A Proof-Theoretical Study on Logics with Constructible Falsity

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Abstract

Constructible falsity $\sim A$, also called strong negation, is an alternative to Heyting's negation $\neg A$ ($\leftrightarrow (A \to \bot)$) in intuitionistic logics. In this paper we give the proofs for Kripke completeness of the basic logic $\mathsf{N}_\neg$ and its five variations. Among them the most novel results are about the logics with what we call omniscience axiom, $\neg\neg(A \vee \sim A)$. We present two different proofs based on tree-sequents: one is by an embedding of classical logic, and the other is by an extended version of tree-sequent.

1 Introduction

Intuitionistic logic Int introduced by Heyting is a realization as a formal system of Brouwer's intuitionism as to mathematics. However, its negation $\neg A$, which is equivalent to $A \to \bot$ and called *Heyting's negation* to make distinct from $\sim A$, is subject to criticisms that it is not constructive enough. For instance, it seems a natural demand from constructivists' point of view that in order to show $\neg \forall x A$, we must know a concrete object t such that $\neg A[t/x]$ holds. But this is not the case with Heyting's negation.

$\sim A$, called *constructible falsity* or *strong negation*, was introduced independently by Nelson [Nel49] and Markov [Mar50]. $\sim A$ is axiomatized in the Hilbert-style system as follows:

$$
\begin{array}{l}
A \to (\sim A \to B) \\
\sim(A \wedge B) \leftrightarrow \sim A \vee \sim B \quad \sim(A \vee B) \leftrightarrow \sim A \wedge \sim B \\
\sim(A \to B) \leftrightarrow A \wedge \sim B \quad \sim\sim A \leftrightarrow A \quad \sim\neg A \leftrightarrow A \\
\sim\forall x A \leftrightarrow \exists x \sim A \quad \sim\exists x A \leftrightarrow \forall x \sim A
\end{array}
$$

It is easy to verify that $\sim$ is free from such criticisms as stated above.

The logic[*] N (N stands for Nelson's logic) is the $\neg$-free fragment of Int plus constructible falsity $\sim$ axiomatized as above. The basic logic considered in this paper is the logic $\mathsf{N}_\neg$, N plus Heyting's negation $\neg$.

We will also consider the variations of $\mathsf{N}_\neg$ which are obtained by adding some of the following characters:

- D *Constant domain axiom*, $\forall x(A(x) \vee B) \to \forall x A(x) \vee B$. Here x have no free occurrences in B. Its name comes from the fact that a Kripke model for Int which makes this axiom valid has a constant domain, i.e. the same domain for every possible world. This is one of the axioms which characterize intermediate (or super-intuitionistic) logics.

- O $\neg\neg(A \vee {\sim}A)$, which the authors would like to call *omniscience axiom*. This axiom is needless to say peculiar to the logics with both $\sim$ and $\neg$, and as we shall show later, it is interpreted as a reference to Kripke model that *every possible world has an omniscient world which approves it.* This axiom may also be considered as one of the weaker versions of *the law of excluded middle*, $A \vee {\sim}A$[†].

- P This indicates omitting the axiom $A \to ({\sim}A \to B)$, and as a result we have paraconsistency.

With these three characters we obtain $8(= 2^3)$ variations of $\mathsf{N}_\neg$, which form the lattice (the upperright the stronger) presented below (Fig. 1).

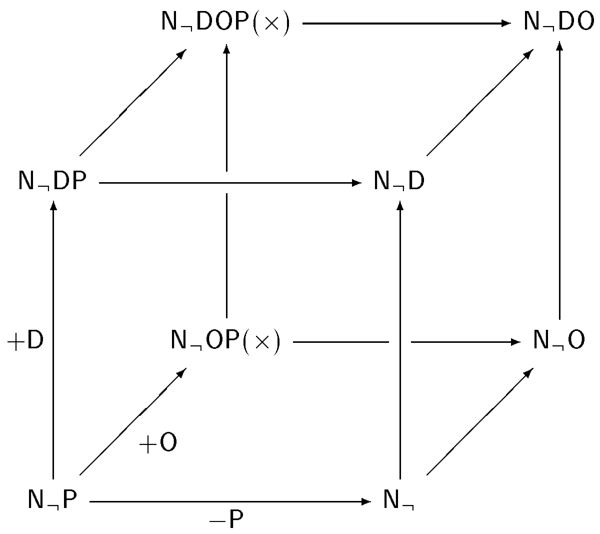

Fig. 1: $\mathsf{N}_\neg$-family

In the literature [AN84] ND, NP or NDP ($\neg$-free fragment of $\mathsf{N}_\neg\mathsf{D}$, $\mathsf{N}_\neg\mathsf{P}$ or $\mathsf{N}_\neg\mathsf{DP}$) is referred to as N^+, N^- or N^{+-}, respectively.

[*]In this paper the word *logic* is used for a pair of a formal language and the set of formulas of that language which are admitted as theorems. To define the set of theorems, we will adopt Gentzen-style sequent systems.

[†]$\neg A \vee \neg\neg A$ is said to be the *weak law of excluded middle* and characterizes intermediate logics.

In this paper we prove Kripke completeness of logics $\mathsf{N}_\neg$, $\mathsf{N}_\neg\mathsf{D}$, $\mathsf{N}_\neg\mathsf{P}$, $\mathsf{N}_\neg\mathsf{DP}$, $\mathsf{N}_\neg\mathsf{O}$ and $\mathsf{N}_\neg\mathsf{DO}$.

For each logic we introduce a Gentzen-style sequent calculus, Kripke-type possible world semantics, and give the correspondence between them, i.e. the completeness theorem[‡]. For the proof of completeness, we adopt the general method *tree-sequent*. Roughly stated, a tree-sequent is a finite tree each of whose nodes are associated with a sequent. Since it simulates the shape of Kripke models for variations of Int, we can easily obtain a counter-model for an unprovable sequent by expanding the corresponding tree-sequent, hence the proof of completeness. In the literature [Kas99] applications for Int, some intermediate logics, modal logics, and relevant logics are presented.

For the logics $\mathsf{N}_\neg\mathsf{O}$ and $\mathsf{N}_\neg\mathsf{DO}$, we can hardly prove their Kripke completeness by simply applying the tree-sequent method, because omniscience axiom may be regarded as a reference to "upper bounds" of Kripke models. Novel results in this paper lie in the proofs for them; we present two different methods to consider the logics with O. One uses an embedding of classical logic Cl into $\mathsf{N}_\neg\mathsf{O}$, and its idea is that an omniscient possible world can be induced by a Cl-model (which is usually called a *structure*). The other proof is by an extended version of tree-sequent which the authors would like to call *tree-sequent with guardians*.

Kripke completeness of the logic $\mathsf{N}_\neg\mathsf{OP}$ or $\mathsf{N}_\neg\mathsf{DOP}$ remain unconsidered in this paper. It is hard to find an embedding of classical logic into those two logics, so that our first method fails. And the method of tree-sequent with guardians also fails, while tree-sequent can treat the logics $\mathsf{N}_\neg\mathsf{P}$ and $\mathsf{N}_\neg\mathsf{DP}$; in fact, the authors are not certain as to which type of models $\mathsf{N}_\neg\mathsf{OP}$ or $\mathsf{N}_\neg\mathsf{DOP}$ is complete. In any case, they require more inspection[§].

We describe the notations used in this paper.

The relation $\equiv$ denotes syntactical equivalence. For example, $A \wedge B$ and $B \wedge A$ are logically equivalent in those logics considered in this paper. However, $A \wedge B \not\equiv B \wedge A$.

$A[y/x]$ is a substitution, i.e. a formula obtained by substituting every free occurrence of x in A by y. It is not preferable that by substituting x by y a new bound variable comes to existence, which is the case when a free occurrence of x is in a scope of $\forall y$. We avoid such cases by taking variants, i.e. substituting bound variables.

As we do in the Tarski-type semantics for Cl, in defining semantics we will introduce constants each of which designates a certain individual u. This kind of constant is said to be the *name* of u, and denoted by $\underline{u}$.

We will sometimes denote a formula by $A(x)$ to emphasize that x has free occurrences only in A, as we just did to demonstrate constant domain axiom. However, more often we do not add (x) but do put explicitly as "x has no free occurrence in $\ldots$".

For a finite set of formulas $\Gamma = \{A_1, \ldots, A_m\}$, the formula $\bigwedge \Gamma$ (or $\bigvee \Gamma$) is defined as $A_1 \wedge \ldots \wedge A_m$ (or $A_1 \vee \ldots \vee A_m$). If $\Gamma = \emptyset$, then it is $\top$ (or $\bot$), which is an abbreviation of $A \to A$ (or $\neg(A \to A)$, respectively).

[‡]Some other cut-free sequent systems are presented by Ishimoto [Ish70], and Kripke completeness of Σ (N plus constants) is presented by Tanaka [Tan80] whose proof is done differently from ours.

[§]When we consider a logic with both $\neg$ and $\sim$ in it, it seems reasonable to admit $\sim\neg A \leftrightarrow A$ as an axiom. However, this results in the fact that the formula $\neg A \to (\sim\neg A \to B)$ is provable even without the axiom $A \to (\sim A \to B)$, which designates that any paraconsistent variation of $\mathsf{N}_\neg$ can have only limited paraconsistency.

2 Logic $\mathsf{N}_\neg$

2.1 Sequent System $\mathsf{GN}_\neg$

Here we introduce the Gentzen-style sequent system for the logic $\mathsf{N}_\neg$.

The language of the logic $\mathsf{N}_\neg$ consists of the following symbols:

- countably many variables, $x_1, x_2, \ldots$;
- countably many m-ary predicate symbols for each $m \in \mathbb{N}$, $p^1_m, p^2_m, \ldots$;
- logical connectives, $\wedge, \neg, \to, \sim, \forall$.

$\vee$ and $\exists$ are introduced as defined symbols:

$$A \vee B \equiv {\sim}({\sim}A \wedge {\sim}B) \qquad \exists x A \equiv {\sim}\forall x {\sim}A$$

$A \leftrightarrow B$ is an abbreviation of $(A \to B) \wedge (B \to A)$.

We do not consider constants or function symbols, which makes the arguments simpler without essential loss of generality.

Terms and formulas of $\mathsf{N}_\neg$ are composed in the same way as those of Cl, and note that $\sim$ is unary.

A *sequent* is defined as an ordered pair of finite sets of formulas, therefore the rule of exchange or contraction is omitted.

Now we present the axioms and the inference rules of the Gentzen-style sequent system $\mathsf{GN}_\neg$ for the logic $\mathsf{N}_\neg$:

$$\frac{}{A \Rightarrow A}\ (\text{axiom 1}) \qquad \frac{}{A, {\sim}A \Rightarrow}\ (\text{axiom 2})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ (\text{weakening})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut})$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Sigma \Rightarrow \Pi}{A \to B, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\to\text{L}) \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\to\text{R})_{\text{S}}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\text{L}) \qquad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A}\ (\neg\text{R})_{\text{S}}$$

$$\frac{A[y/x], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ (\forall\text{L}) \qquad \frac{\Gamma \Rightarrow A[z/x]}{\Gamma \Rightarrow \forall x A}\ (\forall\text{R})_{\text{S, VC}}$$

$$\frac{{\sim}A, \Gamma \Rightarrow \Delta \quad {\sim}B, \Gamma \Rightarrow \Delta}{{\sim}(A \wedge B), \Gamma \Rightarrow \Delta}\ ({\sim}\wedge\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}A, {\sim}B}{\Gamma \Rightarrow \Delta, {\sim}(A \wedge B)}\ ({\sim}\wedge\text{R})$$

$$\frac{A, {\sim}B, \Gamma \Rightarrow \Delta}{{\sim}(A \to B), \Gamma \Rightarrow \Delta}\ ({\sim}{\to}\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, {\sim}B}{\Gamma \Rightarrow \Delta, {\sim}(A \to B)}\ ({\sim}{\to}\text{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{{\sim}\neg A, \Gamma \Rightarrow \Delta}\ ({\sim}\neg\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, {\sim}\neg A}\ ({\sim}\neg\text{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{{\sim}{\sim}A, \Gamma \Rightarrow \Delta}\ ({\sim}{\sim}\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, {\sim}{\sim}A}\ ({\sim}{\sim}\text{R})$$

$$\frac{\sim A[z/x], \Gamma \Rightarrow \Delta}{\sim\forall x A, \Gamma \Rightarrow \Delta} \ (\sim\forall\mathrm{L})_{\mathrm{VC}} \qquad \frac{\Gamma \Rightarrow \Delta, \sim A[y/x]}{\Gamma \Rightarrow \Delta, \sim\forall x A} \ (\sim\forall\mathrm{R})$$

Here the subscript S indicates the condition that the succedent of the conclusion consist of only one formula, and VC the *eigenvariable condition*, i.e. the eigenvariable z must not appear in the conclusion.

A formula A is *provable* in $\mathsf{GN}_\neg$, $\mathsf{GN}_\neg \vdash A$, if $\mathsf{GN}_\neg \vdash \ \Rightarrow A$.

Now we consider logical equivalence in $\mathsf{N}_\neg$ which require some remarks.

DEFINITION 2.1 (LOGICAL EQUIVALENCE IN $\mathsf{N}_\neg$)
Let A and B be formulas of $\mathsf{N}_\neg$. A and B are *logically equivalent*, $A \cong_{\mathsf{N}_\neg} B$, if $\mathsf{GN}_\neg \vdash A \leftrightarrow B$ and $\mathsf{GN}_\neg \vdash \sim A \leftrightarrow \sim B$.

THEOREM 2.2 (THE EQUIVALENCE THEOREM)
Let $C[\,]$ be a formula of $\mathsf{N}_\neg$ where the special atomic formula $[\,]$ is allowed to appear, and $C[A]$ a formula obtained by substituting every occurence of $[\,]$ by the formula A. If $A \cong_{\mathsf{N}_\neg} B$, Then $C[A] \cong_{\mathsf{N}_\neg} C[B]$.

Proof. By the induction on the construction of $C[\,]$. ∎

REMARK 2.3
It is common in a logic L which does not have $\sim$ to define the logical equivalence $A \cong_{\mathsf{L}} B$ by $\mathsf{L} \vdash A \leftrightarrow B$. However, this is not enough to allow the equivalence theorem for the logics with $\sim$ as we shall see in lemma 2.8.

It is required naturally that the meaning of a formula A is invariant as to substitution of bound variables.

LEMMA 2.4
If z has no free occurrences in A, then $\forall z A[z/x] \cong_{\mathsf{N}_\neg} \forall x A$, i.e. variants are logically equivalent in $\mathsf{N}_\neg$.

2.2 Kripke-Type Semantics for $\mathsf{N}_\neg$

The Kripke-type possible world semantics for $\mathsf{N}_\neg$ is introduced as a natural extension of that for Int, which is one of the reasons why logics with constructible falsity are worthy of consideration.

Let $(M, \leq)$ be a poset, W a non-empty set, U be a map of M into $\mathfrak{P}W$, satisfying

- $U(a) \neq \emptyset$ for all $a \in M$;
- $a \leq b$ implies $U(a) \subseteq U(b)$.

