Limiting Negations in Bounded-Depth Circuits: An Extension of Markov's Theorem

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Abstract

From a theorem of Markov, the minimum number of negation gates in a circuit sufficient to compute any collection of Boolean functions on n variable is $\ell = \lceil \log(n+1) \rceil$. Santha and Wilson [SIAM Journal of Computing 22(2):294–302 (1993)] showed that in some classes of bounded-depth circuits ℓ negation gates are no longer sufficient for some explicitly defined Boolean function. In this paper, we consider a general class of bounded-depth circuits in which each gate computes an arbitrary monotone Boolean function or its negation. Our purpose is to extend the theorem of Markov for such a general class of circuits. We first show that a lower bound shown by Santha and Wilson becomes an extension of Markov's lower bound by a small refinement. Then, we present tight upper bounds on the number of negations for computing an arbitrary collection of Boolean functions.

Keywords: Computational complexity; Circuit complexity; Negation-limited circuits

1 Introduction

Despite the importance of lower bounds on the circuit complexity of explicit problems, the best bounds known are only linear. However, good lower bounds are known for the complexity of monotone circuits, where negations are forbidden. In order to extend the results on monotone circuits to the general circuit model, a natural intermediate step is the study of circuits with the limited number of negations, and this motivates the study on the complexity of negation-limited circuits. For example, the negation-limited circuit complexity of inverters [2, 4], merging [1] and the relationship between the number of negations and circuit size [6] are studied.

When we consider the monotone circuit complexity, we have to consider only a class of monotone functions. Once negations are permitted in the circuit, we can also consider a class of non-monotone functions. Before studying the complexity of such a class, a natural question arises: how many negations are necessary to compute particular functions?

Markov [3] answered this question without any complexity theoretical consideration. From a theorem of Markov, the minimum number of negation gates in a circuit sufficient to compute an arbitrary Boolean function on n variables is $\lceil \log(n+1) \rceil$. The result of Fischer [4] shows that every function which can be computed by a polynomial-size logarithmic-depth circuit containing an aribitrary number of negations can be computed by a polynomial-size logarithmic-depth circuit containing only $\lceil \log(n+1) \rceil$ negations.

Santha and Wilson [5] considered polynomial-size constant-depth circuits and showed that $\lceil \log(n+1) \rceil$ negations are no longer sufficient. For threshold circuits and AC^0 circuits, they proved that several Boolean functions computable in constant depth cannot be computed in constant depth with $o(n^\epsilon)$ negations, for all $\epsilon > 0$. These results were derived from a lower bound on the number of negations in a general class of circuits in which each gate computes an arbitrary monotone Boolean function or its negation.

In this paper, we consider such a general class of circuits. Our purpose is to extend the theorem of Markov for such a general class of circuits. We study the necessary and sufficient number of negations for computing an arbitrary collection of Boolean functions. We first show that, by a small refinement, the lower bound shown by Santha and Wilson becomes an extension of Markov's lower bound, i.e., it matches Markov's lower bounds when depth of circuits are sufficiently large. Then, we show for an arbitrary collection F of Boolean functions a tight upper bound of the number of negations for computing F.

The rest of this paper is organized as follows. After giving some preliminaries, lower bounds and upper bounds for the number of negations for computing an arbitrary collection of Boolean functions are shown respectively in Section 3 and Section 4.

2 Preliminaries

A chain $\alpha = (\alpha^0, \ldots, \alpha^k)$ for some $1 \le k \le n$ is a vector of $\alpha^i = (\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_n) \in \{0,1\}^n$ such that $\alpha^i \le \alpha^{i+1}$ (i.e., $\alpha^i_j \le \alpha^{i+1}_j$ for each $1 \le j \le n$). For a Boolean function f, an index $i \in \{1, 2, \ldots, k\}$ is a *decrease* of f over chain α if $1 = f(\alpha^{i-1}) > f(\alpha^i) = 0$. For a collection $F = \{f_1, \ldots, f_m\}$ of Boolean functions, an index $i \in \{1, 2, \ldots, k\}$ is a *decrease* of F over chain α if i is a decrease of f_j over chain α for some $1 \le j \le m$. By $d(F, \alpha)$ we denote the set of decreases of F over chain α. By $d(F)$ we denote the maximum value of $\#d(F, \alpha)$ among all chains α, where $\#S$ denotes the cardinarity of a set S. A collection of Boolean functions F is called *monotone* if $d(F) = 0$.

