by the weakening rule. In (2), it is easy to show that the list is also an essential antecedents list for $\Gamma; \Pi \Rightarrow \Delta; \Sigma$.

(Case 4): The last inference is

$$\frac{\Gamma, \Pi' \Rightarrow \Delta, \Sigma}{\Gamma;\ \alpha\!:\!A, \Pi' \Rightarrow \Delta;\ \Sigma}\ \text{(weakening left)}$$

where $\Pi = (\alpha\!:\!A, \Pi')$. Since $\Pi' \not\gtrsim \Delta$, the induction hypothesis for the upper sequent $\Gamma; \Pi' \Rightarrow \Delta; \Sigma$ is available: (1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or (2) there is an essential antecedents list for the upper sequent. (1) is the required condition. In (2), it is easy to show that the list is also an essential antecedents list for $\Gamma; \Pi \Rightarrow \Delta; \Sigma$.

(Case 5): The last inference is

$$\frac{\Gamma, \Pi \Rightarrow \Delta', \Sigma}{\Gamma;\ \Pi \Rightarrow \Delta', \alpha\!:\!A;\ \Sigma}\ \text{(weakening right)}$$

where $\Delta = (\Delta', \alpha\!:\!A)$. This case is similar to the Case 3. (Note that $\Pi \not\gtrsim \Delta'$.)

(Case 6): The last inference is

$$\frac{\Gamma, \Pi \Rightarrow \Delta, \Sigma'}{\Gamma;\ \Pi \Rightarrow \Delta;\ \Sigma', \alpha\!:\!A}\ \text{(weakening right)}$$

where $\Sigma = (\Sigma', \alpha : A)$. If $\Sigma' \neq \emptyset$, this is similar to the Case 4. If $\Sigma' = \emptyset$, the condition (1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$ holds since Lemma 3.2.

(Case 7): The last inference is

$$\frac{\alpha\!:\!A, \alpha\!:\!A, \Gamma', \Pi \Rightarrow \Delta, \Sigma}{\alpha\!:\!A, \Gamma';\ \Pi \Rightarrow \Delta;\ \Sigma}\ \text{(contraction left)}$$

where $\Gamma = (\alpha : A, \Gamma')$. The induction hypothesis for the upper sequent $\alpha\!:\!A, \alpha\!:\!A, \Gamma'; \Pi \Rightarrow \Delta; \Sigma$ is available: (1) $\mathbf{LM}X \vdash \alpha\!:\!A, \alpha\!:\!A, \Gamma' \Rightarrow \Delta$; or (2) there is an essential antecedents list for the upper sequent. In (1), we have $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$ by the contraction rule. In (2), it is easy to show that the list is also an essential antecedents list for $\Gamma; \Pi \Rightarrow \Delta; \Sigma$.

(Case 8): The last inference is

$$\frac{\Gamma, \alpha\!:\!A, \alpha\!:\!A, \Pi' \Rightarrow \Delta, \Sigma}{\Gamma;\ \alpha\!:\!A, \Pi' \Rightarrow \Delta;\ \Sigma}\ \text{(contraction left)}$$

where $\Pi = (\alpha\!:\!A, \Pi')$. The induction hypothesis for the upper sequent $\Gamma; \alpha\!:\!A, \alpha\!:\!A, \Pi' \Rightarrow \Delta; \Sigma$ is available: (1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or (2) there is an essential antecedents list for the upper sequent. (1) is the required condition. In (2), it is easy to show that the list is also an essential antecedents list for $\Gamma; \Pi \Rightarrow \Delta; \Sigma$.

(Case 9): The last inference is

$$\frac{\Gamma, \Pi \Rightarrow \Delta', \alpha\!:\!A, \alpha\!:\!A, \Sigma}{\Gamma;\ \Pi \Rightarrow \Delta', \alpha\!:\!A;\ \Sigma}\ \text{(contraction right)}$$

where $\Delta = (\Delta', \alpha\!:\!A)$. This case is similar to the Case 7.

(Case 10): The last inference is

$$\frac{\Gamma, \Pi \Rightarrow \Delta, \Sigma', \alpha{:}A, \alpha{:}A}{\Gamma;\ \Pi \Rightarrow \Delta;\ \Sigma', \alpha{:}A}\ \text{(contraction right)}$$

where $\Sigma = (\Sigma', \alpha{:}A)$. This case is similar to the Case 8.

(Case 11): The last inference is

$$\frac{\begin{array}{c}\vdots\ P \\ \Gamma', \Pi \Rightarrow \beta{:}A\end{array} \quad \begin{array}{c}\vdots\ Q \\ \alpha\cup\beta{:}B, \Gamma', \Pi \Rightarrow \Delta, \Sigma\end{array}}{\alpha{:}A{\to}B, \Gamma';\ \Pi \Rightarrow \Delta;\ \Sigma}\ (\to \text{left})$$

where $\Gamma = (\alpha{:}A{\to}B, \Gamma')$.

(Subcase 11-1): $\{\alpha\cup\beta{:}B\} \precsim \Delta$. In this case, we have $\Pi \not\precsim \{\beta{:}A\}$ because otherwise $\Pi \precsim \Delta$ holds. Then we have

(†) $\mathbf{LM}X \vdash \Gamma' \Rightarrow \beta{:}A$

by applying the Lemma 3.2 to P. On the other hand, the induction hypothesis is available for Q in which the dropped part is $\langle \alpha\cup\beta{:}B, \Gamma'; \Delta\rangle$. Thus we have

(1) $\mathbf{LM}X \vdash \alpha\cup\beta{:}B, \Gamma' \Rightarrow \Delta$; or

(2) there is an essential antecedents list $\langle \Gamma_1; \ldots; \Gamma_n\rangle$ for $\alpha\cup\beta{:}B, \Gamma'; \Pi \Rightarrow \Delta; \Sigma$.

In (1), we have $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$ by (†) and the rule "$\to$ left". In (2), the list is also an essential antecedents list for $\Gamma; \Pi \Rightarrow \Delta; \Sigma$. This is proved as follows. (2-1) $\mathcal{L}(\Gamma_1, \ldots, \Gamma_n) \subseteq \mathcal{L}(\Gamma, \Delta)$ is verified by the induction hypothesis and the condition $\{\alpha\cup\beta{:}B\} \precsim \Delta$. (2-2) $\mathbf{LM}X \vdash \Gamma_i, \Pi \Rightarrow \Sigma$ $(i = 1, \ldots, n)$ is just the induction hypothesis. (2-3) The derivability of $\mathcal{R}^{\langle \Gamma_1; \cdots; \Gamma_n\rangle}_{\langle \Gamma\rangle\langle \Delta\rangle}$ is shown by

$$\frac{\begin{array}{c}\vdots\ (\dagger) \\ \Gamma' \Rightarrow \beta{:}A\end{array} \quad \begin{array}{c}\Gamma_i, \Phi \Rightarrow \Psi\ (i = 1, \ldots, n) \\ \vdots\ \mathcal{R}^{\langle \Gamma_1; \cdots; \Gamma_n\rangle}_{\langle \alpha\cup\beta:B, \Gamma'\rangle\langle \Delta\rangle}\ \text{(i.h.)} \\ \alpha\cup\beta{:}B, \Gamma', \Phi \Rightarrow \Psi, \Delta\end{array}}{\alpha{:}A{\to}B, \Gamma', \Phi \Rightarrow \Psi, \Delta.}\ (\text{weakening}, \to \text{left})$$

(Subcase 11-2): $\{\alpha\cup\beta{:}B\} \not\precsim \Delta$. In this case, we have $(\alpha\cup\beta{:}B, \Pi) \not\precsim \Delta$, and the induction hypothesis is available for Q in which the dropped part is $\langle \Gamma'; \Delta\rangle$. Thus we have

(1) $\mathbf{LM}X \vdash \Gamma' \Rightarrow \Delta$; or

(2) there is an essential antecedents list $\langle \Gamma_1; \cdots; \Gamma_n\rangle$ for $\Gamma'; \alpha\cup\beta{:}B, \Pi \Rightarrow \Delta; \Sigma$.

In (1), we have $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$ by the weakening rule. In (2), the list

$$\mathcal{A} = \langle \Gamma_1, \Gamma\ ;\ \Gamma_2, \Gamma\ ;\ \cdots\ ;\ \Gamma_n, \Gamma\rangle$$

is an essential antecedents list for $\Gamma; \Pi \Rightarrow \Delta; \Sigma$. This is proved as follows. (2-1) $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\Gamma, \Delta)$ is obvious by the induction hypothesis. (2-2) For $i = 1, \ldots, n$, the fact $\mathbf{LM}X \vdash \Gamma_i, \Gamma, \Pi \Rightarrow \Sigma$ is shown by

$$\frac{\begin{array}{c}\vdots\ P \\ \Gamma', \Pi \Rightarrow \beta{:}A\end{array} \quad \begin{array}{c}\vdots\ \text{i.h.} \\ \Gamma_i, \alpha\cup\beta{:}B, \Pi \Rightarrow \Sigma\end{array}}{\Gamma_i, \alpha{:}A{\to}B, \Gamma', \Pi \Rightarrow \Sigma.}\ (\text{weakening}, \to \text{left})$$

(2-3) The derivability of $\mathcal{R}^{\langle\mathcal{A}\rangle}_{\langle\Gamma\rangle\langle\Delta\rangle}$ is shown by

$$\dfrac{\dfrac{\Gamma_i,\Gamma,\Phi\Rightarrow\Psi\ (i=1,\ldots,n)}{\vdots\ \mathcal{R}^{\langle\Gamma_1;\cdots;\Gamma_n\rangle}_{\langle\Gamma'\rangle\langle\Delta\rangle}\ \text{(i.h.)}}}{\dfrac{\Gamma',\Gamma,\Phi\Rightarrow\Psi,\Delta}{\Gamma,\Phi\Rightarrow\Psi,\Delta.}\ \text{(contraction)}}$$

(Case 12): The last inference is

$$\dfrac{\begin{matrix}\vdots\ P\\ \Gamma,\Pi'\Rightarrow\beta{:}A\end{matrix}\quad\begin{matrix}\vdots\ Q\\ \Gamma,\alpha\cup\beta{:}B,\Pi'\Rightarrow\Delta,\Sigma\end{matrix}}{\Gamma;\ \alpha{:}A{\to}B,\Pi'\Rightarrow\Delta;\ \Sigma}\ (\to\text{ left})$$

where $\Pi=(\alpha{:}A{\to}B,\Pi')$. In this case, we have $(\alpha\cup\beta{:}B,\Pi')\not\precsim\Delta$ because $(\beta{:}A{\to}B,\Pi')\not\precsim\Delta$, and the induction hypothesis is available for Q in which the dropped part is $\langle\Gamma;\Delta\rangle$. Thus we have

(1) $\mathbf{LM}X\vdash\Gamma\Rightarrow\Delta$; or

(2) there is an essential antecedents list $\langle\Gamma_1;\cdots;\Gamma_n\rangle$ for $\Gamma;\alpha\cup\beta{:}B,\Pi'\Rightarrow\Delta;\Sigma$.

(1) is just the required condition. In (2), the list

$$\mathcal{A}=\langle\Gamma_1,\Gamma\ ;\ \Gamma_2,\Gamma\ ;\ \cdots\ ;\ \Gamma_n,\Gamma\rangle$$

is an essential antecedents list for $\Gamma;\Pi\Rightarrow\Delta;\Sigma$. This is proved as follows. (2-1) $\mathcal{L}(\mathcal{A})\subseteq\mathcal{L}(\Gamma,\Delta)$ is obvious by the induction hypothesis. (2-2) For $i=1,\ldots,n$, the fact $\mathbf{LM}X\vdash\Gamma_i,\Gamma,\Pi\Rightarrow\Sigma$ is shown by

$$\dfrac{\begin{matrix}\vdots\ P\\ \Gamma,\Pi'\Rightarrow\beta{:}A\end{matrix}\quad\begin{matrix}\vdots\ \text{i.h.}\\ \Gamma_i,\alpha\cup\beta{:}B,\Pi'\Rightarrow\Sigma\end{matrix}}{\Gamma_i,\Gamma,\alpha{:}A{\to}B,\Pi'\Rightarrow\Sigma.}\ (\text{weakening, }\to\text{ left})$$

(2-3) The derivability of $\mathcal{R}^{\langle\mathcal{A}\rangle}_{\langle\Gamma\rangle\langle\Delta\rangle}$ is shown by

$$\dfrac{\dfrac{\Gamma_i,\Gamma,\Phi\Rightarrow\Psi\ (i=1,\ldots,n)}{\vdots\ \mathcal{R}^{\langle\Gamma_1;\cdots;\Gamma_n\rangle}_{\langle\Gamma\rangle\langle\Delta\rangle}\ \text{(i.h.)}}}{\dfrac{\Gamma,\Gamma,\Phi\Rightarrow\Psi,\Delta}{\Gamma,\Phi\Rightarrow\Psi,\Delta.}\ \text{(contraction)}}$$

(Case 13): The last inference is

$$\dfrac{\Gamma,\Pi,\{a\}{:}A\Rightarrow\alpha\cup\{a\}{:}B}{\Gamma;\ \Pi\Rightarrow\ ;\ \alpha{:}A{\to}B}\ (\to\text{ right})$$

where $\Sigma=\{\alpha{:}A{\to}B\}$ and $\Delta=\emptyset$. In this case, $\langle\Gamma\rangle$ is an essential antecedents list ($n=1$) for $\Gamma;\Pi\Rightarrow;\alpha{:}A{\to}B$.