For every predicate symbol p (we assume p is m-ary), we define two interpretations of p in $a \in M$, $p^{I^+(a)}$ and $p^{I^-(a)}$, as subsets of $U(a)^m$, satisfying

- $a \leq b$ implies $p^{I^+(a)} \subseteq p^{I^+(b)}$ and $p^{I^-(a)} \subseteq p^{I^-(b)}$;
- $p^{I^+(a)} \cap p^{I^-(a)} = \emptyset$.

Then the quintuple $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ is said to be an $\mathsf{N}_\neg$-*model*.

An element a in M is said to be a *possible world*, and can be thought of as a stage of our knowledge. $a \leq b$ can be read as b is more advanced, knows more than a. $U(a)$ is called a *domain* of a, being the set of individuals recognized by a.

Given an $\mathsf{N}_\neg$-model $\mathfrak{M}$, we can obtain the two relations between $a \in M$ and a closed formula A, $a \models^+ A$ and $a \models^- A$, by extending two interpretations I^+ and I^- in the procedure shown below. $a \models^+ A$ or $a \models^- A$ may be interpreted as "*a verifies A*" or "*a refutes A*" respectively (this terminology the authors hope reflects the idea of constructivism).

$a \models^+ A$ and $a \models^- A$ are defined inductively on the construction of the closed formula A:

$$
\begin{array}{lcl}
a \models^+ p(\underline{u_1}, \ldots, \underline{u_m}) & \iff & (u_1, \ldots, u_m) \in p^{I^+(a)}\ ; \\
a \models^- p(\underline{u_1}, \ldots, \underline{u_m}) & \iff & (u_1, \ldots, u_m) \in p^{I^-(a)}\ ; \\
a \models^+ A \wedge B & \iff & a \models^+ A \quad \text{and} \quad a \models^+ B\ ; \\
a \models^- A \wedge B & \iff & a \models^- A \quad \text{or} \quad a \models^- B\ ; \\
a \models^+ A \to B & \iff & \text{for every } b \geq a, \quad b \models^+ A \quad \text{implies} \quad b \models^+ B\ ; \\
a \models^- A \to B & \iff & a \models^+ A \quad \text{and} \quad a \models^- B\ ; \\
a \models^+ \neg A & \iff & \text{for every } b \geq a, \quad b \not\models^+ A\ ; \\
a \models^- \neg A & \iff & a \models^+ A\ ; \\
a \models^+ {\sim} A & \iff & a \models^- A\ ; \\
a \models^- {\sim} A & \iff & a \models^+ A\ ; \\
a \models^+ \forall x A & \iff & \text{for every } b \geq a \text{ and every } u \in U(b)\ , \quad b \models^+ A[\underline{u}/x]\ ; \\
a \models^- \forall x A & \iff & \text{for some } u \in U(a), \quad a \models^- A[\underline{u}/x]\ .
\end{array}
$$

The case where neither $a \models^+ A$ nor $a \models^- A$ holds is interpreted as "*a cannot tell the truth of A*" and denoted by $a \not\models^{+-} A$.

A formula A of $\mathsf{N}_\neg$ is *valid in an* $\mathsf{N}_\neg$-*model* $\mathfrak{M} = (M, \leq, U, I^+, I^-)$, $\mathfrak{M} \models A$, if $a \models^+ \forall \overrightarrow{x} A$ for every $a \in M$, where $\forall \overrightarrow{x} A$ is a universal closure of A. A is *valid*, $\mathsf{N}_\neg \models A$, if A is valid in every $\mathsf{N}_\neg$-model.

A sequent $\Gamma \Rightarrow \Delta$ of $\mathsf{GN}_\neg$ is *valid* (or *valid in* $\mathfrak{M}$), $\mathsf{N}_\neg \models \Gamma \Rightarrow \Delta$ (or $\mathfrak{M} \models \Gamma \Rightarrow \Delta$), if the formula $(\bigwedge \Gamma) \to (\bigvee \Delta)$ is valid (or valid in $\mathfrak{M}$).

The result of the following lemma is necessary for $a \in M$ to be considered as a stage of knowledge, and is verified easily by the induction on the construction of a formula A.

Lemma 2.5
Let A a formula of $\mathsf{N}_\neg$, $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ be an $\mathsf{N}_\neg$-model, $a, b \in M$ and $a \leq b$. Then $a \models^+ A$ (or $a \models^- A$) implies $b \models^+ A$ (or $b \models^- A$, respectively).

The next lemma is proved by the induction on the construction of a formula A, the base case of which is verified by the condition of an $\mathsf{N}_\neg$-model $p^{I^+(a)} \cap p^{I^-(a)} = \emptyset$. This is not the case for the logics with the letter P, i.e. those which allow paraconsistency.

Lemma 2.6
Let A a formula of $\mathsf{N}_\neg$, $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ be an $\mathsf{N}_\neg$-model and $a \in M$. Then it is impossible that both $a \models^+ A$ and $a \models^- A$ hold.

The soundness of the sequent system $\mathsf{GN}_\neg$ with respect to the Kripke-type semantics defined above is also easily proved by the induction on the derivation in $\mathsf{GN}_\neg$.

THEOREM 2.7 (KRIPKE SOUNDNESS OF $\mathsf{GN}_\neg$)
If $\mathsf{GN}_\neg \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{N}_\neg \models \Gamma \Rightarrow \Delta$.

From the soundness theorem we can obtain the following result which justifies the definition of logical equivalence in $\mathsf{N}_\neg$ (see remark 2.3).

COROLLARY 2.8
$\mathsf{GN}_\neg \vdash A \leftrightarrow B$ *does not necessarily imply* $\mathsf{GN}_\neg \vdash {\sim}A \leftrightarrow {\sim}B$.

Proof. Let p and q be distinct 0-ary predicate symbols, then it is easy to verify $\mathsf{GN}_\neg \vdash (p \wedge {\sim}p) \leftrightarrow (q \wedge {\sim}q)$.

Here let $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ be an $\mathsf{N}_\neg$-model where $M = \{0\}$, $0 \models^+ p$, $0 \not\models^{+-} q$. Then $\mathfrak{M} \not\models {\sim}(p \wedge {\sim}p) \leftrightarrow {\sim}(q \wedge {\sim}q)$, hence $\mathsf{GN}_\neg \not\vdash {\sim}(p \wedge {\sim}p) \leftrightarrow {\sim}(q \wedge {\sim}q)$ by the contraposition of the theorem. ∎

Now we can give some examples of provable / unprovable formulas in $\mathsf{GN}_\neg$. We present them in comparison with Cl or Int. In the table presented below, $\neg$ is denoted by $\sim$ for Cl or Int.

	Cl	Int	$\mathsf{N}_\neg$
$A \vee {\sim}A$	○	×	×
$(A \rightarrow \perp) \rightarrow {\sim}A$	○	○	×
${\sim}(A \wedge B) \rightarrow {\sim}A \vee {\sim}B$	○	×	○
${\sim}{\sim}A \rightarrow A$	○	×	○
${\sim}\forall x{\sim}A \rightarrow \exists xA$	○	×	○
${\sim}(A \wedge {\sim}A)$	○	○	×
$(A \rightarrow B) \rightarrow ({\sim}B \rightarrow {\sim}A)$	○	○	×
$({\sim}B \rightarrow {\sim}A) \rightarrow (A \rightarrow B)$	○	×	×

2.3 Tree-Sequent System $\mathsf{TN}_\neg$

Here we introduce $\mathsf{TN}_\neg$, the tree-sequent system for $\mathsf{N}_\neg$, and consider the relation between $\mathsf{TN}_\neg$ and the Gentzen-style system $\mathsf{GN}_\neg$.

In the next subsection, we shall prove Kripke completeness of $\mathsf{GN}_\neg$ using tree-sequents. In general, a formal deductive system (of Hilbert-style, Gentzen-style, tableau method, or else) is designed to derive a formula which is valid in any model of the corresponding semantics. This request can be stated differently, as "a formal deductive system should derive a formula *for which we have no counter models*". This may help us understand how the idea of tree-sequents is obtained or justified.

In variations of Int, we can assume a Kripke model to form a tree by taking every path in an original model as a new node. Thus if we would like a unit of deduction to simulate the shape of Kripke models, its shape should be a tree.

A *tree-sequent* $\mathcal{T}$ of $\mathsf{TN}_\neg$ is a finite tree each of whose nodes is associated with its own $\Gamma \overset{\alpha}{\Rightarrow} \Delta$, where $\Gamma \Rightarrow \Delta$ is a sequent of $\mathsf{GN}_\neg$, α a finite set of variables, and the following conditions are satisfied:

- Let a be an arbitrary node of $\mathcal{T}$, $a_0(= 0)$, a_1, …, $a_n(= a)$ a path from the root 0 of $\mathcal{T}$ to a, and $\Gamma_i \overset{\alpha_i}{\Rightarrow} \Delta_i$ associated to a_i. Then α_0, α_1, …, α_n are disjoint.

The (disjoint) union $\alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_n$ is said to be the set of *available variables* at the node a.

- Every free variable which appears in the sequent associated to a is available at a.

If $\Gamma \overset{\alpha}{\Rightarrow} \Delta$ is associated to a node a (we will denote this by $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$), then α is the set of variables which are available at a for the first time in tracing from the root.

We present an example of a tree-sequent of $\mathsf{TN}_\neg$:

$$
\begin{array}{ccc}
 & & \overset{\{z\}}{\Rightarrow} r(z,y) \\
 & & | \\
\neg(p(x) \vee {\sim}p(x)) \overset{\emptyset}{\Rightarrow} p(x) \vee {\sim}p(x) & & \forall z(q(y) \vee r(z,y)) \overset{\{y\}}{\Rightarrow} q(y) \vee \forall z r(z,y) \\
\quad\diagdown & & \diagup\quad \\
 & \overset{\{x\}}{\Rightarrow} \neg\neg(p(x) \vee {\sim}p(x)) &
\end{array}
$$

This tree-sequent can also be denoted in the following styles:

$$
\dfrac{\neg(p(x) \vee {\sim}p(x)) \overset{\emptyset}{\Rightarrow} p(x) \vee {\sim}p(x) \quad \dfrac{\overset{\{z\}}{\Rightarrow} r(z,y)}{\forall z(q(y) \vee r(z,y)) \overset{\{y\}}{\Rightarrow} q(y) \vee \forall z r(z,y)}}{\overset{\{x\}}{\Rightarrow} \neg\neg(p(x) \vee {\sim}p(x))}
$$

or

$$
\Big[\ \overset{\{x\}}{\Rightarrow} \neg\neg(p(x) \vee {\sim}p(x)) \ \mid \ [\, \neg(p(x) \vee {\sim}p(x)) \overset{\emptyset}{\Rightarrow} p(x) \vee {\sim}p(x) \,] \quad [\, \forall z(q(y) \vee r(z,y)) \overset{\{y\}}{\Rightarrow} q(y) \vee \forall z r(z,y) \ \mid \overset{\{z\}}{\Rightarrow} r(z,y) \,] \ \Big]
$$

We will adopt the last one for the economy of space, i.e.

$$
\frac{\mathcal{T}_1 \quad \cdots \quad \mathcal{T}_m}{\Gamma \overset{\alpha}{\Rightarrow} \Delta} \quad \text{is denoted by} \quad [\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \mathcal{T}_1 \ldots \mathcal{T}_m].
$$

However, this style hardly clarify the structure of a tree-sequent. It will help us understand better to rewrite in the first style.

The tree-sequent system $\mathsf{TN}_\neg$ derives tree-sequents using the following axioms and inference rules:

$$
\frac{}{\cdots [A \overset{\alpha}{\Rightarrow} A \mid \cdots} \text{(axiom 1)} \qquad \frac{}{\cdots [A, {\sim}A \overset{\alpha}{\Rightarrow} \mid \cdots} \text{(axiom 2)}
$$

For instance, (axiom 1) is interpreted that if a sequent of the form $A \Rightarrow A$ is associated to any node of $\mathcal{T}$, then $\mathsf{TN}_\neg \vdash \mathcal{T}$.

$$
\frac{\cdots [\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots [\Sigma, \Gamma \overset{\alpha}{\Rightarrow} \Delta, \Pi \mid \cdots} \text{(weakening)}
$$

This is interpreted as we can add any formula to the antecedent or the succedent of a sequent associated to any node.

$$
\frac{\cdots [\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots \quad \cdots [A, \Sigma \overset{\alpha}{\Rightarrow} \Pi \mid \cdots}{\cdots [\Gamma, \Sigma \overset{\alpha}{\Rightarrow} \Delta, \Pi \mid \cdots} \text{(cut)}
$$

Here the tree-structure and the nodes not displayed are the same among the conclusion and the two hypotheses¶.

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots [A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots] \cdots] \cdots}{\cdots[A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots [\Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots] \cdots] \cdots}\ (\text{drop L})$$

Here a formula A in the antecedent of a node $b : A, \Sigma \overset{\beta}{\Rightarrow} \Delta$ is *dropped* to the antecedent of a sequent associated to the mother of b.

$$\frac{\cdots[A, B, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[A \wedge B, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ (\wedge\text{L})$$

Here the rule $(\wedge\text{L})$ of $\mathsf{GN}_\neg$ is applied to one node (more precisely, a sequent associated to one node), and the tree-structure and the nodes not displayed are the same between the hypothesis and the conclusion.

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots \quad \cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, B \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \wedge B \mid \cdots}\ (\wedge\text{R})$$

Here the tree-structure and the nodes not displayed are the same among the conclusion and the two hypotheses. This kind of remark is to be attached to every inference rule presented below.

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots \quad \cdots[B, \Sigma \overset{\alpha}{\Rightarrow} \Pi \mid \cdots}{\cdots[A \rightarrow B, \Gamma, \Sigma \overset{\alpha}{\Rightarrow} \Delta, \Pi \mid \cdots}\ (\rightarrow\text{L})$$

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots [A \overset{\emptyset}{\Rightarrow} B] \cdots] \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \rightarrow B \mid \cdots}\ (\rightarrow\text{R})_{\mathrm{t}}$$

Here the daughter $A \overset{\emptyset}{\Rightarrow} B$ (which must be a leaf) of the node $\Gamma \overset{\alpha}{\Rightarrow} \Delta$ in the hypothesis is *trimmed*. The subscript t will be added to the rules of this type, i.e. a leaf is trimmed.

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots}{\cdots[\neg A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ (\neg\text{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots [A \overset{\emptyset}{\Rightarrow}\ ] \cdots] \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, \neg A \mid \cdots}\ (\neg\text{R})_{\mathrm{t}}$$

$$\frac{\cdots[A[y/x], \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[\forall x A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ (\forall\text{L})$$

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots [\ \overset{\{z\}}{\Rightarrow} A[z/x]\,] \cdots] \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, \forall x A \mid \cdots}\ (\forall\text{R})_{\mathrm{t}}$$

z is not available at the node associated with $\Gamma \overset{\alpha}{\Rightarrow} \Delta$, thus z does not appear as a free variable in $\Gamma \Rightarrow \Delta$.

¶Cut rule is necessary for the proofs of lemma 2.9 and its counterparts for other logics, which are used only when omniscience axiom is involved. Hence just to prove the completeness of the logics without the letter O, we do not need cut rule.