From the definition of decreases, the following lemmas can be obtained immediately.

Lemma 1 *Let $F = \{f_1, \ldots, f_m\}$ be an arbitrary collection of Boolean functions, and let g be an arbitrary monotone function of m variables. Then, for every chain α,*

$$d(\{g(f_1, \ldots, f_m)\}, \alpha) \subseteq d(F, \alpha).$$

Lemma 2 *Let f be an arbitrary Boolean function. Then, for every chain α,*

$$\#d(\{\neg f\}, \alpha) \le \#d(\{f\}, \alpha) + 1.$$

A *circuit* is a directed acyclic network in which inputs and gates are connected by directed wire. In-degree of an input must be 0, and in-degree of each gate is an arbitrary positive integer (i.e., unbounded). The *size* of a circuit is the number of gates in it, and the *depth* of a circuit is the length of a longest directed path in it. We say an input or a gate is in *level* i, for some non-negative integer i, if the length of a longest directed path to the input or the gate is i. It follows that all inputs are in level 0.

In this paper, we consider bounded-depth circuits in which each gate computes an arbitrary monotone Boolean function or its negation, i.e., each gate is unbounded fan-in. A gate is called a *monotone gates* if it computes a monotone Boolean function, and a gate is called a *negation gates* if it computes the negation of a monotone Boolean function.

3 Lower Bound on the Number of Negations

In this section, we show a lower bound on the number of negation gates in a depth-D circuit computing an arbitrary collection of Boolean functions. Let F be an arbitrary collection of Boolean functions. The result on the lower bound shown by Santha and Wilson [5] can be rewritten in terms of $d(F)$ as follows.

Proposition 3 *Every depth-D circuit computing F contains at least $D(d(F)+1)^{1/D} - D$ negation gates.*

Proof. Let C be a depth-D circuit computing F, and let G^ℓ, for $0 \le \ell \le D$, be the collection of Boolean functions each of which is computed by some gate in level ℓ', for some $\ell' \le \ell$, of C. Obviously, we have $F \subseteq G^D$, because each $f \in F$ is computed by some gate in C. It follows that $d(F, \alpha) \le d(G^D, \alpha)$ for every chain α, and thus, $d(F) \le d(G^D)$.

Suppose C contains ν negation gates, and let ν_ℓ, for $1 \le \ell \le D$, be the number of negation gates at level ℓ in C, i.e., $\nu = \sum_{\ell=1}^{D} \nu_\ell$. Level 0 consists of inputs $x_1, \ldots, x_n$, where $d(\{x_i\}) = 0$ for all $1 \le i \le n$, and thus $d(G^0) = 0$. From Lemma 1, for $1 \le \ell \le D$, $d(\{f\}, \alpha) \subseteq d(G^{\ell-1}, \alpha)$ if f is a function computed by a monotone gate in level ℓ, and from Lemma 2, $\#d(\{f\}, \alpha) \le \#d(G^{\ell-1}, \alpha) + 1$ if f is a function computed by a negation gate in level ℓ. Since there are ν_ℓ negation gates in level ℓ, we have $d(G^\ell) \le (\nu_\ell + 1)(d(G^{\ell-1}) + 1) - 1$. Therefore, we have $d(G^D) \le \prod_{\ell=1}^{D}(\nu_\ell + 1) - 1$, and it implies

$$d(F) \le \prod_{\ell=1}^{D}(\nu_\ell + 1) - 1. \tag{1}$$

The product $\prod_{\ell=1}^{D}(\nu_\ell + 1)$ is maximized when $\nu_\ell = \nu/D$ for each $1 \le \ell \le D$, it follows that the number ν of negation gates in C is at least $D(d(F)+1)^{1/D} - D$. ∎

However, notice that the lower bound $D(d(F)+1)^{1/D} - D$ tends to 0 when D grows, i.e., it does match the theorem of Markov. A small refinement of Proposition 3 can be made.