(Case 14): The last inference is

$$\dfrac{\alpha{:}A_i,\Gamma',\Pi\Rightarrow\Delta,\Sigma}{\alpha{:}A_1{\wedge}A_2,\Gamma';\ \Pi\Rightarrow\Delta;\ \Sigma}\ (\wedge\text{ left})$$

where $\Gamma=(\alpha:A_1{\wedge}A_2,\Gamma')$ and $i=1$ or 2. The induction hypothesis for the upper sequent $\alpha{:}A_i,\Gamma';\Pi\Rightarrow\Delta;\Sigma$ is available: (1) $\mathbf{LM}X\vdash\alpha{:}A_i,\Gamma'\Rightarrow\Delta$; or (2) there is an essential antecedents list for the upper sequent. In (1), we have $\mathbf{LM}X\vdash\Gamma\Rightarrow\Delta$ by the rule "$\wedge$ left". In (2), it is easy to show that the list is also an essential antecedents list for $\Gamma;\Pi\Rightarrow\Delta;\Sigma$.

(Case 15): The last inference is

$$\frac{\varGamma, \alpha\!:\!A_i, \varPi' \Rightarrow \varDelta, \varSigma}{\varGamma;\ \alpha\!:\!A_1\wedge A_2, \varPi' \Rightarrow \varDelta;\ \varSigma}\ (\wedge\ \text{left})$$

where $\varPi = (\alpha : A_1\wedge A_2, \varPi')$ and $i = 1$ or 2. The induction hypothesis for the upper sequent $\varGamma; \alpha\!:\!A_i, \varPi' \Rightarrow \varDelta; \varSigma$ is available: (1) $\mathbf{LM}X \vdash \varGamma \Rightarrow \varDelta$; or (2) there is an essential antecedents list for the upper sequent. (1) is the required condition. In (2), it is easy to show that the list is also an essential antecedents list for $\varGamma; \varPi \Rightarrow \varDelta; \varSigma$.

(Case 16): The last inference is

$$\frac{\varGamma, \varPi \Rightarrow \alpha\!:\!A \quad \varGamma, \varPi \Rightarrow \alpha\!:\!B}{\varGamma;\ \varPi \Rightarrow\ ;\ \alpha\!:\!A\wedge B}\ (\wedge\ \text{right})$$

where $\varSigma = \{\alpha : A\wedge B\}$ and $\varDelta = \emptyset$. In this case, $\langle\varGamma\rangle$ is an essential antecedents list ($n = 1$) for $\varGamma; \varPi \Rightarrow; \alpha\!:\!A\wedge B$.

(Case 17): The last inference is

$$\frac{\begin{array}{c}\vdots\ P\\ \alpha\!:\!A, \varGamma', \varPi \Rightarrow \varDelta, \varSigma\end{array} \quad \begin{array}{c}\vdots\ Q\\ \alpha\!:\!B, \varGamma', \varPi \Rightarrow \varDelta, \varSigma\end{array}}{\alpha\!:\!A\vee B, \varGamma';\ \varPi \Rightarrow \varDelta;\ \varSigma}\ (\vee\ \text{left})$$

where $\varGamma = (\alpha\!:\!A\vee B, \varGamma')$. The induction hypothesis is available for P in which the dropped part is $\langle\alpha\!:\!A, \varGamma'; \varDelta\rangle$. Thus we have

(A1) $\mathbf{LM}X \vdash \alpha\!:\!A, \varGamma' \Rightarrow \varDelta$; or

(A2) there is an essential antecedents list $\langle\varGamma_1^A; \cdots; \varGamma_m^A\rangle$ for $\alpha\!:\!A, \varGamma'; \varPi \Rightarrow \varDelta; \varSigma$.

Similarly, by the induction hypothesis for Q, we have

(B1) $\mathbf{LM}X \vdash \alpha\!:\!B, \varGamma' \Rightarrow \varDelta$; or

(B2) there is an essential antecedents list $\langle\varGamma_1^B; \cdots; \varGamma_n^B\rangle$ for $\alpha\!:\!B, \varGamma'; \varPi \Rightarrow \varDelta; \varSigma$.

Then we consider four subcases according to the choice of the above conditions.

(Subcase 17-1): (A1) and (B1). We have $\mathbf{LM}X \vdash \varGamma \Rightarrow \varDelta$ by "$\vee$ left".

(Subcase 17-2): (A1) and (B2). We show that the list $\langle\varGamma_1^B; \cdots; \varGamma_n^B\rangle$ is also an essential antecedents list for $\varGamma; \varPi \Rightarrow \varDelta; \varSigma$. The conditions (2-1) and (2-2) are obvious by the induction hypothesis. The condition (2-3) is shown by

$$\frac{\begin{array}{c}\vdots\ \text{(A1)}\\ \alpha\!:\!A, \varGamma' \Rightarrow \varDelta\end{array} \quad \begin{array}{c}\varGamma_i^B, \varPhi \Rightarrow \varPsi \quad (i = 1, \ldots, n)\\ \vdots\ \mathcal{R}^{\langle\varGamma_1^B;\cdots;\varGamma_n^B\rangle}_{\langle\alpha:B,\varGamma'\rangle\langle\varDelta\rangle}\ \text{(i.h.)}\\ \alpha\!:\!B, \varGamma', \varPhi \Rightarrow \varPsi, \varDelta\end{array}}{\alpha\!:\!A\vee B, \varGamma', \varPhi \Rightarrow \varPsi, \varDelta.}\ (\text{weakening}, \vee\ \text{left})$$

(Subcase 17-3): (A2) and (B1). Similar to 17-2.

(Subcase 17-4): (A2) and (B2). We show that the list $\langle\varGamma_1^A; \cdots; \varGamma_m^A; \varGamma_1^B; \cdots; \varGamma_n^B\rangle$ is an essential antecedents list for $\varGamma; \varPi \Rightarrow \varDelta; \varSigma$. The conditions (2-1) and (2-2) are obvious by the induction hypotheses. The condition (2-3) is shown by

$$\frac{\begin{array}{c}\varGamma_i^A, \varPhi \Rightarrow \varPsi \quad (i = 1, \ldots, m)\\ \vdots\ \mathcal{R}^{\langle\varGamma_1^A;\cdots;\varGamma_m^A\rangle}_{\langle\alpha:A,\varGamma'\rangle\langle\varDelta\rangle}\ \text{(i.h.)}\\ \alpha\!:\!A, \varGamma', \varPhi \Rightarrow \varPsi, \varDelta\end{array} \quad \begin{array}{c}\varGamma_i^B, \varPhi \Rightarrow \varPsi \quad (i = 1, \ldots, n)\\ \vdots\ \mathcal{R}^{\langle\varGamma_1^B;\cdots;\varGamma_n^B\rangle}_{\langle\alpha:B,\varGamma'\rangle\langle\varDelta\rangle}\ \text{(i.h.)}\\ \alpha\!:\!B, \varGamma', \varPhi \Rightarrow \varPsi, \varDelta\end{array}}{\alpha\!:\!A\vee B, \varGamma', \varPhi \Rightarrow \varPsi, \varDelta.}\ (\vee\ \text{left})$$

(Case 18): The last inference is

$$\dfrac{\begin{array}{cc} \vdots\, P & \vdots\, Q \\ \Gamma, \alpha\!:\!A, \Pi' \Rightarrow \Delta, \Sigma & \Gamma, \alpha\!:\!B, \Pi' \Rightarrow \Delta, \Sigma \end{array}}{\Gamma;\ \alpha\!:\!A\vee B, \Pi' \Rightarrow \Delta;\ \Sigma}\ (\vee\ \text{left})$$

where $\Pi = (\alpha\!:\!A\vee B, \Pi')$. The induction hypothesis is available for P in which the dropped part is $\langle \Gamma; \Delta\rangle$. Thus we have

(A1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or

(A2) there is an essential antecedents list $\langle \Gamma_1^A; \cdots; \Gamma_m^A\rangle$ for $\Gamma; \alpha\!:\!A, \Pi' \Rightarrow \Delta; \Sigma$.

Similarly, by the induction hypothesis for Q, we have

(B1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or

(B2) there is an essential antecedents list $\langle \Gamma_1^B; \cdots; \Gamma_n^B\rangle$ for $\Gamma; \alpha\!:\!B, \Pi' \Rightarrow \Delta; \Sigma$.

(A1) and (B1) are equivalent, and they are just the required condition (1). Then we assume both (A2) and (B2) hold. Consider the "product" $\langle \Gamma_i^A, \Gamma_j^B \mid 1 \le i \le m, 1 \le j \le n\rangle$ of the two lists; for example,

$$\langle \Gamma_1^A, \Gamma_1^B\ ;\ \Gamma_1^A, \Gamma_2^B\ ;\ \Gamma_1^A, \Gamma_3^B\ ;\ \Gamma_2^A, \Gamma_1^B\ ;\ \Gamma_2^A, \Gamma_2^B\ ;\ \Gamma_2^A, \Gamma_3^B\rangle$$

if $m = 2$ and $n = 3$. We show that this is an essential antecedents list for $\Gamma; \Pi \Rightarrow \Delta; \Sigma$. The condition (2-1) is obvious by the induction hypothesis. The condition (2-2) is shown by

$$\dfrac{\begin{array}{cc} \vdots\ \text{i.h.} & \vdots\ \text{i.h.} \\ \Gamma_i^A, \alpha\!:\!A, \Pi' \Rightarrow \Sigma & \Gamma_j^B, \alpha\!:\!B, \Pi' \Rightarrow \Sigma \end{array}}{\Gamma_i^A, \Gamma_j^B, \alpha\!:\!A\vee B, \Pi' \Rightarrow \Sigma.}\ (\text{weakening}, \vee\ \text{left})$$

The condition (2-3) is shown by

$$\begin{array}{c} \Gamma_i^A, \Gamma_j^B, \Phi \Rightarrow \Psi \quad (i = 1, \ldots, m, j = 1, \ldots, n) \\ \vdots\ \mathcal{R}_{\langle\Gamma\rangle\langle\Delta\rangle}^{\langle \Gamma_1^B; \cdots; \Gamma_n^B\rangle}\ \ (\text{i.h.}) \\ \Gamma_i^A, \Gamma, \Phi \Rightarrow \Psi, \Delta \quad (i = 1, \ldots, m) \\ \vdots\ \mathcal{R}_{\langle\Gamma\rangle\langle\Delta\rangle}^{\langle \Gamma_1^A; \cdots; \Gamma_m^A\rangle}\ \ (\text{i.h.}) \\ \dfrac{\Gamma, \Gamma, \Phi \Rightarrow \Psi, \Delta, \Delta}{\Gamma, \Phi \Rightarrow \Psi, \Delta.}\ (\text{contraction}) \end{array}$$

(Case 19): The last inference is

$$\dfrac{\Gamma, \Pi \Rightarrow \alpha\!:\!A_i}{\Gamma;\ \Pi \Rightarrow\ ;\ \alpha\!:\!A_1\vee A_2}\ (\vee\ \text{right})$$

where $\Sigma = \{\alpha\!:\!A_1\vee A_2\}$, $\Delta = \emptyset$, and $i = 1$ or 2. In this case, $\langle\Gamma\rangle$ is an essential antecedents list ($n = 1$) for $\Gamma; \Pi \Rightarrow; \alpha\!:\!A\vee B$.

This completes the proof of Lemma 3.3. ∎

Lemma 3.4 *Each inference rule of* $\mathbf{LK}X$ *is admissible in* $\mathbf{LM}X$*, for* $X = {}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}$*; that is, for each inference rule*

$$\frac{\mathcal{S}_1}{\mathcal{T}} \quad \textit{or} \quad \frac{\mathcal{S}_1 \quad \mathcal{S}_2}{\mathcal{T}}$$

of $\mathbf{LK}X$*, if* $\mathbf{LM}X \vdash \mathcal{S}_i$ *for all* i*, then* $\mathbf{LM}X \vdash \mathcal{T}$*.*

Proof We show the admissibility of "$\to$ left", "$\to$ right", "$\wedge$ right", and "$\vee$ right" rules of $\mathbf{LK}X$. (The other rules are shared by the two systems.)

[Admissibility of "$\to$ left"] Suppose (i) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \beta : A$, and (ii) $\mathbf{LM}X \vdash \alpha\cup\beta : B, \Pi \Rightarrow \Sigma$ where $\max(\alpha) \leq \max(\beta)$ if $X = {}^{\cup}\mathbf{T}$; then the goal is to show $\mathbf{LM}X \vdash \alpha : A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma$. We apply the Main Lemma 3.3 to (i) in which the dropped part is $\langle \Gamma; \Delta \rangle$; then we have

(1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or

(2) there exists an essential antecedents list $\langle \Gamma_1; \ldots; \Gamma_n \rangle$ for $\Gamma;\ \Rightarrow \Delta;\ \beta : A$.