Moreover, if the Gentzen-style system is proved to be cut-free, then cut rule of the tree-sequent system can also be omitted.

$$\frac{\cdots[{\sim}A,\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots \quad \cdots[{\sim}B,\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}(A \wedge B),\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}{\wedge}\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}A, {\sim}B \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}(A \wedge B) \mid \cdots}\ ({\sim}{\wedge}\mathrm{R})$$

$$\frac{\cdots[A,{\sim}B,\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}(A \to B),\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}{\to}\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots \quad \cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}B \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}(A \to B) \mid \cdots}\ ({\sim}{\to}\mathrm{R})$$

$$\frac{\cdots[A,\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}\neg A,\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}\neg\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}\neg A \mid \cdots}\ ({\sim}\neg\mathrm{R})$$

$$\frac{\cdots[A,\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}{\sim}A,\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}{\sim}\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}{\sim}A \mid \cdots}\ ({\sim}{\sim}\mathrm{R})$$

$$\frac{\cdots[{\sim}A[z/x],\Gamma \overset{\alpha\cup\{z\}}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}\forall xA,\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}\forall\mathrm{L})_{\mathrm{VC}} \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}A[y/x] \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}\forall xA \mid \cdots}\ ({\sim}\forall\mathrm{R})$$

VC, an abbreviation for the variable condition, is that the eigenvariable z must not appear in any node of the conclusion.

Now we will consider the relation between $\mathsf{TN}_\neg$ and $\mathsf{GN}_\neg$, through the following two lemmas.

Lemma 2.9
Let $\mathcal{T}$ be a tree-sequent of $\mathsf{TN}_\neg$. If $\mathcal{T}$ has a node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ such that $\mathsf{GN}_\neg \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{TN}_\neg \vdash \mathcal{T}$.

Proof. By the induction on the derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{GN}_\neg$. It is easy to check the case where the rule or the axiom applied at last is other than $(\ \ )_{\mathrm{S}}$ i.e. those which demand the succedent of the conclusion consist of only one formula. We will only present the case for $(\to\mathrm{R})_{\mathrm{S}}$, $(\neg\mathrm{R})_{\mathrm{S}}$ or $(\forall\mathrm{R})_{\mathrm{S,\,VC}}$.

For $(\to\mathrm{R})_{\mathrm{S}}$, we have in $\mathsf{TN}_\neg$

$$\cfrac{\cfrac{\cfrac{\text{ind. hyp.}}{\cdots[\ \overset{\alpha}{\Rightarrow}\ \mid \cdots[\Gamma, A \overset{\emptyset}{\Rightarrow} B]\cdots]\cdots}}{\cdots[\Gamma \overset{\alpha}{\Rightarrow}\ \mid \cdots[A \overset{\emptyset}{\Rightarrow} B]\cdots]\cdots}\ (\text{drop L})}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} A \to B \mid \cdots}\ (\to\mathrm{R})$$

For $(\neg\mathrm{R})_{\mathrm{S}}$,

$$\cfrac{\cfrac{\cfrac{\text{ind. hyp.}}{\cdots[\ \overset{\alpha}{\Rightarrow}\ \mid \cdots[\Gamma, A \overset{\emptyset}{\Rightarrow}\ ]\cdots]\cdots}}{\cdots[\Gamma \overset{\alpha}{\Rightarrow}\ \mid \cdots[A \overset{\emptyset}{\Rightarrow}\ ]\cdots]\cdots}\ (\text{drop L})}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \neg A \mid \cdots}\ (\neg\mathrm{R})$$

For $(\forall \mathrm{R})_{\mathrm{S,\,VC}}$,

$$\cfrac{\cfrac{\cfrac{\text{ind. hyp.}}{\cdots[\ \overset{\alpha}{\Rightarrow}\ |\cdots[\Gamma \overset{\{z\}}{\Rightarrow} A[z/x]\,]\cdots]\cdots}}{\cdots[\Gamma \overset{\alpha}{\Rightarrow}\ |\cdots[\ \overset{\{z\}}{\Rightarrow} A[z/x]\,]\cdots]\cdots}\ (\text{drop L})}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \forall x A \mid \cdots}\ (\forall \mathrm{R})$$

∎

A *pre-tree-sequent* $\mathcal{T}$ of $\mathsf{TN}_\neg$ satisfies all the conditions of a tree-sequent except the one that if x has a free occurrence in $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ then x is available at a. This concept is necessary for the proof of the next lemma which is done by an induction.

The *translation into a formula* of a pre-tree-sequent $\mathcal{T}$ of $\mathsf{TN}_\neg$, denoted by $\mathcal{T}^f$, is defined inductively on the height of $\mathcal{T}$;

$$[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \mathcal{T}_1 \ldots \mathcal{T}_m]^f :\equiv \forall \vec{\alpha}\big((\bigwedge \Gamma) \to (\bigvee \Delta) \vee \mathcal{T}_1^f \vee \cdots \vee \mathcal{T}_m^f \big)$$

Then the next important lemma is proved, which approves so called the soundness of $\mathsf{TN}_\neg$.

LEMMA 2.10
If $\mathsf{TN}_\neg \vdash \mathcal{T}$*, then* $\mathsf{GN}_\neg \vdash \mathcal{T}^f$*.*

To prove this lemma we will prepare a couple of sublemmas.

SUBLEMMA 2.11
Let $\mathcal{T}$ be a tree-sequent of $\mathsf{TN}_\neg$, a a node of $\mathcal{T}$, $\mathcal{T}'$ a pre-tree-sequent which consists of all the descendants of a, $\mathcal{T}_1', \ldots, \mathcal{T}_k'$ pre-tree-sequents of $\mathsf{TN}_\neg$, and $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be a tree-sequent obtained by substituting $\mathcal{T}'$ by $\mathcal{T}_1', \ldots, \mathcal{T}_k'$, respectively (Fig. 2).

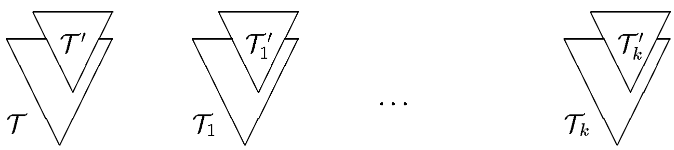

Fig. 2: $\mathcal{T}, \mathcal{T}_1, \ldots, \mathcal{T}_k$

Then $\mathsf{GN}_\neg \vdash \mathcal{T}_1'^f, \ldots, \mathcal{T}_k'^f \Rightarrow \mathcal{T}'^f$ *implies* $\mathsf{GN}_\neg \vdash \mathcal{T}_1^f, \ldots, \mathcal{T}_k^f \Rightarrow \mathcal{T}^f$*.*

SUBLEMMA 2.12
Let $\mathcal{T}$ and $\mathcal{T}'$ the same as those in the sublemma above. Then $\mathsf{GN}_\neg \vdash \mathcal{T}'^f$ *implies* $\mathsf{GN}_\neg \vdash \mathcal{T}^f$*.*

Proof. Both sublemmas are proved easily by the induction on the height of a in $\mathcal{T}$. ∎

Proof of lemma 2.10
By the induction on the derivation of $\mathcal{T}$ in $\mathsf{TN}_\neg$. For the base case, i.e. when $\mathcal{T}$ is an axiom, use sublemma 2.12. For the step cases where $\mathcal{T}$ is derived by an inference rule with a hypothesis (or hypotheses), use sublemma 2.11. ∎

2.4 Kripke Completeness of $\mathsf{GN}_\neg$

Here we present the way how we can construct what we may call a counter $\mathsf{N}_\neg$-model for an unprovable tree-sequent $\mathcal{T}$ of $\mathsf{TN}_\neg$, hence Kripke completeness of $\mathsf{TN}_\neg$. Then Kripke completeness of $\mathsf{GN}_\neg$ is obtained immediately by lemma 2.10.

The sketch is as follows. Let $\mathcal{T}$ be an unprovable tree-sequent of $\mathsf{TN}_\neg$, then we can extend $\mathcal{T}$ to obtain an infinite tree-sequent $\widetilde{\mathcal{T}}$ which is $\mathsf{TN}_\neg$-saturated (we shall define this concept later). A $\mathsf{TN}_\neg$-saturated tree-sequent induces an $\mathsf{N}_\neg$-model $\mathfrak{M}$ (here the set of available variables at a is used as the seed of the domain of a), and this is what we would like to obtain, i.e. a counter-model for $\mathcal{T}$.

We start with defining the concept $\mathsf{TN}_\neg$-saturatedness, which is a natural extension of that for LK. Here an *infinite tree-sequent* of $\mathsf{TN}_\neg$ is a possibly infinite tree each of whose node is associated with its own $\Gamma \overset{\alpha}{\Rightarrow} \Delta$, where Γ and Δ are sets of a possibly infinite number of formulas of $\mathsf{N}_\neg$, and α is a set of a possibly infinite number of variables, and satisfies the same conditions as those of a tree-sequent, as to the availability of variables.

Definition 2.13 ($\mathsf{TN}_\neg$-saturatedness)
An infinite tree-sequent $\mathcal{T}$ of $\mathsf{TN}_\neg$ is $\mathsf{TN}_\neg$-*saturated* if it satisfies the following conditions:

1. Let $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ and $b : \Sigma \overset{\beta}{\Rightarrow} \Pi$ be nodes of $\mathcal{T}$. If b is a descendant of a, then $\Gamma \subseteq \Sigma$;
2. For every node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ of $\mathcal{T}$,
   (a) ($\wedge$L)-saturated : If $A \wedge B \in \Gamma$, then $A \in \Gamma$ and $B \in \Gamma$;
   (b) ($\wedge$R)-saturated : If $A \wedge B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$;
   (c) ($\to$L)-saturated : If $A \to B \in \Gamma$, then $A \in \Delta$ or $B \in \Gamma$;
   (d) $(\to\mathrm{R})_{\mathsf{t}}$-saturated : If $A \to B \in \Delta$, then there exists a descendant $b : \Sigma \overset{\beta}{\Rightarrow} \Pi$ of a such that $A \in \Sigma$ and $B \in \Pi$;
   (e) ($\neg$L)-saturated : If $\neg A \in \Gamma$, then $A \in \Delta$;
   (f) $(\neg\mathrm{R})_{\mathsf{t}}$-saturated : If $\neg A \in \Delta$, then there exists a descendant $b : \Sigma \overset{\beta}{\Rightarrow} \Pi$ of a such that $A \in \Sigma$;
   (g) ($\forall$L)-saturated : If $\forall x A \in \Gamma$, then $A[y/x] \in \Gamma$ for every available variable y at a ;
   (h) $(\forall\mathrm{R})_{\mathsf{t}}$-saturated : If $\forall x A \in \Delta$, then there exists a descendant $b : \Sigma \overset{\beta}{\Rightarrow} \Pi$ of a and a variable y such that $A[y/x] \in \Pi$;
   (i) ($\sim\wedge$L)-saturated : If $\sim(A \wedge B) \in \Gamma$, then $\sim A \in \Gamma$ or $\sim B \in \Gamma$;
   (j) ($\sim\wedge$R)-saturated : If $\sim(A \wedge B) \in \Delta$, then $\sim A \in \Delta$ and $\sim B \in \Delta$;
   (k) ($\sim\to$L)-saturated : If $\sim(A \to B) \in \Gamma$, then $A \in \Gamma$ and $\sim B \in \Gamma$;

(l) $(\sim\!\to\!\mathrm{R})$-saturated : If $\sim\!(A \to B) \in \Delta$, then $A \in \Delta$ or $\sim\! B \in \Delta$;

(m) $(\sim\!\neg\mathrm{L})$-saturated : If $\sim\!\neg A \in \Gamma$, then $A \in \Gamma$;

(n) $(\sim\!\neg\mathrm{R})$-saturated : If $\sim\!\neg A \in \Delta$, then $A \in \Delta$;

(o) $(\sim\sim\!\mathrm{L})$-saturated : If $\sim\sim\! A \in \Gamma$, then $A \in \Gamma$;

(p) $(\sim\sim\!\mathrm{R})$-saturated : If $\sim\sim\! A \in \Delta$, then $A \in \Delta$;

(q) $(\sim\!\forall\mathrm{L})$-saturated : If $\sim\!\forall x A \in \Gamma$, then $\sim\! A[y/x] \in \Gamma$ for some variable y ;

(r) $(\sim\!\forall\mathrm{R})$-saturated : If $\sim\!\forall x A \in \Delta$, then $\sim\! A[y/x] \in \Delta$ for every available variable y at a.

Note that for the rules $(\quad)_{\mathrm{t}}$, descendants are involved.

Now let $\mathcal{T}$ a tree-sequent of $\mathsf{TN}_\neg$ such that there is at least one variable available at its root, and $\mathsf{TN}_\neg \not\vdash \mathcal{T}$. We describe the way $\mathcal{T}$ is extended to a $\mathsf{TN}_\neg$-saturated infinite tree-sequent.

By our definition of the language of $\mathsf{N}_\neg$, there are only countably many formulas of $\mathsf{N}_\neg$, hence we can enumerate them as $B_1, B_2, \ldots$. We arrange this to obtain a new sequence,

$$B_1 \mid B_1, B_2 \mid B_1, B_2, B_3 \mid \ldots$$

which we denote by $A_1, A_2, \ldots$. The point is that every formula of $\mathsf{N}_\neg$ appears infinitely many times in $A_1, A_2, \ldots$. Again by definition, a term of $\mathsf{N}_\neg$ is to be a variable and we can enumerate them as $x_1, x_2, \ldots$.

We extend $\mathcal{T}_0 (:\equiv \mathcal{T})$ step by step and obtain a sequence of (finite) tree-sequents $\mathcal{T}_1, \mathcal{T}_2, \ldots$. The step from $\mathcal{T}_{i-1}$ to $\mathcal{T}_i$ is for the reduction of the formula A_i, described below in detail. It should be paid attention to that each operation preserves the unprovability of the tree-sequent.

1. (inheritance) For every node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ of $\mathcal{T}_{i-1}$ such that $A_i \in \Gamma$, add A_i to the antecedent of each descendant of a. This operation is called an *inheritance*, and preserves unprovability because of the rule (drop L) ;

2. (reduction) According to the shape of A_i, one of the following operations is executed for each node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ of $\mathcal{T}_{i-1}$:

   (a) $A_i \equiv B \wedge C$. If $A_i \in \Gamma$, then add B and C to Γ. This operation preserves unprovability because of the rule $(\wedge\mathrm{L})$ of $\mathsf{TN}_\neg$. If $A_i \in \Delta$, then at least one of the tree-sequents obtained by adding B or C to Δ is unprovable because of the rule $(\wedge\mathrm{R})$. Take the unprovable one as a resulting tree-sequent ;

   (b) $A_i \equiv B \to C$. If $A_i \in \Gamma$, then add B to Δ, or C to Γ, so that the resulting tree-sequent remains unprovable. If $A_i \in \Delta$, then make a new daughter (which is a leaf) of a, $b : B \overset{\emptyset}{\Rightarrow} C$;

   (c) $A_i \equiv \neg B$. If $A_i \in \Gamma$, then add B to Δ. If $A_i \in \Delta$, then make a new daughter $b : B \overset{\emptyset}{\Rightarrow}$ of a ;

   (d) $A_i \equiv \forall x B$. If $A_i \in \Gamma$, then add $B[y/x]$ to Γ, for every y which is available at a and is in $\{x_1, \ldots, x_i\}$. If $A_i \in \Delta$, then take a fresh variable x_m and make a new daughter $b : \overset{\{x_m\}}{\Rightarrow} B[x_m/x]$ of a ;

(e) $A_i \equiv {\sim}(B \wedge C)$. If $A_i \in \Gamma$, add ${\sim}B$ or ${\sim}C$ to Γ, so that the resulting tree-sequent remains unprovable. If $A_i \in \Delta$, add ${\sim}B$ and ${\sim}C$ to Δ;

(f) $A_i \equiv {\sim}(B \to C)$. If $A_i \in \Gamma$, add B and ${\sim}C$ to Γ. If $A_i \in \Delta$, add B or ${\sim}C$ to Δ so that the resulting tree-sequent remains unprovable;

(g) $A_i \equiv {\sim}\neg B$ or $A_i \equiv {\sim}{\sim}B$. If $A_i \in \Gamma$ (or Δ), add B to Γ (or Δ, respectively);

(h) $A_i \equiv {\sim}\forall x B$. If $A_i \in \Gamma$, then take a fresh variable x_m, add ${\sim}B[x_m/x]$ to Γ, and also add x_m to α. If $A_i \in \Delta$, add ${\sim}B[y/x]$ to Δ for every y which is available at a and in $\{x_1, \dots, x_i\}$.