Since each ν_ℓ is integer, when $\nu = \sum_{\ell=1}^{D} \nu_\ell$ is fixed, $\prod_{\ell=1}^{D}(\nu_\ell + 1)$ is maximized if $|\nu_k - \nu_\ell| \le 1$ for every $1 \le k, \ell \le D$. Therefore, from $\nu = \sum_{\ell=1}^{D} \nu_\ell$ and (1), ν is minimized when, for each

$1 \leq \ell \leq D$,

$$\nu_\ell \in \{\lfloor (d(F)+1)^{1/D} \rfloor, \lfloor (d(F)+1)^{1/D} \rfloor + 1\}.$$

Let $\sigma_D : N \to N$ be a function defined as follows.

$$\sigma_D(w) = D\lfloor (w+1)^{1/D} \rfloor - D + k,$$

where k is the smallest integer such that

$$\lceil (w+1)^{1/D} \rceil^k \lfloor (w+1)^{1/D} \rfloor^{D-k} \geq w+1.$$

Then we have the following theorem.

Theorem 4 *Every depth-D circuit computing F contains at least $\sigma_D(d(F))$ negation gates.*

Notice that $\sigma_D(d(F)) = \lceil \log(d(F)+1) \rceil$ for $D \geq \lceil \log(d(F)+1) \rceil$. That is, Markov's lower bound $\lceil \log(d(F)+1) \rceil$ of the number of negation gates can be derived from this theorem.

3.1 Special Cases

For some collections of Boolean functions, larger lower bounds of the number of negation gates can be obtained. Let $INV = \{INV_1, INV_2, \ldots, INV_n\}$ be a collection of Boolean function such that $INV_i(x) = \neg x_i$ for $1 \leq i \leq n$. Notice that $d(INV) = n$, and from Theorem 4, every depth-D circuit computing INV contains at least $\sigma_D(n)$ negation gates. However, it requires more negation gates.

Lemma 5 *Let D be a positive integer. Suppose there exists a depth-$(D+1)$ circuit C computing INV such that C contains ν negation gates. Then, there exists a depth-$(D+1)$ circuit C' computing INV such that C' contains at most ν negation gates and has no negation gate in level $D+1$.*

Proof. Let C be a depth-$(D+1)$ circuit computing INV such that C contains ν negation gates. Suppose there exists a gate in level $D+1$ which does not compute INV_i for any $1 \leq i \leq n$. Then by removing such a gate the resulting circuit also computes INV. Thus, we assume without loss of generality that all gates in level $D+1$ are output gates, i.e., each gates in level $D+1$ computes INV_i for some $1 \leq i \leq n$.

If all gates in level $D+1$ are monotone, then $C' = C$ is a circuit satisfying the lemma. Suppose there exists a negation gate in level $D+1$. Notice that each negation gate in level $D+1$ computes INV_i for some $1 \leq i \leq n$, and such a gate can be replaced by a negation gate with one input x_i and output $\neg x_i$. Then the resulting circuit C' contains the same number of negation gates as C and has no negation gate in level $D+1$. ∎

Let C be a depth-$(D+1)$ circuit computing INV in which all gates in depth $D+1$ are monotone gates, and let G be the collection of Boolean functions each of which is computed by some gate in level ℓ of C, for some $\ell \leq D$. From Lemma 1, we have $d(G) = d(INV) = n$, and from Theorem 4, C contains at least $\sigma_D(n)$ negation gates. Therefore, the following theorem is obtained.