In the case (1), we have $\mathbf{LM}X \vdash \alpha : A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma$ by the weakening rule. In the case (2), we have

$$\dfrac{\dfrac{\vdots\, P_1}{\alpha : A \to B, \Gamma_1, \Pi \Rightarrow \Sigma} \quad \cdots \quad \dfrac{\vdots\, P_n}{\alpha : A \to B, \Gamma_n, \Pi \Rightarrow \Sigma}}{\begin{array}{c}\vdots\ \mathcal{R}^{\langle \Gamma_1; \cdots; \Gamma_n \rangle}_{\langle \Gamma \rangle \langle \Delta \rangle} \\ \alpha : A \to B, \Gamma, \Pi \Rightarrow \Sigma, \Delta\end{array}}$$

where P_i is

$$\dfrac{\dfrac{\vdots\ (2\text{-}2)}{\Gamma_i \Rightarrow \beta : A} \qquad \dfrac{\vdots\ (\text{ii})}{\alpha\cup\beta : B, \Pi \Rightarrow \Sigma}}{\alpha : A \to B, \Gamma_i, \Pi \Rightarrow \Sigma.}\ (\text{weakening}, \to \text{left})$$

[Admissibility of "$\to$ right"] Suppose (i) $\mathbf{LM}X \vdash \{a\} : A, \Gamma \Rightarrow \Delta, \alpha\cup\{a\} : B$ where $a \notin \mathcal{L}(\Gamma, \Delta) \cup \alpha$ if $X = {}^{\cup}\mathbf{R}$, and $a \notin \mathcal{L}(\Gamma, \Delta) \cup \alpha$ and $\max(\alpha) < a$ if $X = {}^{\cup}\mathbf{T}$; then the goal is to show $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha : A \to B$. We apply the Main Lemma 3.3 to (i) in which the dropped part is $\langle \Gamma; \Delta \rangle$ (note that $\{a\} : A \not\precsim \Delta$ holds by the Label Condition); then we have

(1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or

(2) there exists an essential antecedents list $\langle \Gamma_1; \ldots; \Gamma_n \rangle$ for $\Gamma; \{a\} : A \Rightarrow \Delta; \alpha\cup\{a\} : B$.

In the case (1), we have $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha : A \to B$ by the weakening rule. In the case (2), we have

$$\dfrac{\dfrac{\dfrac{\vdots\ (2\text{-}2)}{\Gamma_1, \{a\} : A \Rightarrow \alpha\cup\{a\} : B}}{\Gamma_1 \Rightarrow \alpha : A \to B}\ (\to \text{right}) \quad \cdots \quad \dfrac{\dfrac{\vdots\ (2\text{-}2)}{\Gamma_n, \{a\} : A \Rightarrow \alpha\cup\{a\} : B}}{\Gamma_n \Rightarrow \alpha : A \to B}\ (\to \text{right})}{\begin{array}{c}\vdots\ \mathcal{R}^{\langle \Gamma_1; \cdots; \Gamma_n \rangle}_{\langle \Gamma \rangle \langle \Delta \rangle} \\ \Gamma \Rightarrow \alpha : A \to B, \Delta\end{array}}$$

where the Label Condition is satisfied by the condition (2-1).

[Admissibility of "$\wedge$ right"] Suppose (i) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha : A$, and (ii) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha : B$; then the goal is to show $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha : A \wedge B$. We apply the Main Lemma 3.3 to (i) in which the dropped part is $\langle \Gamma; \Delta \rangle$; then we have

(A1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or

(A2) there exists an essential antecedents list $\langle \Gamma^A_1; \ldots; \Gamma^A_m \rangle$ for $\Gamma;\ \Rightarrow \Delta;\ \alpha : A$.

Similarly, by the Main Lemma 3.3 for (ii), we have

(B1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or

(B2) there exists an essential antecedents list $\langle \Gamma^B_1; \ldots; \Gamma^B_n \rangle$ for $\Gamma;\ \Rightarrow \Delta;\ \alpha : B$.

If (A1) or (B1) holds, then we have $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha\colon A\wedge B$ by the weakening rule. If both (A2) and (B2) hold, then we have

$$\begin{array}{c}
\vdots\ P_{i,j} \\
\Gamma_i^A, \Gamma_j^B \Rightarrow \alpha\colon A\wedge B \quad (i=1,\ldots,m, j=1,\ldots,n) \\
\vdots\ \mathcal{R}^{\langle \Gamma_1^B;\cdots;\Gamma_n^B\rangle}_{\langle\Gamma\rangle\langle\Delta\rangle} \\
\Gamma_i^A, \Gamma \Rightarrow \alpha\colon A\wedge B, \Delta \quad (i=1,\ldots,m) \\
\vdots\ \mathcal{R}^{\langle \Gamma_1^A;\cdots;\Gamma_n^A\rangle}_{\langle\Gamma\rangle\langle\Delta\rangle} \\
\dfrac{\Gamma, \Gamma \Rightarrow \alpha\colon A\wedge B, \Delta, \Delta}{\Gamma \Rightarrow \alpha\colon A\wedge B, \Delta}\ \text{(contraction)}
\end{array}$$

where $P_{i,j}$ is

$$\dfrac{\begin{array}{c}\vdots\ \text{(A2-2)} \\ \Gamma_i^A \Rightarrow \alpha\colon A\end{array} \qquad \begin{array}{c}\vdots\ \text{(B2-2)} \\ \Gamma_j^B \Rightarrow \alpha\colon B\end{array}}{\Gamma_i^A, \Gamma_j^B \Rightarrow \alpha\colon A\wedge B.}\ \text{(weakening, } \wedge \text{ right)}$$

[Admissibility of "$\vee$ right"] Suppose (i) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha\colon Z$ where $Z = A$ or B; then the goal is to show $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha\colon A\vee B$. We apply the Main Lemma 3.3 to (i) in which the dropped part is $\langle \Gamma; \Delta\rangle$; then we have

(1) $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$; or

(2) there exists an essential antecedents list $\langle \Gamma_1; \ldots; \Gamma_n\rangle$ for $\Gamma;\ \Rightarrow \Delta;\ \alpha\colon Z$.

In the case (1), we have $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta, \alpha\colon A\vee B$ by the weakening rule. In the case (2), we have

$$\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{(2-2)} \\ \Gamma_1 \Rightarrow \alpha\colon Z\end{array}}{\Gamma_1 \Rightarrow \alpha\colon A\vee B}\ (\vee\ \text{right}) \quad \cdots \quad \dfrac{\begin{array}{c}\vdots\ \text{(2-2)} \\ \Gamma_n \Rightarrow \alpha\colon Z\end{array}}{\Gamma_n \Rightarrow \alpha\colon A\vee B}\ (\vee\ \text{right})}{\vdots\ \mathcal{R}^{\langle \Gamma_1;\cdots;\Gamma_n\rangle}_{\langle\Gamma\rangle\langle\Delta\rangle}} \\
\Gamma \Rightarrow \alpha\colon A\vee B, \Delta
\end{array}$$

∎

Theorem 3.5 *If* $\mathbf{LK}X \vdash \Gamma \Rightarrow \Delta$, *then* $\mathbf{LM}X \vdash \Gamma \Rightarrow \Delta$, *for* $X = {}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}$.

Proof By induction on the $\mathbf{LK}X$-proof of $\Gamma \Rightarrow \Delta$, using Lemma 3.4. ∎

Theorem 3.6 *If* $\mathbf{LK}X \vdash \Gamma \Rightarrow \alpha\colon A$, *then* $\mathbf{LJ}X \vdash \Gamma \Rightarrow \alpha\colon A$, *for* $X = {}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}$.

Proof By Theorems 3.5 and 3.1. ∎

4 Weakening elimination

In this section we show that $\mathbf{LJ}X$ can be changed into a system without the weakening rule.

The labelled sequent calculi $\mathbf{LI}X$ ($X = {}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}$) is defined as follows.

Axioms: $\alpha{:}A \Rightarrow \alpha{:}A$.

Inference rules:

$$\frac{\alpha{:}A,\ \alpha{:}A,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A,\ \Gamma \Rightarrow \tau{:}F}\ \text{(contraction)}$$

$$\frac{\Gamma \Rightarrow \beta{:}A \qquad \alpha{\cup}\beta{:}B,\ \Pi \Rightarrow \tau{:}F}{\alpha{:}A{\to}B,\ \Gamma,\Pi \Rightarrow \tau{:}F}\ (\to \text{left})\ \text{with the proviso:}$$

(Label Condition): $\max(\alpha) \le \max(\beta)$ if $X = {}^{\cup}\mathbf{T}$.

$$\frac{\{a\}{:}A,\ \Gamma \Rightarrow \alpha{\cup}\{a\}{:}B}{\Gamma \Rightarrow \alpha{:}A{\to}B}\ (\to \text{right})\ \text{with the proviso:}$$

(Label Condition): $a \notin \mathcal{L}(\Gamma) \cup \alpha$ if $X = {}^{\cup}\mathbf{R}$. $a \notin \mathcal{L}(\Gamma) \cup \alpha$ and $\max(\alpha) < a$ if $X = {}^{\cup}\mathbf{T}$.

$$\frac{\alpha{:}A,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A{\wedge}B,\ \Gamma \Rightarrow \tau{:}F}\ (\wedge\ \text{left}) \qquad \frac{\alpha{:}B,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A{\wedge}B,\ \Gamma \Rightarrow \tau{:}F}\ (\wedge\ \text{left})$$

$$\frac{\Gamma \Rightarrow \alpha{:}A \qquad \Pi \Rightarrow \alpha{:}B}{\Gamma,\Pi \Rightarrow \alpha{:}A{\wedge}B}\ (\wedge\ \text{right})$$

$$\frac{\alpha{:}A,\ \Gamma \Rightarrow \tau{:}F \qquad \alpha{:}B,\ \Pi \Rightarrow \tau{:}F}{\alpha{:}A{\vee}B,\ \Gamma,\Pi \Rightarrow \tau{:}F}\ (\vee\ \text{left})$$

$$\frac{\Gamma \Rightarrow \alpha{:}A}{\Gamma \Rightarrow \alpha{:}A{\vee}B}\ (\vee\ \text{right}) \qquad \frac{\Gamma \Rightarrow \alpha{:}B}{\Gamma \Rightarrow \alpha{:}A{\vee}B}\ (\vee\ \text{right})$$

That is to say, $\mathbf{LI}X$ is obtained from $\mathbf{LJ}X$ by eliminating the weakening rule and modifying the other rules so that they can work enough without the weakening rule. For example, the "$\to$ left" rule

$$\frac{\Gamma \Rightarrow \beta{:}A \qquad \alpha{\cup}\beta{:}B,\ \Pi \Rightarrow \tau{:}F}{\alpha{:}A{\to}B,\ \Gamma,\Pi \Rightarrow \tau{:}F}$$

of $\mathbf{LI}X$ is stronger than the form

$$\frac{\Gamma \Rightarrow \beta{:}A \qquad \alpha{\cup}\beta{:}B,\ \Gamma \Rightarrow \tau{:}F}{\alpha{:}A{\to}B,\ \Gamma \Rightarrow \tau{:}F}$$

when the weakening rule is not available.

Lemma 4.1 *If* $\mathbf{LI}X \vdash \Gamma \Rightarrow \alpha{:}A$, *then* $\mathcal{L}(\Gamma) = \alpha$, *where* $X \in \{{}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$.

Proof By induction on $\mathbf{LI}X$-proofs. ∎

Theorem 4.2 *If* $\mathbf{LJ}X \vdash \Gamma \Rightarrow \alpha{:}A$, *then* $\mathbf{LI}X \vdash \Gamma' \Rightarrow \alpha{:}A$ *for some* $\Gamma' \subseteq \Gamma$, *where* $X \in \{{}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$.

Proof By induction on $\mathbf{LJ}X$-proofs, using Lemma 4.1 in the case of "$\to$ right". ∎

The system $\mathbf{LI}X$ will be used for the soundness proof in the next section, where Lemma 4.1 will play an important role for ${}^{\cup}\mathbf{T}$.

5 Soundness

In this section we show the soundness of $\mathbf{LI}X$, and finally we get the equivalence between the five conditions: $A \in X$, $\mathbf{LK}X \vdash \Rightarrow \emptyset{:}A$, $\mathbf{LM}X \vdash \Rightarrow \emptyset{:}A$, $\mathbf{LJ}X \vdash \Rightarrow \emptyset{:}A$, and $\mathbf{LI}X \vdash \Rightarrow \emptyset{:}A$.

Lemma 5.1 *The following hold for any $^{\cup}\mathbf{T}$-model $\langle\langle I, \cdot, \mathbf{e}, \preceq\rangle, V\rangle$ and any $\alpha, \beta \in I$.*

(1) $\alpha \preceq \alpha{\cdot}\beta$.

(2) $\alpha \preceq \beta$ implies $\alpha{\cdot}\beta \preceq \beta$.