Take an infinite tree-sequent $\widetilde{\mathcal{T}}$ of $\mathsf{TN}_\neg$ as a union of $\mathcal{T}_0, \mathcal{T}_1, \dots$, i.e. the tree-structure of $\widetilde{\mathcal{T}}$ is a union of those of $\mathcal{T}_0, \mathcal{T}_1, \dots$, and the infinite sequent or the set of variables associated to each node is again a union of those of $\mathcal{T}_0, \mathcal{T}_1, \dots$.

Then it is easily verified that $\widetilde{\mathcal{T}}$ is $\mathsf{TN}_\neg$-saturated.

Now we will construct an $\mathsf{N}_\neg$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ from $\widetilde{\mathcal{T}}$. Let $(M, \leq)$ be a tree-structure of $\widetilde{\mathcal{T}}$ and $U(a)$ be the set of available variables at a (here the condition that one variable is available at the root of $\mathcal{T}$ assures that $U(a) \neq \emptyset$ for every $a \in M$). For a node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ of $\widetilde{\mathcal{T}}$ and an m-ary predicate symbol p,

$$p^{I^+(a)} := \{(y_1, \dots, y_m) \mid p(y_1, \dots, y_m) \in \Gamma\}$$
$$p^{I^-(a)} := \{(y_1, \dots, y_m) \mid {\sim}p(y_1, \dots, y_m) \in \Gamma\}$$

The model $\mathfrak{M}$ defined above certainly satisfies the conditions of $\mathsf{N}_\neg$-models; for example, the condition $p^{I^+(a)} \cap p^{I^-(a)} = \emptyset$ is satisfied since every finite sub-tree-sequent of $\mathcal{T}$ is unprovable in $\mathsf{TN}_\neg$ and $\mathsf{TN}_\neg$ has an axiom

$$\frac{}{\cdots [p(\overrightarrow{x}), {\sim}p(\overrightarrow{x}) \overset{\alpha}{\Rightarrow} | \cdots} \text{ (axiom 2)}$$

Then the following fact is again easily verified by the induction on the construction of a formula A: for every node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ of $\widetilde{\mathcal{T}}$, $A \in \Gamma$ implies $a \models^+ A[\underline{\overrightarrow{x}}/\overrightarrow{x}]$ in $\mathfrak{M}$ and $A \in \Delta$ implies $a \not\models^+ A[\underline{\overrightarrow{x}}/\overrightarrow{x}]$. Here we used the fact that $\widetilde{\mathcal{T}}$ is $\mathsf{TN}_\neg$-saturated.

With the fact that $\mathcal{T}$ is a sub-tree-sequent of $\widetilde{\mathcal{T}}$, we have proved the following theorem:

Theorem 2.14 (Kripke Completeness of $\mathsf{TN}_\neg$)
Let $\mathcal{T}$ be a tree-sequent of $\mathsf{TN}_\neg$ at whose root at least one variable is available, and $\mathsf{TN}_\neg \nvdash \mathcal{T}$. Then there exists a counter $\mathsf{N}_\neg$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ for $\mathcal{T}$, that is:

- *The tree-structure of $\mathcal{T}$ can be embedded in $(M, \leq)$;*
- *For each node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ of $\mathcal{T}$, $A \in \Gamma$ implies $a \models^+ A[\underline{\overrightarrow{x}}/\overrightarrow{x}]$ in $\mathfrak{M}$, and $A \in \Delta$ implies $a \not\models^+ A[\underline{\overrightarrow{x}}/\overrightarrow{x}]$.*

From this we immediately obtain Kripke completeness of $\mathsf{GN}_\neg$.

Corollary 2.15 (Kripke Completeness of $\mathsf{GN}_\neg$)
For every formula A of $\mathsf{N}_\neg$, $\mathsf{N}_\neg \models A$ implies $\mathsf{GN}_\neg \vdash A$.

Proof. Let $\mathsf{GN}_\neg \nvdash A$, and α be a nonempty finite set of variables which includes all the free variables of A. Then by $[\ \overset{\alpha}{\Rightarrow} A]^f \cong_{\mathsf{N}_\neg} \forall \overrightarrow{\alpha} A$ and the contraposition of lemma 2.10, $\mathsf{TN}_\neg \nvdash [\ \overset{\alpha}{\Rightarrow} A]$. The theorem shows that there exists an $\mathsf{N}_\neg$-model $\mathfrak{M}$ such that $\mathfrak{M} \not\models A$. ∎

3 Logic $\mathsf{N_\neg D}$

In this section we consider the logic $\mathsf{N_\neg D}$, a variation of $\mathsf{N_\neg}$ obtained by adding constant domain axiom $\forall x(A(x) \vee B) \to \forall x A(x) \vee B$. This axiom corresponds to the reference to Kripke models that *every possible world has the same domain in common*, as we shall show later. Kripke completeness of $\mathsf{GN_\neg D}$, the Gentzen-style sequent system for $\mathsf{N_\neg D}$, is shown just as that for $\mathsf{GN_\neg}$, using tree-sequents.

DEFINITION 3.1 (SEQUENT SYSTEM $\mathsf{GN_\neg D}$)
$\mathsf{GN_\neg D}$ is $\mathsf{GN_\neg}$ plus the following axiom:

$$\frac{}{\Rightarrow \forall x(A(x) \vee B) \to \forall x A(x) \vee B}\ (\text{axiom D})$$

The following is a well-known result in the intermediate logic CD and also easily verified.

LEMMA 3.2
Let $\mathsf{GN_\neg D'}$ *be the system obtained from* $\mathsf{GN_\neg}$ *by substituting the rule* $(\forall \mathrm{R})_{\mathsf{S,VC}}$ *by*

$$\frac{\Gamma \Rightarrow \Delta, A[z/x]}{\Gamma \Rightarrow \Delta, \forall x A}\ (\forall \mathrm{R})_{\mathsf{VC}}$$

i.e. allow plural formulas in the succedent. Then $\mathsf{GN_\neg D'}$ *is equivalent to* $\mathsf{GN_\neg D}$.

DEFINITION 3.3 ($\mathsf{N_\neg D}$-MODEL)
An $\mathsf{N_\neg}$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ is an $\mathsf{N_\neg D}$-*model* if U is a constant map.

We define the validity, $\mathsf{N_\neg D} \models A$, just like that of $\mathsf{N_\neg}$. Then the soundness of $\mathsf{GN_\neg D}$ is shown by the induction on the derivation.

THEOREM 3.4 (KRIPKE SOUNDNESS OF $\mathsf{GN_\neg D}$)
If $\mathsf{GN_\neg D} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{N_\neg D} \models \Gamma \Rightarrow \Delta$.

Now we introduce $\mathsf{TN_\neg D}$, the tree-sequent system for $\mathsf{N_\neg D}$. A *tree-sequent* $\mathcal{T}$ of $\mathsf{TN_\neg D}$ is a finite tree each of whose nodes is associated with a sequent $\Gamma \Rightarrow \Delta$ of $\mathsf{GN_\neg}$ (or $\mathsf{GN_\neg D}$. Each will do since they have the same language in common). This definition is different from that of $\mathsf{TN_\neg}$ in that the indication of a set of variables is omitted here, because in $\mathsf{TN_\neg D}$ we need not consider whether a variable is available or not, which worked as a seed of a domain in the proofs above.

The axioms and inference rules of $\mathsf{TN_\neg D}$ are the same as those of $\mathsf{TN_\neg}$, except that they have no consideration about availability of variables, and the rule $(\forall \mathrm{R})_{\mathsf{t}}$ (which trims a leaf) of $\mathsf{TN_\neg}$ is substituted by

$$\frac{\cdots [\Gamma \Rightarrow \Delta, A[z/x] \mid \cdots}{\cdots [\Gamma \Rightarrow \Delta, \forall x A \mid \cdots}\ (\forall \mathrm{R})_{\mathsf{VC}}$$

. The rule adopted here may be justified by lemma 3.2.

Next we consider the relation between $\mathsf{TN_\neg D}$ and $\mathsf{GN_\neg D}$, just as that between $\mathsf{TN_\neg}$ and $\mathsf{GN_\neg}$.

LEMMA 3.5
Let $\mathcal{T}$ *be a tree-sequent of* $\mathsf{TN_\neg D}$. *If* $\mathcal{T}$ *has a node* $a : \Gamma \Rightarrow \Delta$ *such that* $\mathsf{GN_\neg D} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{TN_\neg D} \vdash \mathcal{T}$.

Proof. Just like the case of $\mathsf{TN_\neg}$, i.e. lemma 2.9. ∎

DEFINITION 3.6 (TRANSLATION)
A *translation into a formula* of a tree-sequent $\mathcal{T}$ of $\mathsf{TN}_\neg\mathsf{D}$, denoted by $\mathcal{T}^f$, is a universal closure of $\mathcal{T}^p$, which in turn is defined inductively on the height of $\mathcal{T}$;

$$[\Gamma \Rightarrow \Delta \mid \mathcal{T}_1 \dots \mathcal{T}_m]^p :\equiv (\bigwedge \Gamma) \to (\bigvee \Delta) \vee \mathcal{T}_1^p \vee \dots \vee \mathcal{T}_m^p$$

Here the concept pre-tree-sequent is not necessary because it was for considering availability of variables.

LEMMA 3.7
If $\mathsf{TN}_\neg\mathsf{D} \vdash \mathcal{T}$*, then* $\mathsf{GN}_\neg\mathsf{D} \vdash \mathcal{T}^f$*.*

Again we prepare a couple of sublemmas, the proofs for which are easy by induction just like those for lemmas 2.11 and 2.12.

SUBLEMMA 3.8
Let $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{T}_1', \dots, \mathcal{T}_k'$ *and* $\mathcal{T}_1, \dots, \mathcal{T}_k$ *be just as those of lemma 2.11. Then* $\mathsf{GN}_\neg\mathsf{D} \vdash \mathcal{T}_1'^p, \dots, \mathcal{T}_k'^p \Rightarrow \mathcal{T}'^p$ *implies* $\mathsf{GN}_\neg\mathsf{D} \vdash \mathcal{T}_1^p, \dots, \mathcal{T}_k^p \Rightarrow \mathcal{T}^p$*.*

SUBLEMMA 3.9
Let $\mathcal{T}$ *and* $\mathcal{T}'$ *the same as those in the sublemma above. Then* $\mathsf{GN}_\neg\mathsf{D} \vdash \mathcal{T}'^f$ *implies* $\mathsf{GN}_\neg\mathsf{D} \vdash \mathcal{T}^f$*.*

Proof of lemma 3.7
By the induction on the derivation of $\mathcal{T}$ in $\mathsf{TN}_\neg\mathsf{D}$. The proof is carried out just like that of lemma 2.10, and it suffices to check the case where $\mathcal{T}$ is inferred by the rule $(\forall\mathrm{R})_{\mathsf{VC}}$ of $\mathsf{TN}_\neg\mathsf{D}$. In fact, we can not apply sublemma 3.8 which was used in the case of $\mathsf{TN}_\neg$, therefore we will make use of the character of $\mathsf{N}_\neg\mathsf{D}$ instead.

Suppose $\mathsf{GN}_\neg\mathsf{D} \vdash (\cdots[\Gamma \Rightarrow \Delta, A[z/x] \mid \cdots)^f$ as an induction hypothesis. Then $(\cdots[\Gamma \Rightarrow \Delta, A[z/x] \mid \cdots)^f$ is in the form

$$\forall \overrightarrow{x} \forall z(B_1 \to C_1 \vee (B_2 \to C_2 \vee (\cdots \vee (B_n \to C_n \vee A[z/x]) \cdots)))$$

, where z has no free occurrence in any of B_i or C_i. Then we repeatedly apply the following logical equivalences which are easily verified:

$$\forall x(B \to C(x)) \cong_{\mathsf{N}_\neg\mathsf{D}} B \to \forall x C(x) \quad \forall x(C \vee D(x)) \cong_{\mathsf{N}_\neg\mathsf{D}} C \vee \forall x D(x)$$

The second one is unique to logics with constant domain axiom. Thus we have the logical equivalence,

$$\begin{aligned}
&(\cdots[\Gamma \Rightarrow \Delta, A[z/x] \mid \cdots)^f \\
&\qquad \equiv \forall \overrightarrow{x} \forall z(B_1 \to C_1 \vee (B_2 \to C_2 \vee (\cdots \vee (B_n \to C_n \vee A[z/x]) \cdots))) \\
&\qquad \cong_{\mathsf{N}_\neg\mathsf{D}} \forall \overrightarrow{x}(B_1 \to C_1 \vee (B_2 \to C_2 \vee (\cdots \vee (B_n \to C_n \vee \forall x A) \cdots))) \\
&\qquad \equiv (\cdots[\Gamma \Rightarrow \Delta, \forall x A \mid \cdots)^f
\end{aligned}$$

By the induction hypothesis and the cut rule, we obtain $\mathsf{GN}_\neg\mathsf{D} \vdash (\cdots[\Gamma \Rightarrow \Delta, \forall x A \mid \cdots)^f$. ∎

DEFINITION 3.10 ($\mathsf{TN}_\neg\mathsf{D}$-SATURATEDNESS)
An infinite tree-sequent $\mathcal{T}$ of $\mathsf{TN}_\neg\mathsf{D}$ is $\mathsf{TN}_\neg\mathsf{D}$-*saturated* if $\mathcal{T}$ satisfies the conditions of $\mathsf{TN}_\neg$-saturatedness (see definition 2.13) where the word *available* is omitted and the condition (2h) $(\forall\mathrm{R})_{\mathrm{t}}$-saturated is substituted by:

($\forall$R)-saturated : If $\forall x A \in \Delta$, then $A[y/x] \in \Delta$ for some variable y;

Now we can prove Kripke completeness extending an unprovable tree-sequent of $\mathsf{TN_\neg D}$ into saturation to obtain a counter model, as we did in $\mathsf{TN_\neg}$.