Theorem 6 *There exists a collection F of Boolean functions such that every depth-$(D+1)$ circuit computing F contains at least $\sigma_D(d(F))$ negation gates.*

4 Upper Bound on the Number of Negations

In this section, we show upper bounds on the number of negation gates in bounded-depth circuits. First we show that, for each non-negative integer ν, there exists a collection F of Boolean functions which can be computed by a depth-D circuit which contains $\sigma_D(d(F)) = \nu$ negation gates. Then we show that every collection of Boolean functions F can be computed by a depth-$(D+1)$ circuit which contains $\sigma_D(d(F))$ negation gates.

Let m be a positive integer, and let $\nu_1, \nu_2, \ldots, \nu_D \in \{\lfloor (m+1)^{1/D} \rfloor, \lfloor (m+1)^{1/D} \rfloor + 1\}$ satisfying $\sum_{\ell=1}^{D} \nu_\ell = \sigma_D(m)$, and for simplicity, let $\nu_{D+1} = 0$. Let $P_\ell = \prod_{k=\ell}^{D+1} (\nu_k + 1)$ for each $1 \leq \ell \leq D+1$. For $1 \leq \ell \leq D$ and $1 \leq j \leq \nu_\ell$, let g_j^ℓ be a Boolean function defined as follows.

$$g_j^\ell(x) = 1 \quad \text{iff} \quad \left(\sum_{i=1}^{m} h_i(x) \right) \bmod P_\ell < j P_{\ell+1},$$

where each h_i, for $1 \leq i \leq m$, is a monotone Boolean function. Let $G = \{g_j^\ell \mid 1 \leq \ell \leq D, 1 \leq j \leq \nu_\ell\}$.

Notice from the definition of σ_D that we have $P_1 \geq m+1$, and thus

$$g_j^1(x) = 1 \quad \text{iff} \quad \sum_{i=1}^{m} h_i(x) < jP_2. \tag{2}$$

Then, for each $1 \leq j \leq \nu_1$, $jP_2 \leq \sum_{i=1}^{m} h_i(x) < (j+1)P_2$ implies $g_1^1(x) = \cdots = g_j^1(x) = 0$ and $g_{n+1}^1(x) = \cdots = g_{\nu_1}^1(x) = 1$, and obviously, $(\sum_{i=1}^{m} h_i(x)) \bmod P_2 = \sum_{i=1}^{m} h_i(x) - jP_2$. Hence,

$$\begin{aligned} \left(\sum_{i=1}^{m} h_i(x)\right) \bmod P_2 &= \sum_{i=1}^{m} h_i(x) - P_2 \sum_{j=1}^{\nu_1} \neg g_j^1(x) \\ &= \sum_{i=1}^{m} h_i(x) + P_2 \sum_{j=1}^{\nu_1} g_j^1(x) - \nu_1 P_2. \end{aligned}$$

By the similar argument, we have

$$\begin{aligned} \left(\sum_{i=1}^{m} h_i(x)\right) \bmod P_3 &= \left(\sum_{i=1}^{m} h_i(x)\right) \bmod P_2 - P_3 \sum_{j=1}^{\nu_2} \neg g_j^2(x) \\ &= \sum_{i=1}^{m} h_i(x) + P_2 \sum_{j=1}^{\nu_1} g_j^1(x) - \nu_1 P_2 + P_3 \sum_{j=1}^{\nu_2} g_j^2(x) - \nu_2 P_3, \end{aligned}$$

and generally, we have, for $2 \leq \ell \leq D+1$,

$$\left(\sum_{i=1}^{m} h_i(x)\right) \bmod P_\ell = \sum_{i=1}^{m} h_i(x) + \sum_{k=1}^{\ell-1} P_{k+1} \sum_{j=1}^{\nu_k} g_j^k(x) - \sum_{k=1}^{\ell-1} \nu_k P_{k+1}. \tag{3}$$

Hence, for $2 \leq \ell \leq D$ and $1 \leq j \leq \nu_\ell$,

$$g_j^\ell(x) = 1 \quad \text{iff} \quad \sum_{i=1}^{m} h_i(x) + \sum_{k=1}^{\ell-1} P_{k+1} \sum_{j=1}^{\nu_k} g_j^k(x) - \sum_{k=1}^{\ell-1} \nu_k P_{k+1} < jP_{\ell+1}. \tag{4}$$