Proof (1) $\mathbf{e}{\cdot}\alpha \preceq \beta{\cdot}\alpha$. (2) $\alpha{\cdot}\beta \preceq \beta{\cdot}\beta = \beta$. ∎

Let M be a $^{\cup}\mathbf{T}$-model $\langle\langle I, \cdot, \mathbf{e}, \preceq\rangle, V\rangle$, and $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a finite set of positive integers such that $a_1 < \ldots < a_n$. A mapping f from $\mathcal{A}$ to I is called an *interpretation of $\mathcal{A}$ on M* if

$$f(a_1)\cdot\cdots\cdot f(a_{i-1}) \preceq f(a_i)$$

holds for any i such that $2 \le i \le n$. If M is a $^{\cup}\mathbf{R}$-model $\langle\langle I, \cdot, \mathbf{e}\rangle, V\rangle$, then any mapping from $\mathcal{A}$ to I is said to be an interpretation. For each label $\tau = \{x_1, x_2, \ldots, x_m\} \subseteq \mathcal{A}$ where $x_1, \ldots, x_m$ are mutually distinct, we define an element $\tau^f \in I$ by

$$\{x_1, x_2, \ldots, x_m\}^f = \mathbf{e}{\cdot}f(x_1){\cdot}f(x_2)\cdot\cdots\cdot f(x_m).$$

Lemma 5.2 *If* $\max(\alpha) \le \max(\beta)$*, then $\alpha^f \preceq \beta^f$, where f is an interpretation of a finite set $\mathcal{A} \supseteq \alpha \cup \beta$ on a $^{\cup}\mathbf{T}$-model $\langle\langle I, \cdot, \mathbf{e}, \preceq\rangle, V\rangle$.*

Proof (Case 1): $\alpha = \emptyset$. In this case, $\alpha^f \preceq \beta^f$ is obvious because $\emptyset^f = \mathbf{e}$.

(Case 2): $\alpha = \{a_1, \ldots, a_m\}$ where $a_1 < \cdots < a_m$ and $m \ge 1$. In this case β is also a non-empty set. Then suppose that $\beta = \{b_1, \ldots, b_n\}$ where $b_1 < \cdots < b_n$ and $n \ge 1$, and that $\{x \in \mathcal{A} \mid x < b_n\} = \{c_1, \ldots, c_k\}$ where $c_1 < \cdots < c_k$. Using Lemma 5.1 and the definition of interpretation (i.e., $f(c_1)\cdots f(c_k) \preceq f(b_n)$), we have

$$\begin{array}{rcl}
\alpha^f & = & f(a_1)\cdot\cdots\cdot f(a_m) \\
& \preceq & \left\{\begin{array}{ll} f(c_1)\cdot\cdots\cdot f(c_k) & \text{if } a_m < b_n \\ f(c_1)\cdot\cdots\cdot f(c_k){\cdot}f(b_n) & \text{if } a_m = b_n \end{array}\right. \\
& \preceq & f(b_n) \preceq f(b_1)\cdot\cdots\cdot f(b_n) = \beta^f
\end{array}$$

∎

Let X be $^{\cup}\mathbf{R}$ or $^{\cup}\mathbf{T}$, M be an X-model, $\mathcal{S} = (\alpha_1{:}A_1, \ldots, \alpha_m{:}A_m \Rightarrow \tau{:}F)$ be a labelled sequent, and f be an interpretation of $\mathcal{L}(\mathcal{S})(= \alpha_1 \cup \cdots \cup \alpha_m \cup \tau)$ on M. We say that $\mathcal{S}$ is *valid in M with respect to f* if and only if at least one of the following conditions holds:

$$(\alpha_1^f \not\models_M A_1), \cdots, (\alpha_m^f \not\models_M A_m), (\tau^f \models_M F).$$

Theorem 5.3 (Soundness of $\mathbf{LI}^{\cup}\mathbf{R}$/$\mathbf{LI}^{\cup}\mathbf{T}$) *Let $X \in \{^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$. If a labelled sequent $\mathcal{S}$ is provable in $\mathbf{LI}X$, then $\mathcal{S}$ is valid in M with respect to f for any X-model M and any interpretation f of $\mathcal{L}(\mathcal{S})$ on M. (In particular, if $\mathbf{LI}X \vdash \Rightarrow \emptyset{:}F$, then $F \in X$.)*

Proof By induction on the $\mathbf{LI}X$-proof of $\mathcal{S}$. We show the nontrivial cases.

(Case 1) The last inference is

$$\frac{\Gamma \Rightarrow \beta{:}A \qquad \alpha{\cup}\beta{:}B,\ \Delta \Rightarrow \tau{:}F}{\alpha{:}A{\to}B,\ \Gamma, \Delta \Rightarrow \tau{:}F}\ (\to \text{left})$$

where $\Gamma = (\gamma_1 : C_1, \ldots, \gamma_c : C_c)$ and $\Delta = (\delta_1 : D_1, \ldots, \delta_d : D_d)$. We call the left upper sequent $\mathcal{S}_l$ and the right upper sequent $\mathcal{S}_r$. We assume that there exist an X-model M and an interpretation f of $\mathcal{L}(\mathcal{S})$ on M such that $\mathcal{S}$ is not valid in M w.r.t. f; then we will show that $\mathcal{S}_l$ or $\mathcal{S}_r$ is not valid in M. By the assumption, we have

(1) $\alpha^f \models_M A \to B$;

(2) $\gamma_i^f \models_M C_i$ $(i = 1, \dots, c)$;

(3) $\delta_i^f \models_M D_i$ $(i = 1, \dots, d)$; and

(4) $\tau^f \not\models_M F$.

Note that $\mathcal{L}(\mathcal{S}_l) \subseteq \mathcal{L}(\mathcal{S})$ and $\mathcal{L}(\mathcal{S}_r) = \mathcal{L}(\mathcal{S})$; these are derived by Lemma 4.1.

(Subcase 1-1): $\beta^f \not\models_M A$. In this case, the condition (2) implies that $\mathcal{S}_l$ is not valid in M w.r.t. f' where f' is the restriction of f on $\mathcal{L}(\mathcal{S}_l)$. Note that the condition of being an interpretation is preserved through the restriction of the domain.

(Subcase 1-2): $\beta^f \models_M A$. Suppose that $\alpha = \{x_1^0, \dots, x_k^0, x_1^1, \dots, x_a^1\}$, $\beta = \{x_1^0, \dots, x_k^0, x_1^2, \dots, x_b^2\}$ where $x_i^s \neq x_j^t$ if $s \neq t$ or $i \neq j$. The condition (1) and Lemma 5.2 (when $X = {}^{\cup}\mathbf{T}$) imply $\alpha^f \cdot \beta^f \models_M B$, which is equivalent to $(\alpha \cup \beta)^f \models_M B$ because

$$\begin{aligned}\alpha^f \cdot \beta^f &= \mathbf{e} \cdot f(x_1^0) \cdot \dots \cdot f(x_k^0) \cdot f(x_1^1) \cdot \dots \cdot f(x_a^1) \cdot \mathbf{e} \cdot f(x_1^0) \cdot \dots \cdot f(x_k^0) \cdot f(x_1^2) \cdot \dots \cdot f(x_b^2) \\ &= \mathbf{e} \cdot f(x_1^0) \cdot \dots \cdot f(x_k^0) \cdot f(x_1^1) \cdot \dots \cdot f(x_a^1) \cdot f(x_1^2) \cdot \dots \cdot f(x_b^2) \;=\; (\alpha \cup \beta)^f .\end{aligned}$$

This and the conditions (3) and (4) imply that $\mathcal{S}_r$ is not valid in M w.r.t. f.

(Case 2): The last inference is

$$\frac{\{a\}\!:\!A,\ \Gamma \Rightarrow \alpha \cup \{a\}\!:\!B}{\Gamma \Rightarrow \alpha\!:\!A \to B}\ (\to \text{right})$$

where $\Gamma = (\gamma_1 : C_1, \dots, \gamma_c : C_c)$. We call the upper sequent $\mathcal{S}'$. We assume that there exist an X-model $M = \langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V\rangle$ and an interpretation f of $\mathcal{L}(\mathcal{S})$ on M such that $\mathcal{S}$ is not valid in M w.r.t. f; then we will show that $\mathcal{S}'$ is not valid in M w.r.t. some interpretation. By the assumption, we have

(1) $\gamma_i^f \models_M C_i$ $(i = 1, \dots, c)$; and

(2) $\alpha^f \not\models_M A \to B$.

By (2), there exists an element $\xi \in I$ such that

(3) $\xi \models_M A$;

(4) $\alpha^f \cdot \xi \not\models_M B$; and

(5) $\alpha^f \preceq \xi$ when $X = {}^{\cup}\mathbf{T}$.

The Label Condition and Lemma 4.1 imply

(6) $\mathcal{L}(\mathcal{S}') = \alpha \cup \{a\}$ and $a \notin \alpha$; and

(7) $a > \max(\alpha)$ when $X = {}^{\cup}\mathbf{T}$.

We define a function f' from $\mathcal{L}(\mathcal{S}')$ to I by extending f with $f(a) = \xi$. Then f' is an interpretation of $\mathcal{L}(\mathcal{S}')$ on M (this is guaranteed by (5), (6) and (7) if $X = {}^{\cup}\mathbf{T}$), and (1), (3) and (4) imply that $\mathcal{S}'$ is not valid in M w.r.t. f'. ∎

By Theorems 2.2, 3.1, 3.5, (3.6,) 4.2, and 5.3, we establish the completeness and soundness of all the systems:

Theorem 5.4 (Main Theorem for ${}^{\cup}\mathbf{R}/{}^{\cup}\mathbf{T}$) *Let $X \in \{{}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$. The following conditions are equivalent.*

- $A \in X$ *(i.e., A is valid in any X-model).*
- $\mathbf{LK}X \vdash \Rightarrow \emptyset : A$.
- $\mathbf{LM}X \vdash \Rightarrow \emptyset : A$.
- $\mathbf{LJ}X \vdash \Rightarrow \emptyset : A$.
- $\mathbf{LI}X \vdash \Rightarrow \emptyset : A$.

6 Hilbert-style systems

In this section we present Hilbert-style systems for ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$, and we show the completeness and soundness of them.

First we introduce the basic system $\mathbf{HT}_{\to\wedge}$

Axiom schemes of $\mathbf{HT}_{\to\wedge}$:

$$
\begin{array}{l}
A\to A, \\
(A\to B)\to((B\to C)\to(A\to C)), \\
(B\to C)\to((A\to B)\to(A\to C)), \\
(A\to(A\to B))\to(A\to B), \\
(A\wedge B)\to A, \ \ (A\wedge B)\to B, \\
((A\to B)\wedge(A\to C))\to(A\to(B\wedge C)).
\end{array}
$$

Inference rules of $\mathbf{HT}_{\to\wedge}$:

$$
\frac{A \quad A\to B}{B}\ \text{(modus ponens)} \quad \frac{A \quad B}{A\wedge B}\ \text{(adjunction)}
$$

These are axioms and rules for the connectives $\to$ and $\wedge$ in the orthodox Hilbert-style formulation of $\mathbf{T}$.

Lemma 6.1 *The following inference rules are derivable in $\mathbf{HT}_{\to\wedge}$.*

$$
\frac{A\to B \quad B\to C}{A\to C}\ \text{(tr)} \qquad \frac{A\to B}{(C\to A)\to(C\to B)}\ \text{(pref)} \qquad \frac{A\to B}{(B\to C)\to(A\to C)}\ \text{(suff)}
$$

$$
\frac{A\to B}{(A\wedge C)\to B}\ (\wedge\ \text{left}) \qquad \frac{A\to B}{(C\wedge A)\to B}\ (\wedge\ \text{left}) \qquad \frac{A\to B \quad A\to C}{A\to(B\wedge C)}\ (\wedge\ \text{right})
$$

Proof Easy. ∎

Suppose that $\Gamma = \{A_1, \ldots, A_n\}$ is a non-empty multiset of formulas, and that A is a formula $A_1\wedge\cdots\wedge A_n$ where the association of $\wedge$ is arbitrary (e.g., $A = (A_1\wedge(A_2\wedge A_3))\wedge(A_4\wedge A_5)$). Then, we say that A is a *conjunction of* Γ, and it is represented by $\bigwedge\{\Gamma\}$.

Lemma 6.2 *If $\Gamma \supseteq \Delta \neq \emptyset$, then $\mathbf{HT}_{\to\wedge} \vdash \bigwedge\{\Gamma\}\to\bigwedge\{\Delta\}$ for any conjunctions $\bigwedge\{\Gamma\}$ and $\bigwedge\{\Delta\}$.*

Proof Easy, using "$\wedge$ left, right" in Lemma 6.1. ∎

To describe the systems for ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$, we give some more definitions.

Any mapping from the set of positive integers to the set of propositional variables is called *label translation*. Let $\alpha : A$ be a labelled formula and f be a label translation where $\alpha = \{a_1, a_2, \ldots, a_n\}$ and $a_1 < a_2 < \cdots < a_n$. The formula

$$f(a_1) \to \Big(f(a_2) \to \Big(\cdots \to \Big(f(a_{n-1}) \to \Big(f(a_n) \to A \Big) \Big) \cdots \Big) \Big)$$

is called the *translation of* $\alpha : A$ *by* f, and it is denoted by $(\alpha : A)^f$. Note that $(\emptyset : A)^f = A$.