Arrange all the formulas and variables to obtain sequences $A_1, A_2, \dots$ and $x_1, x_2, \dots$ as we did in the last section. Let $\mathcal{T}$ a tree-sequent of $\mathsf{TN_\neg D}$ such that $\mathsf{TN_\neg D} \not\vdash \mathcal{T}$. We extend $\mathcal{T}$ step by step and obtain $\mathcal{T}_0, \mathcal{T}_1, \dots$, where each step $\mathcal{T}_{i-1}$ to $\mathcal{T}_i$ is for the reduction of A_i. The operation carried out at each step is almost the same as that in $\mathsf{TN_\neg}$, being different in the following points:

- The word *available* is omitted;
- The case (2d) is substituted by:

  (2d) $A_i \equiv \forall x B$. If $A_i \in \Gamma$, then add all of $B[x_1/x], \dots, B[x_i/x]$ to Γ. If $A_i \in \Delta$, then take a fresh variable x_m and add $B[x_m/x]$ to Δ. This operation preserves unprovability because of the rule ($\forall$R) of $\mathsf{TN_\neg D}$;

Let $\widetilde{\mathcal{T}}$ be a union of $\mathcal{T}_0, \mathcal{T}_1, \dots$. Then $\widetilde{\mathcal{T}}$ is $\mathsf{TN_\neg D}$-saturated.

From $\widetilde{\mathcal{T}}$, we can construct an $\mathsf{N_\neg D}$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ as follows: let $(M, \leq)$ be a tree-structure of $\widetilde{\mathcal{T}}$, $U(a)$ the set of all the variables for every $a \in M$ (hence $\mathfrak{M}$ is $\mathsf{N_\neg D}$-model, i.e. has the constant domain), and interpretations I^+ and I^- just the same as those for $\mathsf{TN_\neg}$.

Then $\mathfrak{M}$ is a counter model for $\mathcal{T}$, and we obtain the following theorem:

THEOREM 3.11 (KRIPKE COMPLETENESS OF $\mathsf{TN_\neg D}$)
Let $\mathcal{T}$ be a tree-sequent of $\mathsf{TN_\neg D}$ such that $\mathsf{TN_\neg} \not\vdash \mathcal{T}$. Then there exists a counter $\mathsf{N_\neg D}$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ for $\mathcal{T}$, that is:

- *The tree-structure of $\mathcal{T}$ can be embedded in $(M, \leq)$;*
- *For each node $a : \Gamma \Rightarrow \Delta$ of $\mathcal{T}$, $A \in \Gamma$ implies $a \models^+ A[\underline{\overrightarrow{x}}/\overrightarrow{x}]$ in $\mathfrak{M}$, and $A \in \Delta$ implies $a \not\models^+ A[\underline{\overrightarrow{x}}/\overrightarrow{x}]$.*

Kripke completeness of the Gentzen-style system $\mathsf{GN_\neg D}$ is derived immediately as a corollary:

COROLLARY 3.12 (KRIPKE COMPLETENESS OF $\mathsf{GN_\neg D}$)
For every formula A of $\mathsf{N_\neg D}$, $\mathsf{N_\neg D} \models A$ implies $\mathsf{GN_\neg D} \vdash A$.

Proof. Let $\mathsf{GN_\neg D} \not\vdash A$ and $\mathcal{T} :\equiv [\ \Rightarrow A]$. Since $\mathcal{T}^f \cong_{\mathsf{N_\neg D}} \forall \overrightarrow{x} A$ and the contraposition of lemma 3.7, $\mathsf{TN_\neg D} \not\vdash \mathcal{T}$. The theorem shows that there exists an $\mathsf{N_\neg D}$-model $\mathfrak{M}$ such that $\mathfrak{M} \not\models A$. ∎

4 Logics $\mathsf{N_\neg P}$, $\mathsf{N_\neg DP}$

In this section we consider the logics $\mathsf{N_\neg P}$ and $\mathsf{N_\neg DP}$, obtained by omitting the axiom $A \to ({\sim}A \to B)$ in $\mathsf{N_\neg}$ and $\mathsf{N_\neg D}$ which results in paraconsistency.

A logic L is *explosive* if in GL, the Gentzen-style system for L, $\mathsf{GL} \vdash A, \neg A \Rightarrow B$ (it depends on the language of L which one of $\neg$ or $\sim$ we adopt, and in fact we use $\sim$ here). For instance, Cl, Int and $\mathsf{N_\neg}$ are all explosive. L is *paraconsistent* if it is not explosive.

Paraconsistent logics are certainly of considerable use as tools to formalize theories which have some contradiction but are non-trivial. We can give some examples of such theories; the Newton-Leibniz version of calculus, Cantor's naive set theory, or early quantum mechanics. The history and some approaches other than we present in this paper are in the literature [PR84] by Priest and Routley, who adopt relevant logics as an approach to paraconsistency. Whether $\mathsf{N}_\neg\mathsf{P}$ or $\mathsf{N}_\neg\mathsf{DP}$ is useful with respect to the motivation stated above is to be inspected further.

The proof of Kripke completeness of $\mathsf{GN}_\neg\mathsf{P}$ or $\mathsf{GN}_\neg\mathsf{DP}$ is exactly the same as that of $\mathsf{GN}_\neg$ or $\mathsf{GN}_\neg\mathsf{D}$, respectively.

The Gentzen-style sequent system $\mathsf{GN}_\neg\mathsf{P}$ or $\mathsf{GN}_\neg\mathsf{DP}$ is defined by omitting the axiom

$$\frac{}{A, \sim A \Rightarrow}\ \text{(axiom 2)}$$

in $\mathsf{GN}_\neg$ or $\mathsf{GN}_\neg\mathsf{D}$, respectively.

$\mathfrak{M} = (M, \leq, U, I^+, I^-)$ is an $\mathsf{N}_\neg\mathsf{P}$-model (or a $\mathsf{N}_\neg\mathsf{DP}$-model) if it satisfies all the conditions of $\mathsf{N}_\neg$-model (or a $\mathsf{N}_\neg\mathsf{D}$-model) except the one $p^{I^+(a)} \cap p^{I^-(a)} = \emptyset$.

The tree-sequent system $\mathsf{TN}_\neg\mathsf{P}$ or $\mathsf{TN}_\neg\mathsf{DP}$ is obtained by omitting the axiom

$$\frac{}{\cdots [A, \sim A \overset{\alpha}{\Rightarrow} | \cdots}\ \text{(axiom 2)}$$

in $\mathsf{TN}_\neg$ or $\mathsf{TN}_\neg\mathsf{D}$, respectively.

Then we can prove Kripke completeness of $\mathsf{GN}_\neg\mathsf{P}$ or $\mathsf{GN}_\neg\mathsf{DP}$ just like that of $\mathsf{GN}_\neg$ or $\mathsf{GN}_\neg\mathsf{D}$, respectively.

THEOREM 4.1 (KRIPKE SOUNDNESS AND COMPLETENESS OF $\mathsf{GN}_\neg\mathsf{P}$ AND $\mathsf{GN}_\neg\mathsf{DP}$)
Let L *be either* $\mathsf{N}_\neg\mathsf{P}$ *or* $\mathsf{N}_\neg\mathsf{DP}$, *and* A *an arbitrary formula of* L. *Then* $\mathsf{L} \models A$ *if and only if* $\mathsf{GL} \vdash A$.

From the soundness, we can assure that $\mathsf{N}_\neg\mathsf{P}$ and $\mathsf{N}_\neg\mathsf{DP}$ are both paraconsistent:

COROLLARY 4.2
Let L *be either* $\mathsf{N}_\neg\mathsf{P}$ *or* $\mathsf{N}_\neg\mathsf{DP}$. *Then sequents of the form* $A, \sim A \Rightarrow B$ *are not provable in* GL, *in general.*

Proof. Let p and q 0-ary predicate symbols, $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ be an L-model such that $M = \{0\}$, $0 \models^+ p$, $0 \models^- p$, and $0 \not\models^{+-} q$. Then $\mathfrak{M} \not\models p \wedge \sim p \to q$ and by the soundness we have $\mathsf{GL} \not\vdash p, \sim p \Rightarrow q$. ∎

5 Logics $\mathsf{N}_\neg\mathsf{O}$, $\mathsf{N}_\neg\mathsf{DO}$

5.1 First Proof – Kripke Completeness of $\mathsf{GN}_\neg\mathsf{O}$

In this subsection we consider the logic $\mathsf{N}_\neg\mathsf{O}$ (O for *omniscience axiom*, $\neg\neg(A \vee \sim A)$), whose Kripke completeness we can hardly prove by only applying the tree-sequent method we used above. Here we use some novel results about the relation to classical logic.

$\mathsf{GN}_\neg\mathsf{O}$, the Gentzen-style sequent system for $\mathsf{N}_\neg\mathsf{O}$, is $\mathsf{GN}_\neg$ plus the axiom

$$\frac{}{\Rightarrow \neg\neg(A \vee \sim A)}\ \text{(axiom O)}$$

An $\mathsf{N}_\neg$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ is an $\mathsf{N}_\neg\mathsf{O}$-*model* if for every $a \in M$, there exists $a'(\geq a)$ such that:

- a' is maximal in the poset $(M, \leq)$;
- $p^{I^+(a')} \cup p^{I^-(a')} = U(a')^m$ for every predicate symbol p, where m is the arity of p.

From the condition $p^{I^+(a')} \cap p^{I^-(a')} = \emptyset$ of an $\mathsf{N}_\neg$-model, the union $p^{I^+(a')} \cup p^{I^-(a')}$ is disjoint.

LEMMA 5.1
Let $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ be an $\mathsf{N}_\neg$-model, $a' \in M$ satisfy the two conditions itemized above, and A be an arbitrary closed formula of $\mathsf{N}_\neg$. Then $a' \models^+ A$ or $a' \models^- A$, and by lemma 2.6 exactly one of them holds.

Proof. The proof is easy by the induction on the construction of A, a formula of $\mathsf{N}_\neg$ where $\underline{u}$, a name of an individual, is allowed to occur. We present here only the step cases where $A \equiv B \to C$ or $A \equiv \forall x A$.

When $A \equiv B \to C$, assume $a' \not\models^+ A$. We shall show that $a' \models^- A$ holds. If $a' \models^- B$, by lemma 2.5 for every $c \geq a'$ $c \models^- B$, which in turn implies $c \not\models^+ B$ by lemma 2.6. Hence $a' \models^+ B \to C$, which is a contradiction. So we must have $a' \not\models^- B$, which is equivalent to $a' \models^+ B$ by the induction hypothesis. Now $a' \models^+ C$ leads to a contradiction in the same way, so we must have $a' \models^- C$ by the induction hypothesis.

When $A \equiv \forall x B$, assume $a' \not\models^+ A$. Since a' is maximal, there exists $u \in U(a')$ such that $a' \not\models^+ B[\underline{u}/x]$, which implies $a' \models^- B[\underline{u}/x]$ by the induction hypothesis. Hence we have $a' \models^- \forall x B$. ∎

a' may be regarded as a possible world which is *omniscient*, in the sence of lemma 5.1. This is the origin of the name of the axiom.

In order to prove Kripke soundness of $\mathsf{GN}_\neg\mathsf{O}$, we have only to verify that the formula $\neg\neg(A \vee {\sim}A)$ is valid in every $\mathsf{N}_\neg\mathsf{O}$-model, which is obvious by lemma 5.1.

THEOREM 5.2 (KRIPKE SOUNDNESS OF $\mathsf{GN}_\neg\mathsf{O}$)
If $\mathsf{GN}_\neg\mathsf{O} \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{N}_\neg\mathsf{O} \models \Gamma \Rightarrow \Delta$.

$\mathsf{TN}_\neg\mathsf{O}$, the tree-sequent system for the logic $\mathsf{N}_\neg\mathsf{O}$, is obtained from $\mathsf{TN}_\neg$ by adding an axiom

$$\frac{}{\cdots [\overset{\alpha}{\Rightarrow} \neg\neg(A \vee {\sim}A) \mid \cdots} \text{ (axiom O)}$$

. Then the same relations as those between $\mathsf{GN}_\neg$ and $\mathsf{TN}_\neg$ hold between $\mathsf{GN}_\neg\mathsf{O}$ and $\mathsf{TN}_\neg\mathsf{O}$, as we shall see in the following lemmas.

LEMMA 5.3
Let $\mathcal{T}$ be a tree-sequent of $\mathsf{TN}_\neg\mathsf{O}$. If $\mathcal{T}$ has a node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ such that $\mathsf{GN}_\neg\mathsf{O} \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{TN}_\neg\mathsf{O} \vdash \mathcal{T}$.

Proof. By the induction on the derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{GN}_\neg\mathsf{O}$. ∎

We define the *translation into a formula* of a tree-sequent $\mathcal{T}$ of $\mathsf{TN}_\neg\mathsf{O}$, $\mathcal{T}^f$, as the same one as that of $\mathsf{TN}_\neg$.

LEMMA 5.4
If $\mathsf{TN}_\neg\mathsf{O} \vdash \mathcal{T}$, then $\mathsf{GN}_\neg\mathsf{O} \vdash \mathcal{T}^f$.

Proof. Similarly to the proof of lemma 2.10. ∎

We make a remark about the language of Cl. All of $\vee$, $\rightarrow$ and $\exists$ can be introduced as defined symbols in Cl; however in this paper we assume the logical connectives of Cl be not only $\wedge$, $\neg$ and $\forall$ but include $\rightarrow$ to make correspondence with $\mathsf{N}_\neg\mathsf{O}$.

In this paper a structure in Tarski-type semantics for Cl is often said to be a Cl-*model*. For a Cl-model $\mathfrak{A}$ we denote the domain of $\mathfrak{A}$ by $|\mathfrak{A}|$, and the interpretation of a predicate symbol p by $p_{\mathfrak{A}}$. We will often denote a formula of Cl by $A_\neg$, in order to make clear that A contains no $\sim$'s. $A_{\sim\neg}$ is a formula of $\mathsf{N}_\neg$ which is obtained by replacing some (possibly all or no) $\neg$'s in A by $\sim$.

Now we prepare a lemma which is a basic idea for the proof of Kripke completeness of $\mathsf{GN}_\neg$; that is, *a* Cl-*model induces an omniscient possible world.*

LEMMA 5.5
Let $\mathfrak{A}$ be a Cl-model, and $\mathfrak{M}_{\mathfrak{A}} = (M, \leq, U, I^+, I^-)$ an $\mathsf{N}_\neg\mathsf{O}$-model such that $M = \{0\}$, $U(0) = |\mathfrak{A}|$, $p^{I^+(0)} = p_{\mathfrak{A}}$ and $p^{I^-(0)} = U(0)^m \backslash p_{\mathfrak{A}}$. Then for an arbitrary closed formula $A_\neg$ of Cl, $\mathfrak{A} \models A_\neg$ iff $0 \models^+ A_{\sim\neg}$ (or equivalently $\mathfrak{M} \models A_{\sim\neg}$). From this and lemma 5.1, we also have $\mathfrak{A} \not\models A_\neg$ iff $0 \models^- A_{\sim\neg}$.

Proof. By the induction on the construction of $A_\neg$. We only present the step case where $A_\neg \equiv \neg B_\neg$.