Let $\mathcal{S} = (\alpha_1 : A_1, \ldots, \alpha_n : A_n \Rightarrow \beta : B)$ be a labelled sequent (in $\mathbf{LJ}X$-style), and f be a label translation. A formula of the form

$$\bigwedge \big\{ (\alpha_1 : A_1)^f, \ldots, (\alpha_n : A_n)^f \big\} \to (\beta : B)^f$$

is called a *translation of* $\mathcal{S}$ *by* f, and it is denoted by $\mathcal{S}^f$. For example

$$\Big((p \to A) \wedge (p \to (q \to B)) \Big) \to \Big(p \to (q \to C) \Big)$$

is a translation of the labelled sequent $\{1\} : A, \{1,2\} : B \Rightarrow \{1,2\} : C$ by f where $f(1) = p$ and $f(2) = q$. We say that f *is good for* $\mathcal{S}$ if and only if

- $a \neq b \Rightarrow f(a) \neq f(b)$, and
- $f(a) \notin \mathcal{V}(\mathcal{S})$

hold for any $a, b \in \mathcal{L}(\mathcal{S}) = \alpha_1 \cup \cdots \cup \alpha_n \cup \beta$ where $\mathcal{V}(\mathcal{S})$ denotes the set of propositional variables appearing in $\mathcal{S}$. In other words, f is good for S if and only if f translates the label elements of $\mathcal{S}$ into mutually distinct fresh variables.

Note that $\mathcal{S}^f$ denotes many formulas in general because of the arbitrariness of the conjunction. We will often treat a statement such as "$X \vdash \mathcal{S}^f$". The following theorem guarantees that such a statement does not cause ambiguity although $\mathcal{S}^f$ is not uniquely determined. (We will tacitly use this fact.)

Theorem 6.3 *Suppose that* $\mathcal{S}$ *is a labelled sequent,* f *is a label translation, and* P *and* Q *are arbitrary translations of* $\mathcal{S}$ *by* f*. Then both the formulas* $P \to Q$ *and* $Q \to P$ *are provable in* $\mathbf{HT}_{\to\wedge}$*. (Therefore, "*$X \vdash P$*" and "*$X \vdash Q$*" are equivalent for any supersystem* X *of* $\mathbf{HT}_{\to\wedge}$*.)*

Proof By Lemma 6.2 and the rule "suff" in Lemma 6.1. ∎

Now we present Hilbert-style systems $\mathbf{H}^{\cup}\mathbf{T}$ for ${}^{\cup}\mathbf{T}$ and $\mathbf{H}^{\cup}\mathbf{R}$ for ${}^{\cup}\mathbf{R}$.

- $\mathbf{H}^{\cup}\mathbf{T} = \mathbf{HT}_{\to\wedge}$ + ("$\vee$ right" axiom) + ("$\vee$ left" rule) + (substitution rule).
- $\mathbf{H}^{\cup}\mathbf{R} = \mathbf{H}^{\cup}\mathbf{T} + \mathbf{C}$.

These axioms and rules are described as follows.

- Axiom schemes "$\vee$ right": $A \to (A \vee B)$, and $B \to (A \vee B)$.
- Inference rule "$\vee$ left":

$$\frac{(\alpha : A, \mathit{\Gamma} \Rightarrow \tau : F)^f \quad (\alpha : B, \mathit{\Delta} \Rightarrow \tau : F)^f}{(\alpha : A \vee B, \mathit{\Gamma}, \mathit{\Delta} \Rightarrow \tau : F)^f} \; (\vee \text{ left})$$

where f is label translation that is good for this last sequent.

- Substitution rule:
$$\frac{A}{A[p{:=}B]}\ \text{(substitution)}$$
where $A[p{:=}B]$ denotes the formula obtained from the formula A by replacing each occurrence of the propositional variable p with the formula B.

- Axiom scheme **C**: $(A{\to}(B{\to}C)){\to}(B{\to}(A{\to}C))$.

Note that the $\{\to,\wedge,\vee\}$-fragments of the orthodox relevant logics **T** and **R** are usually defined by the Hilbert-style formulations:

$$\mathbf{T} = \mathbf{HT}_{\to\wedge} + (\text{“}\vee\text{ right” axiom}) + (\text{“}\vee\text{ left” axiom}) + (\text{distribution axiom}),$$

$$\mathbf{R} = \mathbf{T} + \mathbf{C}$$

where

$\vee$ left: $((A{\to}C){\wedge}(B{\to}C)){\to}((A{\vee}B){\to}C)$,

Distribution: $(A{\wedge}(B{\vee}C)){\to}((A{\wedge}B){\vee}C)$.

These two axioms are provable in the semilattice logic $\mathbf{H}^{\cup}\mathbf{T}$ (this fact can be verified by Theorem 6.9); thus we assert that

$${}^{\cup}\mathbf{T} = \mathbf{T} + (\text{“}\vee\text{ left” rule}) + (\text{substitution rule})$$

$${}^{\cup}\mathbf{R} = \mathbf{R} + (\text{“}\vee\text{ left” rule}) + (\text{substitution rule})$$

or shortly

$${}^{\cup}\mathbf{T} = \mathbf{T} + (\text{“}\vee\text{ left” rule})$$

$${}^{\cup}\mathbf{R} = \mathbf{R} + (\text{“}\vee\text{ left” rule})$$

under the assumption that the orthodox logics contain the substitution rule, which is an admissible rule. Another Hilbert-style formulation for ${}^{\cup}\mathbf{R}$ has been presented by [3], in which the extra inference rule is slightly different from our “$\vee$ left”.

We will show that $\mathbf{H}^{\cup}\mathbf{T}$ and $\mathbf{H}^{\cup}\mathbf{R}$ are complete and sound with respect to the semilattice models (Theorem 6.13).

Lemma 6.4 *The following inference rule is derivable in* $\mathbf{H}^{\cup}\mathbf{R}$*.*

$$\frac{A_1{\to}(\cdots{\to}(A_n{\to}(B{\to}(C{\to}D)))\cdots)}{A_1{\to}(\cdots{\to}(A_n{\to}(C{\to}(B{\to}D)))\cdots)}\ \text{(exchange) where } n \geq 0.$$

The following inference rules are derivable in $\mathbf{H}^{\cup}\mathbf{T}$ *and in* $\mathbf{H}^{\cup}\mathbf{R}$*.*

$$\frac{\bigwedge\{\Gamma\}{\to}A \qquad \bigwedge\{A,\Delta\}{\to}B}{\bigwedge\{\Gamma,\Delta\}{\to}B}\ \text{(cut) where } \Gamma \neq \emptyset.$$

$$\frac{A_1{\to}(\cdots{\to}(A_n{\to}(B{\to}(B{\to}C)))\cdots)}{A_1{\to}(\cdots{\to}(A_n{\to}(B{\to}C))\cdots)}\ \text{(contraction) where } n \geq 0.$$

$$\frac{A{\to}(B{\to}C)}{A{\to}((P{\to}B){\to}(P{\to}C))}\ (\text{pref}^{+}) \qquad \frac{A{\to}(B{\to}C)}{(P{\to}B){\to}(A{\to}(P{\to}C))}\ (\text{pref}^{\times})$$

$$\frac{A{\to}(B{\to}C)}{(A{\wedge}B){\to}C}\ ({\to}\text{ to }\wedge)$$

Proof Using Lemma 6.1. Here we show "cut", "pref$^\times$" and "$\to$ to $\wedge$". The others are easy.

(cut)

$$\dfrac{\begin{array}{c}\bigwedge\{\Gamma\}\to A\\ \vdots\ \wedge\text{ left, right, etc.}\\ \bigwedge\{\Gamma,\Delta\}\to\bigwedge\{A,\Delta\}\end{array}\qquad \bigwedge\{A,\Delta\}\to B}{\bigwedge\{\Gamma,\Delta\}\to B}\ (\mathrm{tr})$$

(pref$^\times$) Suppose $B_C = B\to C$, $P_C = P\to C$, and $X = (A\to B_C)\to((B_C\to P_C)\to(A\to P_C))$.

$$\dfrac{A\to(B\to C)\qquad \dfrac{X\ (\text{axiom})\qquad \dfrac{\dfrac{(P\to B)\to(B_C\to P_C)\ (\text{axiom})}{((B_C\to P_C)\to(A\to P_C))\to((P\to B)\to(A\to P_C))}\ (\text{suff})}{X\to((A\to B_C)\to((P\to B)\to(A\to P_C)))}\ (\text{pref})}{(A\to B_C)\to((P\to B)\to(A\to P_C))}\ (\text{m.p.})}{(P\to B)\to(A\to(P\to C))}\ (\text{m.p.})$$

($\to$ to $\wedge$)

$$\dfrac{\dfrac{\dfrac{A\to(B\to C)}{(A\wedge B)\to(B\to C)}\ (\wedge\text{ left})\qquad \dfrac{(A\wedge B)\to B\ (\text{axiom})}{(B\to C)\to((A\wedge B)\to C)}\ (\text{suff})}{(A\wedge B)\to((A\wedge B)\to C)}\ (\text{tr})}{(A\wedge B)\to C}\ (\text{contr.})$$

∎

In the following, f denotes an arbitrary label translation.

Lemma 6.5 *(1)* $\mathbf{H}^{\cup}\mathbf{T}\vdash((\alpha{:}A)^f\wedge(\alpha{:}B)^f)\to(\alpha{:}A\wedge B)^f$.

(2) $\mathbf{H}^{\cup}\mathbf{T}\vdash(\alpha{:}A\wedge B)^f\to(\alpha{:}A)^f$.

(3) $\mathbf{H}^{\cup}\mathbf{T}\vdash(\alpha{:}A\wedge B)^f\to(\alpha{:}B)^f$.

(4) $\mathbf{H}^{\cup}\mathbf{T}\vdash(\alpha{:}A)^f\to(\alpha{:}A\vee B)^f$.

(5) $\mathbf{H}^{\cup}\mathbf{T}\vdash(\alpha{:}B)^f\to(\alpha{:}A\vee B)^f$.

Proof By induction on the number of elements of α.

(Case 1): $\alpha=\emptyset$. In this case, (1)–(5) are just the axioms of $\mathbf{H}^{\cup}\mathbf{T}$.

(Case 2): $\alpha=\{a_1,a_2,\ldots,a_n\}$ where $a_1<a_2<\cdots<a_n$ and $n\geq 1$. Suppose that $\{a_2,\ldots,a_n\}=\alpha'$ and $f(a_1)=p$. Then we can show (1)–(5), using Lemma 6.1 and the induction hypotheses. For example, (1) is shown as follows.

(i) $\mathbf{H}^{\cup}\mathbf{T}\vdash\Big((p\to(\alpha'{:}A)^f)\wedge(p\to(\alpha'{:}B)^f)\Big)\to\Big(p\to((\alpha'{:}A)^f\wedge(\alpha'{:}B)^f)\Big)$ (axiom)

(ii) $\mathbf{H}^{\cup}\mathbf{T}\vdash\Big((\alpha'{:}A)^f\wedge(\alpha'{:}B)^f\Big)\to(\alpha'{:}A\wedge B)^f$ (induction hypothesis)

(iii) $\mathbf{H}^{\cup}\mathbf{T}\vdash\Big(p\to((\alpha'{:}A)^f\wedge(\alpha'{:}B)^f)\Big)\to\Big(p\to(\alpha'{:}A\wedge B)^f\Big)$ ((ii) and "pref")

(iv) $\mathbf{H}^{\cup}\mathbf{T}\vdash\Big((p\to(\alpha'{:}A)^f)\wedge(p\to(\alpha'{:}B)^f)\Big)\to\Big(p\to(\alpha'{:}A\wedge B)^f\Big)$ ((i), (iii) and "tr")

(This last formula is just the formula of (1)) ∎

Lemma 6.6 *If* $\max(\beta)\geq\max(\gamma)$, *then* $\mathbf{H}^{\cup}\mathbf{T}\vdash(\gamma{:}B\to C)^f\to((\beta{:}B)^f\to(\beta\cup\gamma{:}C)^f)$.

Proof For any label β and γ such that $\max(\beta) \geq \max(\gamma)$, we show

(A) $\mathbf{H}^{\cup}\mathbf{T} \vdash (\gamma\!:\!B{\to}C)^f \to ((\beta\!:\!B)^f \to (\beta{\cup}\gamma\!:\!C)^f)$; and

(B) if $\beta \neq \emptyset$, than $\mathbf{H}^{\cup}\mathbf{T} \vdash (\beta\!:\!B)^f \to ((\gamma\!:\!B{\to}C)^f \to (\beta{\cup}\gamma\!:\!C)^f)$.

((A) is just the desired proposition.) These are proved simultaneously by induction on $|\beta| + |\gamma|$ (i.e., the sum of the number of elements in β and γ), using Lemma 6.4.

(Case 1): $|\beta| + |\gamma| = 0$. (A) is an axiom $(B{\to}C){\to}(B{\to}C)$.

(Case 2): $|\beta| + |\gamma| \geq 1$.

(Subcase 2-1): $\gamma = \emptyset$. Let $\beta = \{b\} \cup \beta'$ where $b = \min(\beta)$ and $b \notin \beta'$. (A) and (B) are shown as follows.

$$\begin{array}{c} (B{\to}C) \to (B{\to}C) \\ \vdots\ (\text{pref}^{+}) \\ (B{\to}C) \to ((\beta\!:\!B)^f \to (\beta\!:\!C)^f) \end{array} \qquad \dfrac{\begin{array}{c} (B{\to}C) \to (B{\to}C) \\ \vdots\ (\text{pref}^{+}) \\ (B{\to}C) \to ((\beta'\!:\!B)^f \to (\beta'\!:\!C)^f) \end{array}}{(\beta\!:\!B)^f \to ((B{\to}C) \to (\beta\!:\!C)^f)}\ (\text{pref}^{\times})$$

(Subcase 2-2): $\gamma \neq \emptyset$. In this case, the hypothesis "$\max(\beta) \geq \max(\gamma)$" implies that $\beta \neq \emptyset$. Let $\beta = \{b\} \cup \beta'$ and $\gamma = \{c\} \cup \gamma'$ where $b = \min(\beta)$, $c = \min(\gamma)$, $b \notin \beta'$ and $c \notin \gamma'$.

(Subcase 2-2-1): $b < c$. In this case we have $\max(\beta') \geq \max(\gamma)$, and the induction hypothesis is available for β' and γ. Then (A) and (B) are shown by applying "pref^{+}" and "pref$^{\times}$", respectively, to the induction hypothesis (A):

$$\mathbf{H}^{\cup}\mathbf{T} \vdash (\gamma\!:\!B{\to}C)^f \to ((\beta'\!:\!B)^f \to (\beta'{\cup}\gamma\!:\!C)^f).$$

(Subcase 2-2-2): $c \leq b$. In this case we have $\max(\beta) \geq \max(\gamma')$, and the induction hypothesis is available for β and γ'. If $c < b$, then (A) and (B) are shown by applying "pref$^{\times}$" and "pref^{+}", respectively, to the induction hypothesis (B):

$$\mathbf{H}^{\cup}\mathbf{T} \vdash (\beta\!:\!B)^f \to ((\gamma'\!:\!B{\to}C)^f \to (\beta{\cup}\gamma'\!:\!C)^f).$$

If $c = b$, then we also apply "contraction"; for example, (A) is obtained as follows.

$$\dfrac{\dfrac{\begin{array}{c} \vdots\ (\text{i.h. (B)}) \\ (\beta\!:\!B)^f \to ((\gamma'\!:\!B{\to}C)^f \to (p{\to}(\beta'{\cup}\gamma'\!:\!C)^f)) \end{array}}{(\gamma\!:\!B{\to}C)^f \to ((\beta\!:\!B)^f \to (p{\to}(p{\to}(\beta'{\cup}\gamma'\!:\!C)^f)))}\ (\text{pref}^{\times})}{(\gamma\!:\!B{\to}C)^f \to ((\beta\!:\!B)^f \to (p{\to}(\beta'{\cup}\gamma'\!:\!C)^f))}\ (\text{contr.})$$

where $p = f(c)$. (Note that $(p{\to}(\beta'{\cup}\gamma'\!:\!C)^f) = (\beta{\cup}\gamma'\!:\!C)^f = (\beta{\cup}\gamma\!:\!C)^f$.) ∎

Lemma 6.7 $\mathbf{H}^{\cup}\mathbf{R} \vdash (\gamma\!:\!B{\to}C)^f \to ((\beta\!:\!B)^f \to (\beta{\cup}\gamma\!:\!C)^f)$ *(for arbitrary β and γ).*

Proof This lemma is proved by induction on $|\beta| + |\gamma|$ similarly to Lemma 6.6, using an inference rule

$$\dfrac{A{\to}(B{\to}C)}{(P{\to}A){\to}(B{\to}(P{\to}C))}$$

which is derivable by "exchange" and "pref$^{\times}$" (or "pref^{+}"). ∎

For a multiset $\Delta = \{H_1, \ldots, H_n\}$ of formulas, we define a multiset $\Delta^{\#}$ of labelled formulas by

$$\Delta^{\#} = \{\emptyset\!:\!H_1{\to}H_1,\ \emptyset\!:\!H_2{\to}H_2,\ \ldots,\ \emptyset\!:\!H_n{\to}H_n\}.$$

Lemma 6.8 (Translation of LIX-proofs into HX-proofs) *Let $X \in \{{}^{\cup}\mathbf{T}, {}^{\cup}\mathbf{R}\}$. If $\mathbf{LI}X \vdash \Gamma \Rightarrow \alpha{:}A$, then there is a finite multiset Δ of formulas that satisfies the following.*

(1) $\mathbf{H}X \vdash (\Delta^{\#}, \Gamma \Rightarrow \alpha{:}A)^f$ for any label translation f that is good for $\Delta^{\#}, \Gamma \Rightarrow \alpha{:}A$.

(2) $\Delta \cup \Gamma$ is non-empty.

(The multiset Δ will be called a history *for $\Gamma \Rightarrow \alpha{:}A$.)*

Proof By induction on the $\mathbf{LI}X$-proof of $\Gamma \Rightarrow \alpha{:}A$. We divide cases according to the last inference of the proof. In any case, the condition (2) is easily verified; so we will show only the condition (1).

(Case1): $\Gamma \Rightarrow \alpha{:}A$ is an axiom $\alpha{:}A \Rightarrow \alpha{:}A$. In this case the empty set is a history.

(Case 2): The last inference is

$$\dfrac{\beta{:}B, \beta{:}B, \Gamma' \Rightarrow \alpha{:}A}{\beta{:}B, \Gamma' \Rightarrow \alpha{:}A.}\ (\text{contraction})$$

We get a history for the upper sequent by the induction hypothesis, and this is also a history for the lower sequent. The condition (1) is shown by the induction hypothesis and the fact

$$\mathbf{H}^{\cup}\mathbf{T} \vdash \bigwedge\{(\beta{:}B)^f, \Pi\} \to \bigwedge\{(\beta{:}B)^f, (\beta{:}B)^f, \Pi\}.$$

(Case 3): The last inference is

$$\dfrac{\Pi \Rightarrow \beta{:}B \qquad \beta\cup\gamma{:}C,\ \Sigma \Rightarrow \alpha{:}A}{\gamma{:}B\to C,\ \Pi, \Sigma \Rightarrow \alpha{:}A}\ (\to \text{left})$$

where $\max(\beta) \geq \max(\gamma)$ if $X = {}^{\cup}\mathbf{T}$. Let $\mathcal{S}_l$, $\mathcal{S}_r$, and $\mathcal{S}$ be the left upper sequent, the right upper sequent, and the lower sequent, respectively. By the induction hypotheses, we get histories Δ_l and Δ_r for $\mathcal{S}_l$ and $\mathcal{S}_r$ respectively. We show that $\Delta_l \cup \Delta_r$ is a history for $\mathcal{S}$. Suppose that f is a label translation that is good for the sequent $(\Delta_l^{\#}, \Delta_r^{\#}, \mathcal{S})$. Then f is good also for the sequent $(\Delta_x^{\#}, \mathcal{S}_x)$ for $x = l, r$. This fact is verified by the property

$$a \in \mathcal{L}(\mathcal{S}_l) \cup \mathcal{L}(\mathcal{S}_r) \Rightarrow a \in \mathcal{L}(\mathcal{S}),$$

which is shown by the use of Lemma 4.1. Then, using Lemma 6.4, we have

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \text{i.h.}\\ (\Delta_l^{\#}, \Pi \Rightarrow \beta{:}B)^f\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \text{Lemma 6.6 or 6.7}\\ (\gamma{:}B\to C)^f \to ((\beta{:}B)^f \to (\beta\cup\gamma{:}C)^f)\end{array}}{((\gamma{:}B\to C)^f \wedge (\beta{:}B)^f) \to (\beta\cup\gamma{:}C)^f}\ (\to\text{to}\wedge)}{(\Delta_l^{\#}, \gamma{:}B\to C, \Pi \Rightarrow \beta\cup\gamma{:}C)^f}\ (\text{cut})(\dagger) \qquad \begin{array}{c}\vdots\ \text{i.h.}\\ (\Delta_r^{\#}, \beta\cup\gamma{:}C, \Sigma \Rightarrow \alpha{:}A)^f\end{array}}{(\Delta_l^{\#}, \Delta_r^{\#}, \gamma{:}B\to C, \Pi, \Sigma \Rightarrow \alpha{:}A)^f.}\ (\text{cut})$$

Note that the use of cut (†) is legal: $\Delta_l^{\#} \cup \Pi \neq \emptyset$ (induction hypothesis (2) for $\mathcal{S}_l$).

(Case 4): The last inference is

$$\dfrac{\{a\}{:}B,\ \Gamma \Rightarrow \alpha\cup\{a\}{:}C}{\Gamma \Rightarrow \alpha{:}B\to C}\ (\to \text{right})$$

where $a \notin \mathcal{L}(\Gamma) \cup \alpha$ (and $\max(\alpha) < a$ if $X = {}^{\cup}\mathbf{T}$). By the induction hypothesis, we get a history Δ for the upper sequent. We show that $\Delta \cup \{B\}$ is a history for the last sequent $\mathcal{S}$. Suppose that f is a label translation that is good for the sequent $((\Delta, B)^{\#}, \mathcal{S})$. We will show

$$\mathbf{H}X \vdash (\Delta^{\#}, \emptyset{:}B\to B, \Gamma \Rightarrow \alpha{:}B\to C)^f.$$

First we take a "fresh" propositional variable p that does not occur in this formula, and we define a label translation f' by

$$\begin{cases} f'(a) = p, \\ f'(x) = f(x), \quad \text{if } x \neq a. \end{cases}$$

This translation f' is good for $\Delta^{\#}, \{a\}:B, \Gamma \Rightarrow \alpha\cup\{a\}:C$, and we have

$$\mathbf{H}X \vdash (\Delta^{\#}, \{a\}:B, \Gamma \Rightarrow \alpha\cup\{a\}:C)^{f'}$$

by the induction hypothesis. Then we have

$$\cfrac{\cfrac{(\Delta^{\#}, \{a\}:B, \Gamma \Rightarrow \alpha\cup\{a\}:C)^{f'}}{(\Delta^{\#}, \emptyset:p\to B, \Gamma \Rightarrow \alpha:p\to C)^{f}}\ (\ddagger)}{(\Delta^{\#}, \emptyset:B\to B, \Gamma \Rightarrow \alpha:B\to C)^{f}}\ (\text{substitution } [p:=B])$$

where $(\ddagger)$ is null if $X = {}^{\cup}\mathbf{T}$ (the upper formula and the lower formula are identical because $a = \max(\alpha\cup\{a\})$) and $(\ddagger)$ is repetition of "exchange" (Lemma 6.4) if $X = {}^{\cup}\mathbf{R}$.

(Case 5): The last inference is

$$\frac{\beta:B_i, \Gamma' \Rightarrow \alpha:A}{\beta:B_1\wedge B_2, \Gamma' \Rightarrow \alpha:A}\ (\wedge \text{ left})$$

where $i = 1$ or 2. We get a history for the upper sequent by the induction hypothesis, and this is also a history for the lower sequent. The condition (1) is shown by the induction hypothesis, Lemma 6.5(2)(3) and Lemma 6.1

(Case 6): The last inference is

$$\frac{\Pi \Rightarrow \alpha:B \quad \Sigma \Rightarrow \alpha:C}{\Pi, \Sigma \Rightarrow \alpha:B\wedge C.}\ (\wedge \text{ right})$$

We get histories Δ_l and Δ_r for the left and right upper sequents respectively by the induction hypotheses. Then $\Delta_l \cup \Delta_r$ is a history for the lower sequent. The condition (1) is shown by the induction hypotheses, Lemma 6.5(1) and Lemma 6.1

(Case 7): The last inference is

$$\frac{\beta:B, \Pi \Rightarrow \alpha:A \quad \beta:C, \Sigma \Rightarrow \alpha:A}{\beta:B\vee C, \Pi, \Sigma \Rightarrow \alpha:A.}\ (\vee \text{ left})$$

We get histories Δ_l and Δ_r for the left and right upper sequents respectively by the induction hypotheses. Then $\Delta_l \cup \Delta_r$ is a history for the lower sequent. The condition (1) is shown by

$$\frac{\overset{\vdots\ \text{i.h.}}{(\Delta_l^{\#}, \beta:B, \Pi \Rightarrow \alpha:A)^f} \quad \overset{\vdots\ \text{i.h.}}{(\Delta_r^{\#}, \beta:C, \Sigma \Rightarrow \alpha:A)^f}}{(\Delta_l^{\#}, \Delta_r^{\#}, \beta:B\vee C, \Pi, \Sigma \Rightarrow \alpha:A)^f.}\ (\vee \text{ left})$$

(Case 8): The last inference is

$$\frac{\Gamma \Rightarrow \alpha:B_i}{\Gamma \Rightarrow \alpha:B_1\vee B_2}\ (\vee \text{ right})$$

where $i = 1$ or 2. We get a history for the upper sequent by the induction hypothesis, and this is also a history for the lower sequent. The condition (1) is shown by the induction hypothesis, Lemma 6.5(4)(5) and Lemma 6.1 ∎

Theorem 6.9 *If* $\mathbf{LI}X \vdash \Rightarrow \emptyset : A$*, then* $\mathbf{H}X \vdash A$*, for* $X = {}^{\cup}\mathbf{T}$ *and* ${}^{\cup}\mathbf{R}$.