When $A_\neg \equiv \neg B_\neg$, $A_{\sim\neg}$ is syntactically equivalent to $\neg B_{\sim\neg}$ or $\sim B_{\sim\neg}$. $\mathfrak{A} \models A_\neg$ iff $\mathfrak{A} \not\models B_\neg$, which is equivalent to $0 \not\models^+ B_{\sim\neg}$ by the induction hypothesis. This is equivalent to $0 \models^+ \neg B_{\sim\neg}$ since 0 has no descendants, and is also equivalent to $0 \models^+ \sim B_{\sim\neg}$ by lemma 5.1. ∎

THEOREM 5.6 (MAIN THEOREM 1. KRIPKE COMPLETENESS OF $\mathsf{GN}_\neg\mathsf{O}$)
If $\mathsf{N}_\neg\mathsf{O} \models A$, then $\mathsf{GN}_\neg\mathsf{O} \vdash A$.

Proof. We describe the sketch of the proof here, and the details shall be shown later as lemmas.

Let $\mathsf{GN}_\neg\mathsf{O} \not\vdash A$, and α be a finite set of variables which is nonempty and includes every free variable of A. By lemma 5.4 and $[\overset{\alpha}{\Rightarrow} A]^f \equiv \forall\overrightarrow{\alpha}A$, $\mathsf{TN}_\neg\mathsf{O} \not\vdash [\overset{\alpha}{\Rightarrow} A]$. Now we extend $[\overset{\alpha}{\Rightarrow} A]$ into $\mathsf{TN}_\neg$-saturation in the same way as we do in $\mathsf{TN}_\neg$. Because $\mathsf{TN}_\neg$ and $\mathsf{TN}_\neg\mathsf{O}$ have the same inference rules in common, the procedure of extension preserves the unprovability of the tree-sequent in $\mathsf{TN}_\neg\mathsf{O}$. Hence as a result we obtain an infinite tree-sequent $\widetilde{\mathcal{T}}$ which is $\mathsf{TN}_\neg$-saturated, and every finite sub-tree-sequent of which is unprovable in $\mathsf{TN}_\neg\mathsf{O}$. A $\mathsf{TN}_\neg$-saturated $\widetilde{\mathcal{T}}$ induces an $\mathsf{N}_\neg$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ which makes A not valid.

We are to add an omniscient world a' to each possible world a of $\mathfrak{M}$, in the following way.

Let $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ the node of $\widetilde{\mathcal{T}}$ which induces an possible world a of $\mathfrak{M}$. Then by lemma 5.3 and that every finite sub-tree-sequent of $\widetilde{\mathcal{T}}$ is unprovable in $\mathsf{TN}_\neg\mathsf{O}$, for every finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ we have $\mathsf{GN}_\neg\mathsf{O} \not\vdash \Gamma' \Rightarrow \Delta'$, i.e. the infinite sequent $\Gamma \Rightarrow \Delta$ of $\mathsf{GN}_\neg\mathsf{O}$ is $\mathsf{GN}_\neg\mathsf{O}$-*consistent*. It is the result of corollary 5.10 that from $\Gamma \Rightarrow \Delta$ we have a LK-*consistent* sequent $\Gamma_\neg \Rightarrow$, i.e. every finite subsequent of which is unprovable in LK. Then by lemma 5.11 we can construct a Cl-model $\mathfrak{A}$ which has a large domain enough to include $U(a)$ and makes every formula in $\Gamma_\neg$ valid. We construct a' from $\mathfrak{A}$ as 0 of $\mathfrak{M}_{\mathfrak{A}}$ in lemma 5.5. Then a' is an omniscient possible world, and we add it to $\mathfrak{M}$ as a new daughter of a.

By carrying out this procedure for every node of $\mathfrak{M}$, we obtain an $\mathsf{N}_\neg\mathsf{O}$-model $\mathfrak{M}'$ such that $\mathfrak{M}' \not\models A$, hence Kripke completeness of $\mathsf{GN}_\neg\mathsf{O}$. It shall be shown in lemma 5.13 that $\mathfrak{M}'$ is a counter model for A in fact. ∎

First we present some derivations which are only allowed in the logics with omniscience axiom, and some related results.

LEMMA 5.7
In the system $\mathsf{GN}_\neg\mathsf{O}$, *the following derivations are allowed:*

$$\dfrac{}{\neg{\sim}A, \neg A \Rightarrow}\ (\spadesuit 1) \qquad \dfrac{{\sim}A, \Gamma \Rightarrow}{\neg A, \Gamma \Rightarrow}\ (\spadesuit 2) \qquad \dfrac{}{A \to B \Rightarrow \neg\neg({\sim}A \vee B)}\ (\spadesuit 3)$$

Proof. The derivations above can be simulated in $\mathsf{GN}_\neg\mathsf{O}$ as follows. Here it is notable that we do not use (axiom 2) $A, {\sim}A \Rightarrow$, hence these are also allowed in the systems $\mathsf{GN}_\neg\mathsf{OP}$ and $\mathsf{GN}_\neg\mathsf{DOP}$ (needless to say in $\mathsf{GN}_\neg\mathsf{DO}$).
($\spadesuit$ 1)

$$\dfrac{\dfrac{\text{axiom O}}{\Rightarrow \neg\neg(A \vee {\sim}A)} \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, \neg{\sim}A, \neg A \Rightarrow}\ (\text{weakening}, \neg\text{L}) \quad \dfrac{{\sim}A \Rightarrow {\sim}A}{{\sim}A, \neg{\sim}A, \neg A \Rightarrow}}{A \vee {\sim}A, \neg{\sim}A, \neg A \Rightarrow}\ (\vee\text{L})}{\neg\neg(A \vee {\sim}A), \neg{\sim}A, \neg A \Rightarrow}\ (\neg\text{R, L})}{\neg{\sim}A, \neg A \Rightarrow}\ (\text{cut})$$

($\spadesuit$ 2)

$$\dfrac{\dfrac{\dfrac{}{\neg A, \neg{\sim}A \Rightarrow}\ (\spadesuit 1)}{\neg A \Rightarrow \neg\neg{\sim}A}\ (\neg\text{R}) \qquad \dfrac{\dfrac{}{{\sim}A, \Gamma \Rightarrow}\ \text{hyp.}}{\neg\neg{\sim}A, \Gamma \Rightarrow}\ (\neg\text{R, L})}{\neg A, \Gamma \Rightarrow}\ (\text{cut})$$

($\spadesuit$ 3)

$$\dfrac{\dfrac{\text{axiom O}}{\Rightarrow \neg\neg(A \vee {\sim}A)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \to B, A \Rightarrow {\sim}A \vee B}\ (\to\text{L})}{A \to B, A, \neg({\sim}A \vee B) \Rightarrow}\ (\neg\text{L}) \quad \dfrac{\dfrac{{\sim}A \Rightarrow {\sim}A}{A \to B, {\sim}A \Rightarrow {\sim}A \vee B}}{A \to B, {\sim}A, \neg({\sim}A \vee B) \Rightarrow}\ (\neg\text{L})}{A \to B, A \vee {\sim}A, \neg({\sim}A \vee B) \Rightarrow}\ (\vee\text{L})}{A \to B, \neg\neg(A \vee {\sim}A) \Rightarrow \neg\neg({\sim}A \vee B)}\ (\neg\text{R}, \text{L}, \text{R})}{A \to B \Rightarrow \neg\neg({\sim}A \vee B)}\ (\text{cut})$$

∎

LEMMA 5.8
Let the sequent system $\mathsf{GN}_\neg\mathsf{O}'$ *be* $\mathsf{GN}_\neg$ *plus an inference rule*

$$\dfrac{{\sim}A, \Gamma \Rightarrow}{\neg A, \Gamma \Rightarrow}\ (\spadesuit 2)$$

. *Then* $\mathsf{GN}_\neg\mathsf{O}' \vdash\ \Rightarrow \neg\neg(A \vee {\sim}A)$, *and with lemma 5.7 we obtain the equivalence of* $\mathsf{GN}_\neg\mathsf{O}$ *and* $\mathsf{GN}_\neg\mathsf{O}'$.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{{\sim}A, {\sim}{\sim}A \Rightarrow}\ (\text{axiom 2})}{{\sim}(A \vee {\sim}A) \Rightarrow}\ ({\sim}\vee\text{L})}{\neg(A \vee {\sim}A) \Rightarrow}\ (\spadesuit 2)}{\Rightarrow \neg\neg(A \vee {\sim}A)}\ (\neg\text{R})$$

∎

Using lemma 5.7 we can prove the main lemma, which is an embedding of LK into $\mathsf{GN}_{\neg}\mathsf{O}$[||]. Again $\Gamma_{\sim\neg}$ is obtained by substituting some $\neg$'s by $\sim$ in Γ.

LEMMA 5.9 (MAIN LEMMA)
$\mathsf{LK} \vdash \Gamma_{\neg} \Rightarrow \Delta_{\neg}$ *if and only if* $\mathsf{GN}_{\neg}\mathsf{O} \vdash \Gamma_{\sim\neg}, \sim\Delta_{\sim\neg} \Rightarrow$. *Moreover, it is also equivalent to* $\mathsf{GN}_{\neg}\mathsf{O} \vdash \Gamma_{\sim\neg}, \neg\Delta_{\sim\neg} \Rightarrow$.

Proof. The second equivalence is obvious by (♠2) and $\mathsf{GN}_{\neg}\mathsf{O} \vdash \sim A \Rightarrow \neg A$.

We consider the first equivalence. The *only if* part of the lemma is shown semantically; Let $\mathfrak{A}$ an arbitrary Cl-model. By the soundness of $\mathsf{GN}_{\neg}\mathsf{O}$ $\mathfrak{M}_{\mathfrak{A}} \models \Gamma_{\sim\neg}, \sim\Delta_{\sim\neg} \Rightarrow$, and by lemma 5.5 we have $\mathfrak{A} \models \Gamma_{\neg}, \neg\Delta_{\neg} \Rightarrow$, hence $\mathfrak{A} \models \Gamma_{\neg} \Rightarrow \Delta_{\neg}$. Then by completeness of LK, $\mathsf{LK} \vdash \Gamma_{\neg} \Rightarrow \Delta_{\neg}$.

We now consider the *if* part. First note that we may restrict axioms (or initial sequents) of LK to the form $p(\overrightarrow{x}) \Rightarrow p(\overrightarrow{x})$. The proof is carried out through the induction on the derivation in LK.

In the following $A_{\sim\neg}$ is denoted by A' to avoid the proof being too wide.

The inference rule of LK applied at last is presented in the left, and the corresponding proof figure in $\mathsf{GN}_{\neg}\mathsf{O}$ in the right:

$$\frac{}{p(\overrightarrow{x}) \Rightarrow p(\overrightarrow{x})}\ (\text{axiom}) \qquad \frac{}{p(\overrightarrow{x}), \sim p(\overrightarrow{x}) \Rightarrow}\ (\text{axiom 2})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ (\text{weakening}) \qquad \frac{\dfrac{\text{ind. hyp.}}{\Gamma', \sim\Delta' \Rightarrow}}{\Sigma', \Gamma', \sim\Delta', \sim\Pi' \Rightarrow}\ (\text{weakening})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{\dfrac{\text{ind. hyp.}}{A', \Gamma', \sim\Delta' \Rightarrow}}{A' \wedge B', \Gamma', \sim\Delta' \Rightarrow}\ (\wedge\text{L})$$

For the rules introducing $\wedge$ or $\forall$, the proof is similar to that of $(\wedge\text{L})$.

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\text{L}) \qquad \frac{\dfrac{\text{ind. hyp.}}{\Gamma', \sim\Delta', \sim A' \Rightarrow}}{\neg A', \Gamma', \sim\Delta' \Rightarrow}\ (\spadesuit 2) \quad \text{and} \quad \frac{\text{ind. hyp.}}{\sim A', \Gamma', \sim\Delta' \Rightarrow}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ (\neg\text{R}) \qquad \frac{\dfrac{\text{ind. hyp.}}{\Gamma', \sim\Delta', A' \Rightarrow}}{\Gamma', \sim\Delta', \sim\neg A' \ (\text{or } \sim\sim A') \Rightarrow}\ (\sim\neg\text{L}) \text{ or } (\sim\sim\text{L})$$

For the rule of LK $\dfrac{\Gamma \Rightarrow \Delta, A \quad B, \Sigma \Rightarrow \Pi}{A \to B, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\to\text{L})$, we have in $\mathsf{GN}_{\neg}\mathsf{O}$

$$\frac{\dfrac{(\spadesuit 3)}{A' \to B' \Rightarrow \neg\neg(\sim A' \vee B')} \qquad \dfrac{\dfrac{\dfrac{\text{ind. hyp.}}{\sim A', \Gamma', \sim\Delta' \Rightarrow} \quad \dfrac{\text{ind. hyp.}}{B', \Sigma', \sim\Pi' \Rightarrow}}{\sim A' \vee B', \Gamma', \Sigma', \sim\Delta', \sim\Pi' \Rightarrow}\ (\vee\text{L})}{\neg\neg(\sim A' \vee B'), \Gamma', \Sigma', \sim\Delta', \sim\Pi' \Rightarrow}\ (\neg\text{R, L})}{A' \to B', \Gamma', \Sigma', \sim\Delta', \sim\Pi' \Rightarrow}\ (\text{cut})$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ (\to\text{R}) \qquad \frac{\dfrac{\text{ind. hyp.}}{\Gamma', \sim\Delta', A', \sim B' \Rightarrow}}{\Gamma', \sim\Delta', \sim(A' \to B') \Rightarrow}\ (\sim\to\text{L})$$

[||] In the literature [Tan80] an embedding of the $\to$-free fragment of Cl into N is presented.

■

COROLLARY 5.10
Let $A_\neg$ be a formula of Cl *obtained by replacing every $\sim$'s by $\neg$ in a formula A of* $\mathsf{N}_\neg$*. Then for every sequent $\Gamma \Rightarrow \Delta$ of* $\mathsf{GN_\neg O}$*,* $\mathsf{GN_\neg O} \nvdash \Gamma \Rightarrow \Delta$ *implies* $\mathsf{LK} \nvdash \Gamma_\neg \Rightarrow$.

Proof. Take the emptyset as $\Delta_{\sim\neg}$ and use the lemma. ■

The next lemma certifies that we can construct a Cl-model which will be the seed of an omniscient possible world.

LEMMA 5.11
Let $\Gamma \Rightarrow \Delta$ be an infinite sequent of LK *which is consistent, that is, for every finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ we have* $\mathsf{LK} \nvdash \Gamma' \Rightarrow \Delta'$*. Then there exists a* Cl*-model $\mathfrak{A}$ such that:*

1. *$|\mathfrak{A}|$ includes the set of all variables;*
2. *$\mathfrak{A} \models A[\underline{\overrightarrow{x}}/\overrightarrow{x}]$ for every $A \in \Gamma$;*
3. *$\mathfrak{A} \not\models B[\underline{\overrightarrow{x}}/\overrightarrow{x}]$ for every $B \in \Delta$.*

Proof. First increase variables twofold, adding x_i' for each original variable x_i. Then an infinite number of variables $x_1', x_2', \ldots$ have no occurrence in $\Gamma \Rightarrow \Delta$, which is a sequent of the original formal language. Now with the condition that $\Gamma \Rightarrow \Delta$ is consistent we can extend it to a LK-saturated infinite sequent $\widetilde{\Gamma} \Rightarrow \widetilde{\Delta}$, which induces a Cl-model $\mathfrak{A}$ where $|\mathfrak{A}| = \{x_1, x_1', x_2, x_2', \ldots\}$ and the conditions 1. and 2. are satisfied. Hence $\mathfrak{A}$ is what we would like to obtain. ■

REMARK 5.12
The reason why we cannot apply our method to the logic $\mathsf{N_\neg DO}$ lies in the lemma above. Given an *infinite* consistent sequent $\Gamma \Rightarrow \Delta$, we must have infinitely many extra variables to extend it to LK-saturation, which result in the extension of the domain.