Proof Suppose $\mathbf{LI}X \vdash \Rightarrow \emptyset : A$. By Lemma 6.8, there is a history $\Delta = \{H_1, \ldots, H_n\}$ $(n \geq 1)$ such that

$$\mathbf{H}X \vdash (\Delta^{\#} \Rightarrow \emptyset : A)^f = \bigwedge\{H_1 \rightarrow H_1, \ldots, H_n \rightarrow H_n\} \rightarrow A.$$

Then A is provable in $\mathbf{H}X$ by the axioms $H_i \rightarrow H_i$ and the rules adjunction and modus ponens. ∎

To show the soundness of $\mathbf{H}X$, we define a notation concerning models. Let $\langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V\rangle$ be a ${}^{\cup}\mathbf{R}({}^{\cup}\mathbf{T})$-model. Given a propositional variable p, we define $V(p)$ to be the subset of I such that p holds; that is,

$$V(p) = \{\alpha \in I \mid (\alpha, p) \in V\}.$$

Lemma 6.10 *Suppose that* R *is a formula* $r_1 \rightarrow (r_2 \rightarrow (\cdots \rightarrow (r_k \rightarrow Q) \cdots))$ *where* $r_1, \ldots, r_k$ *are propositional variables not occurring in* Q*. Suppose also that* M *and* M' *are* ${}^{\cup}\mathbf{R}({}^{\cup}\mathbf{T})$*-models such that*

- $M = \langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V\rangle$;
- $M' = \langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V'\rangle$;
- $V'(r_i) \subseteq V(r_i)$ *for* $i = 1, \ldots, k$*; and*
- $V'(p) = V(p)$*, for* $p \neq r_i$.

If $\beta \models_M R$*, then* $\beta \models_{M'} R$ *(*β *is an arbitrary element of* I*).*

Proof By induction on k. (Note that $\beta \models_M Q$ if and only if $\beta \models_{M'} Q$.) ∎

Lemma 6.11 *Suppose that* M *is a* ${}^{\cup}\mathbf{R}({}^{\cup}\mathbf{T})$*-model* $\langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V\rangle$ *and* $q_1, \ldots, q_k$ *are propositional variables such that*

- $|V(q_i)| \leq 1$ *for* $i = 1, \ldots, k$*; and*
- $\beta \models_M q_1 \rightarrow (q_2 \rightarrow (\cdots \rightarrow (q_k \rightarrow (B_1 \vee B_2)) \cdots))$.

Then,

$$\beta \models_M q_1 \rightarrow (q_2 \rightarrow (\cdots \rightarrow (q_k \rightarrow B_x)) \cdots))$$

holds for some $x \in \{1, 2\}$.

Proof Easy. ∎

Theorem 6.12 *If* $\mathbf{H}X \vdash A$*, then* $A \in X$ *(i.e.,* A *is valid in any* X*-model), for* $X = {}^{\cup}\mathbf{T}$ *and* ${}^{\cup}\mathbf{R}$.

Proof By induction on the $\mathbf{H}X$-proof of A. We show the nontrivial case: A is inferred by

$$\frac{(\alpha : B_1, \Gamma \Rightarrow \tau : F)^f \quad (\alpha : B_2, \Delta \Rightarrow \tau : F)^f}{(\alpha : B_1 \vee B_2, \Gamma, \Delta \Rightarrow \tau : F)^f.} \ (\vee \text{ left})$$

We will show that *if both of the upper formulas belong to* X*, then so does the lower formula.* First we define

- $\Gamma = (\gamma_1 : C_1, \ldots, \gamma_c : C_c)$ and $\Delta = (\delta_1 : D_1, \ldots, \delta_d : C_d)$;

- $\tau = \{t_1, \ldots, t_n\}$ and $\tau \cup \alpha \cup \gamma_1 \cup \cdots \cup \gamma_c \cup \delta_1 \cup \cdots \cup \delta_d = \{t_1, \ldots, t_n, t_{n+1}, \ldots, t_m\}$ where $t_i \neq t_j$ if $i \neq j$; and
- $f(t_i) = p_i$ $(i = 1, \ldots, m)$.

Then the condition "f is good for the sequent" implies the following.

- $p_i \neq p_j$ if $i \neq j$.
- p_i does not occur in $(B_1 \vee B_2, C_1, \ldots, C_c, D_1, \ldots, D_d, F)$.

Suppose $(\alpha : B_1 \vee B_2, \Gamma, \Delta \Rightarrow \tau : F)^f \notin X$; that is, there is an X-model $M = \langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V\rangle$ such that

$$\mathbf{e} \not\models_M \bigwedge\left\{(\alpha : B_1 \vee B_2)^f, \Gamma^f, \Delta^f\right\} \to (\tau : F)^f$$

where $\Gamma^f = ((\gamma_1 : C_1)^f, \ldots, (\gamma_c : C_c)^f)$ and $\Delta^f = ((\delta_1 : D_1)^f, \ldots, (\delta_d : D_d)^f)$. This means that $\exists\beta, \exists\tau_1, \ldots, \exists\tau_n \in I$,

(0) $\beta\cdot\tau_1\cdot\cdots\cdot\tau_{k-1} \preceq \tau_k$ for $k = 1, \ldots, n$ (if $X = {}^{\cup}\mathbf{T}$);

(1) $\beta \models_M \bigwedge\left\{(\alpha : B_1 \vee B_2)^f, \Gamma^f, \Delta^f\right\}$;

(2) $\tau_i \models_M p_i$ for $i = 1, \ldots, n$; and

(3) $\beta\cdot\tau_1\cdot\cdots\cdot\tau_n \not\models_M F$.

Then we define a model $M' = \langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V'\rangle$ such that

- $V'(p_i) = \{\tau_i\}$ for $i = 1, \ldots, n$;
- $V'(p_j) = \emptyset$ for $j = n+1, \ldots, m$; and
- $V'(r) = V(r)$ for $r \neq p_i$.

Using (1) and Lemmas 6.10 and 6.11, we have

(1′) $\beta \models_{M'} \bigwedge\left\{(\alpha : B_x)^f, \Gamma^f, \Delta^f\right\}$ for some $x \in \{1, 2\}$.

Moreover we have

(2′) $\tau_i \models_{M'} p_i$ for $i = 1, \ldots, n$; and

(3′) $\beta\cdot\tau_1\cdot\cdots\cdot\tau_n \not\models_{M'} F$.

Then at least one of the upper formulas of "$\vee$ left" is out of X. ∎

Theorem 6.13 (Completeness and soundness of $\mathbf{H}^{\cup}\mathbf{T}/\mathbf{H}^{\cup}\mathbf{R}$) $\mathbf{H}X \vdash A$ *if and only if* $A \in X$*, for* $X = {}^{\cup}\mathbf{T}$ *and* ${}^{\cup}\mathbf{R}$.

Proof By Theorems 5.4, 6.9, and 6.12. ∎

7 Contractionless logics

In this section we investigate the contractionless variants of ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$, which we call ${}^{\cup}\mathbf{RW}$ and ${}^{\cup}\mathbf{TW}$. The word "contractionless" means that the "contraction axiom"

$$(A{\to}(A{\to}B)){\to}(A{\to}B)$$

(this formula is usually called $\mathbf{W}$) is not valid in ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$.

In [4, 5, 8], two kinds of models were introduced for ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$:

(1) *Commutative monoid model*: We define a ${}^{\cup}\mathbf{RW_m}$-model (${}^{\cup}\mathbf{TW_m}$-model) to be a structure $\langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V\rangle$ with the same conditions as ${}^{\cup}\mathbf{R}$-model (${}^{\cup}\mathbf{T}$-model) except the idempotence postulate: $\alpha{\cdot}\alpha = \alpha$; in other words, $\langle I, \cdot, \mathbf{e}\rangle$ is a commutative monoid. (The subscript "m" represents "commutative Monoid".) The relation $\models$ between I and the set of formulas is defined by the same way as ${}^{\cup}\mathbf{R}$-model (${}^{\cup}\mathbf{T}$-model).

(2) *Distributive semilattice model*: We define a ${}^{\cup}\mathbf{RW_s}$-model (${}^{\cup}\mathbf{TW_s}$-model) to be a structure $\langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V\rangle$ with the same conditions as ${}^{\cup}\mathbf{R}$-model (${}^{\cup}\mathbf{T}$-model) and an additional condition:

$\alpha \sqsubseteq (\beta{\cdot}\gamma)$ only if there exist α_1 and α_2 such that $\alpha_1{\cdot}\alpha_2 = \alpha$, $\alpha_1 \sqsubseteq \beta$, and $\alpha_2 \sqsubseteq \gamma$, where $\sigma \sqsubseteq \tau$ is defined by $\sigma{\cdot}\tau = \tau$.

In other words, $\langle I, \cdot, \mathbf{e}\rangle$ is a "distributive Semilattice" (from which the subscript "s" comes). In this model, the definition of validity of implication is changed as follows.

$$\alpha \models A{\to}B \iff \begin{array}{l}\forall\beta(\succeq \alpha, \text{for } {}^{\cup}\mathbf{TW_s}) \text{ [if } \alpha \text{ and } \beta \text{ are disjoint and } \beta \models A,\\ \text{then } \alpha{\cdot}\beta \models B],\end{array}$$

where α and β are said to be *disjoint* if the condition

$$\gamma \sqsubseteq \alpha \text{ and } \gamma \sqsubseteq \beta \text{ only if } \gamma = \mathbf{e}$$

holds for any $\gamma \in I$. The validity for atomic formulas, conjunction, and disjunction is same as ${}^{\cup}\mathbf{R}({}^{\cup}\mathbf{T})$-models.

Then the logics ${}^{\cup}\mathbf{RW_m}$, ${}^{\cup}\mathbf{TW_m}$, ${}^{\cup}\mathbf{RW_s}$ and ${}^{\cup}\mathbf{TW_s}$ are defined to be sets of formulas:

$${}^{\cup}\mathbf{RW_m} = \{A \mid A \text{ is valid in any } {}^{\cup}\mathbf{RW_m}\text{-model.}\}$$

$${}^{\cup}\mathbf{TW_m} = \{A \mid A \text{ is valid in any } {}^{\cup}\mathbf{TW_m}\text{-model.}\}$$

$${}^{\cup}\mathbf{RW_s} = \{A \mid A \text{ is valid in any } {}^{\cup}\mathbf{RW_s}\text{-model.}\}$$

$${}^{\cup}\mathbf{TW_s} = \{A \mid A \text{ is valid in any } {}^{\cup}\mathbf{TW_s}\text{-model.}\}$$

We will show ${}^{\cup}\mathbf{RW_m} = {}^{\cup}\mathbf{RW_s}$ and ${}^{\cup}\mathbf{TW_m} = {}^{\cup}\mathbf{TW_s}$. (The former equation had been proved in [4] by a different way from ours; the latter had been "partially" proved (for the $\{\to, \wedge\}$-fragments) in [8].) Then these logics will be also called ${}^{\cup}\mathbf{RW}$ and ${}^{\cup}\mathbf{TW}$ without the subscripts.

We introduce labelled sequent calculi for these logics.

For ${}^{\cup}\mathbf{RW_m}$ and ${}^{\cup}\mathbf{TW_m}$, labels are *not sets but multisets* of natural numbers. Except for this difference, the labelled sequent calculi $\mathbf{LK}X$, $\mathbf{LM}X$, $\mathbf{LJ}X$, $\mathbf{LI}X$ for $X = {}^{\cup}\mathbf{RW_m}$ and ${}^{\cup}\mathbf{TW_m}$ are defined by the same axioms and rules as in Sections 2–4. (The subscript "m" also represents "Multiset".) Of course, the operation $\cup$ at the rules "$\to$ left/right" is read as the multisets union.

For ${}^{\cup}\mathbf{RW_s}$ and ${}^{\cup}\mathbf{TW_s}$, labels are *sets* of natural numbers, like ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$. The labelled sequent calculi $\mathbf{LK}X$, $\mathbf{LM}X$, $\mathbf{LJ}X$, $\mathbf{LI}X$ for $X = {}^{\cup}\mathbf{RW_s}$ and ${}^{\cup}\mathbf{TW_s}$ are defined by the same axioms and rules as in Sections 2–4 except that we impose an additional condition

$$\alpha \cap \beta = \emptyset$$

on all the "$\rightarrow$ left" rules. (The subscript "s" also represents "Set".)

All the arguments in Sections 2–4 work for ${}^{\cup}\mathbf{RW_m}$, ${}^{\cup}\mathbf{TW_m}$, ${}^{\cup}\mathbf{RW_s}$, and ${}^{\cup}\mathbf{TW_s}$, with slight modifications described as follows.

- The label models for ${}^{\cup}\mathbf{RW_m}/{}^{\cup}\mathbf{TW_m}$ consist of the set of finite *multisets* of positive integers, while they are *sets* for ${}^{\cup}\mathbf{RW_s}/{}^{\cup}\mathbf{TW_s}$.
- The clause (1) in the definitions of ${}^{\cup}\mathbf{RW_s}/{}^{\cup}\mathbf{TW_s}$-saturatedness is changed:

(${}^{\cup}\mathbf{RW_s}$) If $[\alpha{:}A{\rightarrow}B \in \Gamma$ and $\alpha \cap \beta = \emptyset]$, then $[\beta{:}A \in \Delta$ or $\alpha{\cup}\beta{:}B \in \Gamma]$.

(${}^{\cup}\mathbf{TW_s}$) If $[\alpha{:}A{\rightarrow}B \in \Gamma$, $\max(\alpha) \leq \max(\beta)$, and $\alpha \cap \beta = \emptyset]$, then $[\beta{:}A \in \Delta$ or $\alpha{\cup}\beta{:}B \in \Gamma]$.

On the other hand, the definitions of ${}^{\cup}\mathbf{RW_m}/{}^{\cup}\mathbf{TW_m}$-saturatedness are equivalent to ${}^{\cup}\mathbf{R}/{}^{\cup}\mathbf{T}$-saturatedness except that the operation $\cup$ is read as the *multisets union*.

- In the definition of the relation $\precsim$ for ${}^{\cup}\mathbf{RW_m}/{}^{\cup}\mathbf{TW_m}$, "$\alpha \subseteq \beta$" means "$\forall x\ [x \in \alpha \Rightarrow x \in \beta]$". ($\alpha$ and β are multiset, but the multiplicity of elements is not essential.)
- In Lemma 4.1 for ${}^{\cup}\mathbf{RW_m}/{}^{\cup}\mathbf{TW_m}$, "$\mathcal{L}(\Gamma) = \alpha$" means that "$\forall x\ [x \in \mathcal{L}(\Gamma) \iff x \in \alpha]$". ($\mathcal{L}(\Gamma)$ is a *set* and α is a *multiset.*)

Then we have the following.

Theorem 7.1 *Let* $X \in \{{}^{\cup}\mathbf{RW}, {}^{\cup}\mathbf{TW}\}$.

(1) $(A \in X_\mathbf{m}) \Longrightarrow (\mathbf{LK}X_\mathbf{m} \vdash \Rightarrow \emptyset{:}A) \Longrightarrow (\mathbf{LM}X_\mathbf{m} \vdash \Rightarrow \emptyset{:}A) \Longrightarrow (\mathbf{LJ}X_\mathbf{m} \vdash \Rightarrow \emptyset{:}A) \Longrightarrow (\mathbf{LI}X_\mathbf{m} \vdash \Rightarrow \emptyset{:}A)$.

(2) $(A \in X_\mathbf{s}) \Longrightarrow (\mathbf{LK}X_\mathbf{s} \vdash \Rightarrow \emptyset{:}A) \Longrightarrow (\mathbf{LM}X_\mathbf{s} \vdash \Rightarrow \emptyset{:}A) \Longrightarrow (\mathbf{LJ}X_\mathbf{s} \vdash \Rightarrow \emptyset{:}A) \Longrightarrow (\mathbf{LI}X_\mathbf{s} \vdash \Rightarrow \emptyset{:}A)$.

Proof Following Sections 2–4 with the above modification. ∎

In the rest of this section, we will show

$$(\mathbf{LI}X_\mathbf{m} \vdash \Rightarrow \emptyset{:}A) \Longrightarrow (\mathbf{LI}X_\mathbf{s} \vdash \Rightarrow \emptyset{:}A) \Longrightarrow (A \in X_\mathbf{m} \text{ and } A \in X_\mathbf{s}).$$

This and Theorem 7.1 imply that all the conditions in Theorem 7.1 are mutually equivalent.

Theorem 7.2 *If* $\mathbf{LI}X_\mathbf{m} \vdash \Rightarrow \emptyset{:}A$*, then* $\mathbf{LI}X_\mathbf{s} \vdash \Rightarrow \emptyset{:}A$*, for* $X = {}^{\cup}\mathbf{RW}$ *and* ${}^{\cup}\mathbf{TW}$.

Proof If a label α in $\mathbf{LI}X_\mathbf{m}$ contains an integer twice or more, then we say α is a *proper multiset label.* The following fact is easily verified by induction on the $\mathbf{LI}X_\mathbf{m}$-proofs.

If $\mathbf{LI}X_\mathbf{m} \vdash \Gamma \Rightarrow \tau{:}A$ and if Γ contains a proper multiset label, then τ is also a proper multiset.

Using this fact, we can show the following.

If a proper multiset label appears in a proof of $\Gamma \Rightarrow \tau{:}A$ in $\mathbf{LI}X_\mathbf{m}$, then τ is a proper multiset.

This implies Theorem 7.2 because $\mathbf{LI}X_{\mathbf{s}}$ is equivalent to "$\mathbf{LI}X_{\mathbf{m}}$ without proper multiset labels". ∎

Let M be a ${}^{\cup}\mathbf{TW}_{\mathbf{m}}$-model $\langle\langle I, \cdot, \mathbf{e}, \preceq\rangle, V\rangle$, and $\mathcal{A}$ be a finite set of positive integers. The *interpretations of* $\mathcal{A}$ *on* M are defined by the same condition as in Section 5. Moreover, the element $\tau^f \in I$ (τ is a subset of $\mathcal{A}$ and f is an interpretation) and the notion "a labelled sequent $\mathcal{S}$ is *valid* in M with respect to f" are defined in the same way as Section 5.

Lemma 7.3 (cf. 5.2) *If* $\max(\alpha) < \max(\beta)$*, then* $\alpha^f \preceq \beta^f$*, where* α *and* β *are sets of positive integers and* f *is an interpretation of a set* $\mathcal{A} \supseteq \alpha \cup \beta$ *on a* ${}^{\cup}\mathbf{TW}_{\mathbf{m}}$*-model* $\langle\langle I, \cdot, \mathbf{e}, \preceq\rangle, V\rangle$*.*

Proof Similar to the proof of Lemma 5.2. Note that Lemma 5.1(1) also holds for ${}^{\cup}\mathbf{TW}_{\mathbf{m}}$-models, while (2) is not necessary because "$a_m = b_n$" (in the proof of Lemma 5.2) does not happen. ∎

Theorem 7.4 (Soundness of $\mathbf{LI}^{\cup}\mathbf{RW}_{\mathbf{s}}/\mathbf{LI}^{\cup}\mathbf{TW}_{\mathbf{s}}$ for commutative monoid models) *Let* $X \in \{{}^{\cup}\mathbf{RW}, {}^{\cup}\mathbf{TW}\}$*. If a labelled sequent* $\mathcal{S}$ *is provable in* $\mathbf{LI}X_{\mathbf{s}}$*, then* $\mathcal{S}$ *is valid in* M *with respect to* f *for any* $X_{\mathbf{m}}$*-model* M *and any interpretation* f *of* $\mathcal{L}(\mathcal{S})$ *on* M*. (In particular, if* $\mathbf{LI}X_{\mathbf{s}} \vdash \Rightarrow \emptyset : F$*, then* $F \in X_{\mathbf{m}}$*.)*

Proof Similar to the proof of Lemma 5.3. (We use Lemma 7.3 instead of 5.2.) ∎

Let M be a ${}^{\cup}\mathbf{RW}_{\mathbf{s}}$-model $\langle\langle I, \cdot, \mathbf{e}\rangle, V\rangle$ or a ${}^{\cup}\mathbf{TW}_{\mathbf{s}}$-model $\langle\langle I, \cdot, \mathbf{e}, \preceq\rangle, V\rangle$, and $\mathcal{A}$ be a set of positive integers. A mapping f from $\mathcal{A}$ to I is called an *s-interpretation of* $\mathcal{A}$ *on* M if f is an interpretation and the condition

$x \neq y \Longrightarrow f(x)$ and $f(y)$ are disjoint

holds for any $x, y \in \mathcal{A}$. The element $\tau^f \in I$ and the notion "a labelled sequent $\mathcal{S}$ is *valid* in M with respect to f" are defined in the same way as Section 5.

Lemma 7.5 *The following hold for any* ${}^{\cup}\mathbf{RW}_{\mathbf{s}}$*(*${}^{\cup}\mathbf{TW}_{\mathbf{s}}$*)-model* $\langle\langle I, \cdot, \mathbf{e}(, \preceq)\rangle, V\rangle$ *and any* $\alpha, \beta, \beta_1, \beta_2 \in I$*.*

(1) $\sqsubseteq$ *is transitive, and* $\alpha \sqsubseteq \alpha{\cdot}\beta$*.*

(2) $\mathbf{e}$ *and* α *are disjoint.*

(3) α *and* $(\beta_1{\cdot}\beta_2)$ *are disjoint if and only if* α *and* β_1 *are disjoint and* α *and* β_2 *are disjoint.*

Proof (1) and (2) are easy.

(3, if-part) Suppose that α and $(\beta_1{\cdot}\beta_2)$ are not disjoint; that is $\gamma \sqsubseteq \alpha$ and $\gamma \sqsubseteq (\beta_1{\cdot}\beta_2)$ for some $\gamma \neq \mathbf{e}$. By the definition of distributive semilattice, there exist γ_1 and γ_2 such that $\gamma = (\gamma_1{\cdot}\gamma_2)$ and $\gamma_i \sqsubseteq \beta_i$ for $i = 1, 2$. Then there is a number $k \in \{1, 2\}$ such that $\gamma_k \neq \mathbf{e}$ because $\gamma \neq \mathbf{e}$, and α and β_k are not disjoint because $\gamma_k \sqsubseteq (\gamma_1{\cdot}\gamma_2) \sqsubseteq \alpha$.

(3, only-if-part) Suppose that α and β_k are not disjoint for $k = 1$ or 2; that is $\gamma \sqsubseteq \alpha$ and $\gamma \sqsubseteq \beta_k$ for some $\gamma \neq \mathbf{e}$. Then α and $(\beta_1{\cdot}\beta_2)$ are not disjoint because $\gamma \sqsubseteq \beta_k \sqsubseteq (\beta_1{\cdot}\beta_2)$. ∎

Lemma 7.6 *Let* M *be a* ${}^{\cup}\mathbf{RW}_{\mathbf{s}}$*-model* $\langle\langle I, \cdot, \mathbf{e}\rangle, V\rangle$ *or a* ${}^{\cup}\mathbf{TW}_{\mathbf{s}}$*-model* $\langle\langle I, \cdot, \mathbf{e}, \preceq\rangle, V\rangle$*,* f *be an s-interpretation on* M*,* α *and* β *be sets of positive integers, and* ξ *be an element of* I*.*

(1) If $\alpha \cap \beta = \emptyset$*, then* α^f *and* β^f *are disjoint.*

(2) If α^f *and* ξ *are disjoint, then* $f(x)$ *and* ξ *are disjoint for all* $x \in \alpha$*.*

Proof (1) When both α and β are nonempty, Lemma 7.5(3)(if-part) shows that α^f and β^f are disjoint (note that $f(x)$ and $f(y)$ are disjoint for any $x \in \alpha$ and any $y \in \beta$). If either α or β is empty, then α^f and β^f are disjoint because of Lemma 7.5(2).

(2) By Lemma 7.5(3)(only-if-part). ∎

Theorem 7.7 (Soundness of $\mathbf{LI}^{\cup}\mathbf{RW_s}/\mathbf{LI}^{\cup}\mathbf{TW_s}$ for distributive semilattice models) *Let $X \in \{{}^{\cup}\mathbf{RW}, {}^{\cup}\mathbf{TW}\}$. If a labelled sequent $\mathcal{S}$ is provable in $\mathbf{LI}X_\mathbf{s}$, then $\mathcal{S}$ is valid in M with respect to f for any $X_\mathbf{s}$-model M and any s-interpretation f of $\mathcal{L}(\mathcal{S})$ on M. (In particular, if $\mathbf{LI}X_\mathbf{s} \vdash \Rightarrow \emptyset : F$, then $F \in X_\mathbf{s}$.)*

Proof Similar to the proof of Lemma 5.3. (In the Subcase 1-2, we invoke Lemma 7.6(1). In the Case 2, the assertion "f' is an s-interpretation of $\mathcal{L}(\mathcal{S}')$ on M" needs Lemma 7.6(2).) ∎

By Theorems 7.1, 7.2, 7.4 and 7.7, we have the following.

Theorem 7.8 (Main Theorem for ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$) *Let $X \in \{{}^{\cup}\mathbf{RW}, {}^{\cup}\mathbf{TW}\}$. The following conditions are equivalent.*

- $A \in X_\mathbf{m}$. • $A \in X_\mathbf{s}$.
- $\mathbf{LK}X_\mathbf{m} \vdash \Rightarrow \emptyset : A$. • $\mathbf{LK}X_\mathbf{s} \vdash \Rightarrow \emptyset : A$.
- $\mathbf{LM}X_\mathbf{m} \vdash \Rightarrow \emptyset : A$. • $\mathbf{LM}X_\mathbf{s} \vdash \Rightarrow \emptyset : A$.
- $\mathbf{LJ}X_\mathbf{m} \vdash \Rightarrow \emptyset : A$. • $\mathbf{LJ}X_\mathbf{s} \vdash \Rightarrow \emptyset : A$.
- $\mathbf{LI}X_\mathbf{m} \vdash \Rightarrow \emptyset : A$. • $\mathbf{LI}X_\mathbf{s} \vdash \Rightarrow \emptyset : A$.

Finally we mention Hilbert-style axiomatization. We do not yet find Hilbert-style systems for ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$. The argumant of Section 6 does not work for them: In the proof of Lemma 6.8 (Case 3), we use "$\to$ to $\wedge$", which is derived with the axiom **W**. It seems that a difficulty is $(A\wedge(A\to B))\to B \notin {}^{\cup}\mathbf{RW}, {}^{\cup}\mathbf{TW}$.

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