Now we prove the last lemma.

LEMMA 5.13
Let A, $\mathfrak{M}'$, $\mathfrak{M}$, a', a and so on be the same as in the proof of theorem 5.6. Then $\mathfrak{M}'$ is certainly an $\mathsf{N_\neg O}$*-model and $\mathfrak{M}' \not\models A$.*

Proof. First we show that $\mathfrak{M}'$ is an $\mathsf{N}_\neg$-model. As we saw in lemma 5.11, $U(a) \subseteq U(a')$. If $\overrightarrow{x} \in p^{I^+(a)}$ (or $p^{I^-(a)}$), then by the definition of $\mathfrak{M}$, $p(\overrightarrow{x})$ (or $\sim p(\overrightarrow{x})$) is in Γ where $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ is a node of $\widetilde{\mathcal{T}}$. Now by the definition of $\mathfrak{A}$ which is a counter model for the sequent $\Gamma_\neg \Rightarrow$ of LK, $\mathfrak{A} \models p(\underline{\overrightarrow{x}})$ (or $\mathfrak{A} \models \neg p(\underline{\overrightarrow{x}})$), which in turn implies that $a' \models^+ p(\underline{\overrightarrow{x}})$ (or $a' \models^- p(\underline{\overrightarrow{x}})$, respectively) by lemma 5.5. Hence we have obtained $p^{I^+(a)} \subseteq p^{I^+(a')}$ and $p^{I^-(a)} \subseteq p^{I^-(a')}$.

Since we added an omniscient world a' for every $a \in M$ as a new daughter, $\mathfrak{M}'$ is an $\mathsf{N_\neg O}$-model.

Now it suffices to show $\mathfrak{M}' \not\models A$, which is reduced to the following:

> For every node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ of $\widetilde{\mathcal{T}}$, $B \in \Gamma$ (or $B \in \Delta$) implies $a \models^+ B[\underline{\overrightarrow{x}}/\overrightarrow{x}]$ (or $a \not\models^+ B[\underline{\overrightarrow{x}}/\overrightarrow{x}]$, respectively) in $\mathfrak{M}'$.

Please note that in the above statement a is considered as a possible world of $\mathfrak{M}'$, not of $\mathfrak{M}$. This is verified as follows. If $B \in \Gamma$, by the definition of a' we have $a' \models^+ B[\underline{\vec{x}}/\vec{x}]$, hence the addition of a' does not affect the fact that $a \models^+ B[\underline{\vec{x}}/\vec{x}]$.

Assume $B \in \Delta$. It is an important fact here that the cases where $a \not\models^+ B$ is affected by the addition of a new daughter a' are limited to those where the logical connective $\neg$, $\to$ or $\forall$ is involved; i.e. where in order to verify whether $\models^+$ holds or not we must see the descendants of a. And on top of it, to show $a \not\models^+ B[\underline{\vec{x}}/\vec{x}]$ in such cases what we must verify is "*there is one world* where ...". Since $\mathfrak{M}$ was already the model where $a \not\models^+ B[\underline{\vec{x}}/\vec{x}]$, it also holds in $\mathfrak{M}'$.

For example, assume $a \not\models^+ \neg B$ in $\mathfrak{M}$ for a closed formula B. Then there is a descendant b of a in $\mathfrak{M}$ where $b \models^+ B$, and b is also in $\mathfrak{M}'$. Hence we have $a \not\models^+ \neg B$ in $\mathfrak{M}'$, too. ∎

5.2 Another Proof – For $\mathsf{GN_\neg O}$ and $\mathsf{GN_\neg DO}$

In this subsection we present another proof of Kripke completeness of $\mathsf{GN_\neg O}$. As we saw above, we cannot apply the method in the previous subsection to $\mathsf{GN_\neg DO}$; that is because we construct omniscient possible worlds *after we have finished* constructing a counter $\mathsf{N_\neg}$-model. In this way we cannot take a fresh variable which is necessary to make an omniscient world, without increasing variables.

The method presented here overcomes this kind of difficulty; it constructs both a counter $\mathsf{N_\neg}$-model and omniscient worlds simultaneously.

In the proof we use a *tree-sequent with guardians*, often abbreviated as $\mathrm{TS_g}$. Roughly speaking, $\mathrm{TS_g}$ is a tree-sequent of $\mathsf{TN_\neg O}$ each of whose nodes has an extra sequent of $\mathsf{GN_\neg O}$ and a finite set of variables; each node has two sequents $\Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi$. The second sequent $\Sigma \overset{\beta}{\Rightarrow} \Pi$ is said to be a *guardian sequent*, or simply a *guardian*, of that node.

Now we explain the idea of $\mathrm{TS_g}$. To construct an omniscient world a' for each possible world a of a counter $\mathsf{N_\neg}$-model after extending an unprovable tree-sequent of $\mathsf{N_\neg O}$, it is necessary to have a record which formula must be verified or refuted at a', and this information is written in the guardian sequent of a. In other words, the guardian sequent of a is the seed of the omniscient world for a.

We present the precise definition:

DEFINITION 5.14
A *tree-sequent with guardians* ($\mathrm{TS_g}$) $\mathcal{G}$ is a finite tree each of whose node a is associated with its own two sequents of $\mathsf{GN_\neg O}$ and two finite sets of variables, $\Gamma \overset{\alpha}{\Rightarrow} \Delta$ and $\Sigma \overset{\beta}{\Rightarrow} \Pi$ (we will denote this fact by $(a : \Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi)$), which satisfies the following condition:

> Let $\mathcal{G}^t$ be a tree obtained by adding a new daughter $(a_g : \Sigma \overset{\beta}{\Rightarrow} \Pi \uparrow \ \Rightarrow \ )$ to each node $(a : \Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi)$, and then omitting all the guardians (Fig. 3). Then $\mathcal{G}^t$ is a tree-sequent of $\mathsf{TN_\neg}$; that is, $\mathcal{G}^t$ satisfies the conditions as to the availability of variables.

For example, $\alpha \cap \beta = \emptyset$ for $(a : \Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi)$.

$\Gamma \overset{\alpha}{\Rightarrow} \Delta$ is said to be a *left-sequent*, or simply *sequent* of a, and $\Sigma \overset{\beta}{\Rightarrow} \Pi$ a *guardian sequent*, or simply *guardian* of a. The tree-sequent $\mathcal{G}^t$ is a *translation into tree-sequent*

$$
\boxed{\mathcal{G}}\quad
\begin{array}{c}
\cdots \\
\backslash \\
\Gamma \stackrel{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi \\
\backslash \\
\cdots
\end{array}
\qquad\qquad
\boxed{\mathcal{G}^t}\quad
\begin{array}{ccc}
\cdots & & \Sigma \stackrel{\beta}{\Rightarrow} \Pi \\
\backslash & & / \\
& \Gamma \stackrel{\alpha}{\Rightarrow} \Delta & \\
& \backslash & \\
& \cdots &
\end{array}
$$

Fig. 3: $\mathcal{G}$ and $\mathcal{G}^t$

of $\mathcal{G}$. A variable x is *available* at the node a in $\mathcal{G}$ if it is available at a in the translation $\mathcal{G}^t$. x is *available at* a_g in $\mathcal{G}$, where a is a node of $\mathcal{G}$, if x is available at the daughter a_g of a in $\mathcal{G}^t$.

Now we introduce the $\mathrm{TS_g}$-system $\mathsf{T_gN_\neg O}$ for the logic $\mathsf{N_\neg O}$:

$$
\frac{}{\cdots[A \stackrel{\alpha}{\Rightarrow} A \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\text{axiom 1})
\qquad
\frac{}{\cdots[A, {\sim}A \stackrel{\alpha}{\Rightarrow} \ \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\text{axiom 2})
$$

$$
\frac{}{\cdots[\Gamma \stackrel{\alpha}{\Rightarrow} \Delta \uparrow A \stackrel{\alpha}{\Rightarrow} A \mid \cdots}\ (\text{axiom g1})
\qquad
\frac{}{\cdots[\Gamma \stackrel{\alpha}{\Rightarrow} \Delta \uparrow A, {\sim}A \stackrel{\alpha}{\Rightarrow} \ \mid \cdots}\ (\text{axiom g2})
$$

$$
\frac{}{\cdots[\Gamma \stackrel{\alpha}{\Rightarrow} \Delta \uparrow \stackrel{\beta}{\Rightarrow} A \vee {\sim}A \mid \cdots}\ (\text{axiom gO})
$$

Inference rules

> (weakening), (drop L), ($\wedge$L), ($\wedge$R), ($\to$L), $(\to\mathrm{R})_{\mathrm{t}}$, ($\neg$L), $(\neg\mathrm{R})_{\mathrm{t}}$, ($\forall$L), $(\forall\mathrm{R})_{\mathrm{t}}$, (${\sim}\wedge$L), (${\sim}\wedge$R), (${\sim}{\to}$L), (${\sim}{\to}$R), (${\sim}\neg$L), (${\sim}\neg$R), (${\sim}{\sim}$L), (${\sim}{\sim}$R), $({\sim}\forall\mathrm{L})_{\mathrm{VC}}$ and (${\sim}\forall$R),

i.e. all the inference rules of $\mathsf{TN_\neg}$ except (cut), are adopted and applied to left-sequents, not to guardians.

$$
\frac{\cdots[\Gamma \stackrel{\alpha}{\Rightarrow} \Delta \uparrow A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots[A, \Gamma \stackrel{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\text{enlightenment L})
$$

The rules (g$\cdots$) involve only guardians but no left-sequents. For them we will indicate only guardians.

$$
\frac{\cdots \uparrow \Gamma \stackrel{\beta}{\Rightarrow} \Delta \mid \cdots}{\cdots \uparrow \Sigma, \Gamma \stackrel{\beta}{\Rightarrow} \Delta, \Pi \mid \cdots}\ (\text{g-weakening})
$$

$$
\frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \mid \cdots \qquad \cdots \uparrow A, \Phi \stackrel{\beta}{\Rightarrow} \Psi \mid \cdots}{\cdots \uparrow \Sigma, \Phi \stackrel{\beta}{\Rightarrow} \Pi, \Psi \mid \cdots}\ (\text{g-cut})
$$

$$
\frac{\cdots \uparrow A, B, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow A \wedge B, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\text{g}\wedge\text{L})
\qquad
\frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \mid \cdots \qquad \cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, B \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \wedge B \mid \cdots}\ (\text{g}\wedge\text{R})
$$

$$
\frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \mid \cdots \qquad \cdots \uparrow B, \Phi \stackrel{\beta}{\Rightarrow} \Psi \mid \cdots}{\cdots \uparrow A \to B, \Sigma, \Phi \stackrel{\beta}{\Rightarrow} \Pi, \Psi \mid \cdots}\ (\text{g}{\to}\text{L})
\qquad
\frac{\cdots \uparrow A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi, B \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \to B \mid \cdots}\ (\text{g}{\to}\text{R})
$$

$$\frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \mid \cdots}{\cdots \uparrow \neg A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}\neg\mathrm{L}) \qquad \frac{\cdots \uparrow A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, \neg A \mid \cdots}\ (\mathrm{g}\neg\mathrm{R})$$

$$\frac{\cdots \uparrow A[y/x], \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow \forall x A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}\forall\mathrm{L}) \qquad \frac{\cdots \uparrow \Sigma \stackrel{\beta \cup \{z\}}{\Rightarrow} \Pi, A[z/x] \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, \forall x A \mid \cdots}\ (\mathrm{g}\forall\mathrm{R})_{\mathrm{VC}}$$

$$\frac{\cdots \uparrow {\sim}A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots \quad \cdots \uparrow {\sim}B, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}(A \wedge B), \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}{\sim}\wedge\mathrm{L}) \quad \frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, {\sim}A, {\sim}B \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, {\sim}(A \wedge B) \mid \cdots}\ (\mathrm{g}{\sim}\wedge\mathrm{R})$$

$$\frac{\cdots \uparrow A, {\sim}B, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}(A \to B), \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}{\sim}{\to}\mathrm{L}) \quad \frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \mid \cdots \quad \cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, {\sim}B \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, {\sim}(A \to B) \mid \cdots}\ (\mathrm{g}{\sim}{\to}\mathrm{R})$$

$$\frac{\cdots \uparrow A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}\neg A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}{\sim}\neg\mathrm{L}) \qquad \frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, {\sim}\neg A \mid \cdots}\ (\mathrm{g}{\sim}\neg\mathrm{R})$$

$$\frac{\cdots \uparrow A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}{\sim}A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}{\sim}{\sim}\mathrm{L}) \qquad \frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, A \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, {\sim}{\sim}A \mid \cdots}\ (\mathrm{g}{\sim}{\sim}\mathrm{R})$$

$$\frac{\cdots \uparrow {\sim}A[z/x], \Sigma \stackrel{\beta \cup \{z\}}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}\forall x A, \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}{\sim}\forall\mathrm{L})_{\mathrm{VC}} \qquad \frac{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, {\sim}A[y/x] \mid \cdots}{\cdots \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi, {\sim}\forall x A \mid \cdots}\ (\mathrm{g}{\sim}\forall\mathrm{R})$$

In view of the function of a guardian as a seed of an omniscient possible world, the logical rules as to guardians are to be of the form of those in LK, as above.

Now we define the *translation into a formula* of $\mathrm{TS_g}$ $\mathcal{G}$, in an inductive way similar to that of tree-sequent. A *pre*-$\mathrm{TS_g}$ satisfies every condition of $\mathrm{TS_g}$ except the one that if x is free in the sequent of a (or in the guardian of a), then x is available at a (or at a_g, respectively). A *translation into a formula* of a pre-$\mathrm{TS_g}$ $\mathcal{G}$, denoted by $\mathcal{G}^f$, is defined inductively on the height of $\mathcal{G}$:

$$\begin{aligned}&[\Gamma \stackrel{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \stackrel{\beta}{\Rightarrow} \Pi \mid \mathcal{G}_1 \ldots \mathcal{G}_m]^f \\ &\quad :\equiv \forall \overrightarrow{\alpha} \Big((\bigwedge \Gamma) \to (\bigvee \Delta) \ \vee\ \forall \overrightarrow{\beta} \neg\neg \Big((\bigwedge \Sigma) \to (\bigvee \Pi) \Big) \ \vee \mathcal{G}_1^f \vee \cdots \vee \mathcal{G}_m^f \Big)\end{aligned}$$

As is the case for systems of tree-sequent, we can prove the following lemma, which may be regarded as the soundness of $\mathsf{T_gN_\neg O}$:

LEMMA 5.15
If $\mathsf{T_gN_\neg O} \vdash \mathcal{G}$, *then* $\mathsf{GN_\neg O} \vdash \mathcal{G}^f$.

Proof. Since the counterparts of lemma 2.11 and 2.12 are easily verified, we can assume that the node to which an inference rule is applied (or the node which is in the form indicated in an axiom) is nothing but the root. Now we prove the lemma by the induction on the derivation of $\mathcal{G}$ in $\mathsf{T_gN_\neg O}$.

The cases for (axiom 1), (2), (g1) or (g2) are easy.

For the (axiom gO), by omniscience axiom $\mathsf{GN_\neg O} \vdash \neg\neg(A \vee {\sim}A)$.

For the rules which are common in $\mathsf{TN_\neg}$ and $\mathsf{T_gN_\neg O}$ such as (drop L) or (${\sim}\forall$R), the proof is carried out just as we did in the proof of lemma 2.10.

For remaining rules involving guardians, some proofs are obvious, others are complicated. Here we present the latter cases.

Before considering each cases, we prepare the following fact:

$$\mathsf{GN}_{\neg}(\,\mathsf{LJ}\text{ or }\mathsf{GN}_{\neg}\mathsf{O}\,) \vdash \neg\neg(A \vee \neg A) \qquad (\clubsuit)$$

$$
\begin{prooftree}
\AxiomC{$A \Rightarrow A$}
\RightLabel{$(\vee\mathrm{R})$}
\UnaryInfC{$A \Rightarrow A \vee \neg A$}
\RightLabel{$(\neg\mathrm{L})$}
\UnaryInfC{$A, \neg(A \vee \neg A) \Rightarrow$}
\RightLabel{$(\neg\mathrm{R})$}
\UnaryInfC{$\neg(A \vee \neg A) \Rightarrow \neg A$}
\RightLabel{$(\vee\mathrm{R})$}
\UnaryInfC{$\neg(A \vee \neg A) \Rightarrow A \vee \neg A$}
\RightLabel{$(\neg\mathrm{L})$}
\UnaryInfC{$\neg(A \vee \neg A), \neg(A \vee \neg A) \Rightarrow$}
\RightLabel{$(\neg\mathrm{R})$}
\UnaryInfC{$\Rightarrow \neg\neg(A \vee \neg A)$}
\end{prooftree}
$$

For (g$\neg$R), it suffices to show that $\mathsf{GN}_{\neg}\mathsf{O} \vdash \neg\neg(C \wedge A \to D) \Rightarrow \neg\neg(C \to D \vee \neg A)$.

$$
\begin{prooftree}
\AxiomC{$(\clubsuit)$}
\UnaryInfC{$\Rightarrow \neg\neg(A \vee \neg A)$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$C, A, C \wedge A \to D \Rightarrow D \vee \neg A$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$C, \neg A, C \wedge A \to D \Rightarrow D \vee \neg A$}
\RightLabel{$(\vee\mathrm{L})$}
\BinaryInfC{$C, A \vee \neg A, C \wedge A \to D \Rightarrow D \vee \neg A$}
\RightLabel{$(\to\mathrm{R})$}
\UnaryInfC{$A \vee \neg A, C \wedge A \to D \Rightarrow C \to D \vee \neg A$}
\doubleLine
\RightLabel{$(\neg\mathrm{L}, \mathrm{R})$}
\UnaryInfC{$\neg\neg(A \vee \neg A), \neg\neg(C \wedge A \to D) \Rightarrow \neg\neg(C \to D \vee \neg A)$}
\RightLabel{(cut)}
\BinaryInfC{$\neg\neg(C \wedge A \to D) \Rightarrow \neg\neg(C \to D \vee \neg A)$}
\end{prooftree}
$$

For (g$\to$R), the proof is similar to above, using $\mathsf{GN}_{\neg}\mathsf{O} \vdash \neg\neg(A \vee \neg A)$.

For (g$\forall$R), it suffices to show that $\mathsf{GN}_{\neg}\mathsf{O} \vdash \forall z \neg\neg(C \to D \vee A[z/x]) \Rightarrow \neg\neg(C \to D \vee \forall x A)$, where z is not free in C nor D.

$$
\begin{prooftree}
\AxiomC{$C \Rightarrow C$}
\AxiomC{}
\RightLabel{(axiom 2)}
\UnaryInfC{$D, \sim D \Rightarrow$}
\AxiomC{}
\RightLabel{(axiom 2)}
\UnaryInfC{$A[z/x], \sim A[z/x] \Rightarrow$}
\RightLabel{$(\vee\mathrm{L})$}
\BinaryInfC{$D \vee A[z/x], \sim D, \sim A[z/x] \Rightarrow$}
\RightLabel{$(\to\mathrm{L})$}
\BinaryInfC{$C \to D \vee A[z/x], C, \sim D, \sim A[z/x] \Rightarrow$}
\doubleLine
\RightLabel{$(\neg\mathrm{R}, \neg\mathrm{L}, \forall\mathrm{L})$}
\UnaryInfC{$\forall z \neg\neg(C \to D \vee A[z/x]), C, \sim D, \sim A[z/x] \Rightarrow$}
\RightLabel{$(\sim\forall\mathrm{L})_{\mathrm{VC}}$}
\UnaryInfC{$\forall z \neg\neg(C \to D \vee A[z/x]), C, \sim D, \sim \forall x A \Rightarrow$}
\doubleLine
\RightLabel{$(\sim\vee\mathrm{L}, \sim\to\mathrm{L})$}
\UnaryInfC{$\forall z \neg\neg(C \to D \vee A[z/x]), \sim(C \to D \vee \forall x A) \Rightarrow$}
\RightLabel{$(\spadesuit\, 2)$}
\UnaryInfC{$\forall z \neg\neg(C \to D \vee A[z/x]), \neg(C \to D \vee \forall x A) \Rightarrow$}
\RightLabel{$(\neg\mathrm{R})$}
\UnaryInfC{$\forall z \neg\neg(C \to D \vee A[z/x]) \Rightarrow \neg\neg(C \to D \vee \forall x A)$}
\end{prooftree}
$$

∎

We are to extend an unprovable $\mathrm{TS_g}$ into saturation and derive a counter model from it; hence the definition of saturatedness is as follows:

Definition 5.16 ($\mathsf{T_g N_{\neg} O}$-saturatedness)
An infinite $\mathrm{TS_g}$ $\mathcal{G}$ is $\mathsf{T_g N_{\neg} O}$-*saturated* if it satisfies the following conditions:

1. The translation into a tree-sequent, $\mathcal{G}^t$, is $\mathsf{TN}_{\neg}$-saturated;

2. For every node $(a : \Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi)$ and every atomic formula $p(\overrightarrow{x})$, if $\overrightarrow{x}$ is available at a_g then $p(\overrightarrow{x}) \in \Sigma$ or $\sim p(\overrightarrow{x}) \in \Sigma$.

Let $\mathcal{G}$ be a $\mathrm{TS_g}$ such that $\mathsf{T_gN_\neg O} \not\vdash \mathcal{G}$ and at least one variable is available at its root. We will extend $\mathcal{G}$ step by step to obtain a sequence of unprovable $\mathrm{TS_g}$s $\mathcal{G}_0(\equiv \mathcal{G}), \mathcal{G}_1, \ldots,$ just as we did in $\mathsf{TN_\neg}$. The step from $\mathcal{G}_{i-1}$ to $\mathcal{G}_i$ involves A_i ($A_1, A_2, \ldots$ is the same sequence of formula as subsection 2.4), and is stated in detail as follows:

1. Apply the same operations (inheritance and reduction) to the left-sequents of $\mathcal{G}$. For example, if $(a : \Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi)$ and $A_i \in \Gamma$, then add A_i to the antecedent of the left-sequent of each decsendant of a, not involving guardians;

2. (report) For each node $(a : \Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi)$, if $A_i \in \Gamma$ then add A_i to Σ. This operation is called a *report* and unprovability is preserved because of the rule (enlightenment L);

3. (g-reduction) According to the shape of A_i, one of the following operations is executed for each node $(a : \Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi)$:

   (a) $A_i \equiv p(\overrightarrow{x})$. If $\overrightarrow{x}$ is available at a_g, then add $p(\overrightarrow{x})$ or $\sim p(\overrightarrow{x})$ to Σ, so that unprovability is preserved. This is possible; if not, we can derive a contradiction as follows:

$$\dfrac{\dfrac{(\text{axiom gO})}{\cdots \uparrow \overset{\beta}{\Rightarrow} p(\overrightarrow{x}) \vee \sim p(\overrightarrow{x}) \mid \cdots} \quad \dfrac{\cdots \uparrow p(\overrightarrow{x}), \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots \quad \cdots \uparrow \sim p(\overrightarrow{x}), \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow p(\overrightarrow{x}) \vee \sim p(\overrightarrow{x}), \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}\,(\mathrm{g}\vee\mathrm{L})}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}\,(\text{g-cut})$$

   (b) $A_i \equiv B \wedge C$. If $A_i \in \Sigma$, then add both B and C to Σ. Unprovability is preserved by the rule (g$\wedge$L). If $A_i \in \Pi$, then add B or C to Π, so that unprovability is preserved. This is possible by (g$\wedge$R);

   (c) $A_i \equiv B \to C$. If $A_i \in \Sigma$, then add B to Π or C to Σ, so that unprovability is preserved. If $A_i \in \Pi$, then add B to Σ and C to Π;

   (d) $A_i \equiv \neg B$. If $A_i \in \Sigma$ (or Π), then add B to Π (or Σ, respectively);

   (e) $A_i \equiv \forall x B$. If $A_i \in \Sigma$, then add $B[y/x]$ to Σ, for every y which is available at a_g and is in $\{x_1, \ldots, x_i\}$. If $A_i \in \Pi$, then take a fresh variable x_m, add $A[x_m/x]$ to Π and also add x_m to β;

   (f) $A_i \equiv \sim(B \wedge C)$. If $A_i \in \Sigma$, add $\sim B$ or $\sim C$ to Σ, so that unprovability is preserved. If $A_i \in \Pi$, add $\sim B$ and $\sim C$ to Π;

   (g) $A_i \equiv \sim(B \to C)$. If $A_i \in \Sigma$, add B and $\sim C$ to Σ. If $A_i \in \Pi$, add B or $\sim C$ to Π so that unprovability is preserved;

   (h) $A_i \equiv \sim\neg B$ or $A_i \equiv \sim\sim B$. If $A_i \in \Sigma$ (or Π), add B to Σ (or Π, respectively);

   (i) $A_i \equiv \sim\forall x B$. If $A_i \in \Sigma$, then take a fresh variable x_m, add $\sim B[x_m/x]$ to Σ, and also add x_m to β. If $A_i \in \Pi$, add $\sim B[y/x]$ to Π for every y which is available at a_g and in $\{x_1, \ldots, x_i\}$.

Let $\widetilde{\mathcal{G}}$ be the infinite $\mathrm{TS_g}$ which is the union of $\mathcal{G}_0, \mathcal{G}_1, \ldots.$ Then $\widetilde{\mathcal{G}}$ is $\mathsf{T_gN_\neg O}$-saturated. Indeed, it satisfies the condition that $\widetilde{\mathcal{G}}^t$ is $\mathsf{TN_\neg}$-saturated; for example, a node $(a_g : \Sigma \overset{\beta}{\Rightarrow} \Pi)$ of $\widetilde{\mathcal{G}}^t$ satisfies the condition

> $(\forall\mathrm{R})_\mathrm{t}$-saturated : If $\forall x A \in \Sigma$, then there exists a descendant $b : \Phi \overset{\beta}{\Rightarrow} \Psi$ of a_g and a variable y such that $A[y/x] \in \Psi$

by taking a_g itself as b. $\widetilde{\mathcal{G}}$ satisfies the second condition of $\mathsf{T_gN_\neg O}$-saturatedness by the operation (g-reduction) as to atomic formulas.

Since an infinite tree-sequent $\widetilde{\mathcal{G}}^t$ is $\mathsf{TN}_\neg$-saturated, it induces an $\mathsf{N}_\neg$-model $\mathfrak{M}$ in the way stated in subsection 2.4. Moreover, the second condition of $\mathsf{T_gN_\neg O}$-saturatedness yields that $\mathfrak{M}$ is actually an $\mathsf{N_\neg O}$-model, where the omniscient world a' for each world a induced by a_g of $\widetilde{\mathcal{G}}^t$.

The following is easily verified by the induction on the construction of a formula A using the fact that $\widetilde{\mathcal{G}}^t$ is $\mathsf{TN}_\neg$-saturated: for every node $a : \Gamma \overset{\alpha}{\Rightarrow} \Delta$ of $\mathcal{G}^t$ ($\mathcal{G}$ is an original $\mathrm{TS_g}$ which is unprovable), $a \in \Gamma$ (or $a \in \Delta$) implies $a \models^+ A[\underline{\vec{x}}/\vec{x}]$ (or $a \not\models^+ A[\underline{\vec{x}}/\vec{x}]$, respectively). Hence we have proved the following theorem:

Theorem 5.17 (Kripke Completeness of $\mathsf{T_gN_\neg O}$)
Let $\mathcal{G}$ be a $\mathrm{TS_g}$ of $\mathsf{T_gN_\neg O}$ at whose root at least one variable is available, and $\mathsf{T_gN_\neg O} \not\vdash \mathcal{G}$. Then there exists a counter $\mathsf{N_\neg O}$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ for $\mathcal{G}$; that is, $\mathfrak{M}$ is a counter model for the tree-sequent $\mathcal{G}^t$ in the sense stated in theorem 2.14.

Corollary 5.18 (Kripke Completeness of $\mathsf{GN_\neg O}$)
For every formula A of $\mathsf{N_\neg O}$, $\mathsf{N_\neg O} \models A$ implies $\mathsf{GN_\neg O} \vdash A$.

Proof. Let a $\mathrm{TS_g}$ $\mathcal{G}$ be $[\ \overset{\alpha}{\Rightarrow} A \uparrow \overset{\emptyset}{\Rightarrow}\ ]$ where $\alpha \neq \emptyset$. By the contraposition of lemma 5.15 we have $\mathsf{T_gN_\neg O} \not\vdash \mathcal{G}$, hence the theorem yields that there exists a counter $\mathsf{N_\neg O}$-model $\mathfrak{M}$ for $\mathcal{G}$, which makes A not valid. ∎

Now we consider the logic $\mathsf{N_\neg DO}$. The Gentzen-style sequent system $\mathsf{GN_\neg DO}$ for the logic $\mathsf{N_\neg DO}$ is $\mathsf{GN_\neg O}$ plus constant domain axiom

$$\frac{}{\Rightarrow \forall x(A(x) \vee B) \to \forall x A(x) \vee B}\ (\text{axiom D})$$

. An $\mathsf{N_\neg O}$-model $\mathfrak{M} = (M, \leq, U, I^+, I^-)$ is an $\mathsf{N_\neg DO}$-*model* if U is a constant map.

After making the same kind of alterations as we did from $\mathsf{TN}_\neg$ to $\mathsf{TN_\neg D}$, the $\mathrm{TS_g}$ method for $\mathsf{N_\neg O}$ stated above can be also applied to the logic $\mathsf{N_\neg DO}$, and we obtain the following theorem:

Theorem 5.19 (Main Theorem 2, Kripke Completeness of $\mathsf{GN_\neg DO}$)
For every formula A of $\mathsf{N_\neg DO}$, $\mathsf{N_\neg DO} \models A$ if and only if $\mathsf{GN_\neg DO} \vdash A$.

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