Moreover, from $P_{D+1} = 1$ and (3), we have

$$\sum_{i=1}^{m} h_i(x) = \sum_{k=1}^{D} \nu_k P_{k+1} - \sum_{k=1}^{D} P_{k+1} \sum_{j=1}^{\nu_k} g_j^k(x). \tag{5}$$

Lemma 7 *The collection $G = \{g_j^\ell \mid 1 \leq \ell \leq D, 1 \leq j \leq \nu_\ell\}$ of Boolean functions defined above can be computed by a depth-D circuit which contains $\sigma_D(m)$ negation gates.*

Proof. A depth-D circuit satisfying the lemma can be constructed as follows. For each $1 \leq \ell \leq D$, the circuit contains exactly ν_ℓ gates in level ℓ, and each of such gates is a negation gate. That is, all gates of the circuits are negation gates.

We show that, for each $1 \leq \ell \leq D$ and $1 \leq j \leq \nu_\ell$, g_j^ℓ can be computed by a negation gate in level ℓ. From (2), each of g_j^1's is the negation of a monotone Boolean function of

$h_1(x), h_2(x), \ldots, h_m(x)$. Since each h_i is a monotone Boolean function of x, each g_j^1 is the negation of a monotone Boolean function of x. Thus, each of g_j^1's can be computed by a negation gate in level 1. From (4), for $2 \le \ell \le D$ each of g_j^ℓ's is the negation of a monotone Boolean function of all $h_i(x)$ for $1 \le i \le m$ and all $g_j^k(x)$'s for $1 \le k \le \ell - 1$, where each $g_j^k(x)$ is computed by some gate in level $k < \ell$. Thus, for $2 \le \ell \le D$, each of g_j^ℓ's can be computed by a negation gate in level ℓ. The proof is completed. ∎

Suppose $m = n$ and each $h_i(x) = x_i$ for $1 \le i \le n$. Then (5) can be rewritten as follows.

$$\sum_{i=1}^{n} x_i = \sum_{k=1}^{D} \nu_k P_{k+1} - \sum_{k=1}^{D} P_{k+1} \sum_{j=1}^{\nu_k} g_j^k(x).$$

Let $\alpha = (\alpha^0, \cdots, \alpha^n)$ be a chain such that $\alpha_i^k = 1$ iff $i \le k$ for each $0 \le k \le n$. Then, we have

$$\sum_{s=1}^{\ell-1} P_{s+1} \sum_{t=1}^{\nu_s} g_t^s(\alpha^{i-1}) > \sum_{s=1}^{\ell-1} P_{s+1} \sum_{t=1}^{\nu_s} g_t^s(\alpha^i)$$

for each $1 \le i \le n$, and thus $d(G, \alpha) = \{1, 2, \ldots, n\}$ (i.e., each i is a decrease of G over α). Therefore, $d(G) = n$. Notice that the depth-D circuit constructed in Lemma 7 computes G and contains exactly $\sigma_D(d(G))$ negation gates. Therefore, we obtain the following theorem.

Theorem 8 *For each non-negative integer ν, there exists a collection F of Boolean functions which can be computed by a depth-D circuit containing $\sigma_D(d(F)) = \nu$ negation gates.*

Now we are ready to show that every collection of Boolean functions F can be computed by a depth-$(D+1)$ circuit which contains $\sigma_D(d(F))$ negation gates.

Theorem 9 *Every collection F of Boolean functions can be computed by a depth-$(D+1)$ circuit which contains exactly $\sigma_D(d(F))$ negation gates.*

Proof. Let $F = \{f_1, \ldots, f_m\}$ be an arbitrary collection of Boolean functions. Let, for each non-negative integer i,

$$S_i = \{a \in \{0,1\}^n \mid \#d(F, \alpha) \le d(F) - i \text{ for all chains } \alpha \text{ with } \alpha^0 = a\}.$$

Then, we have

$$\{0,1\}^n = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_{d(F)} \supseteq S_{d(F)+1} = \emptyset, \tag{6}$$

and $S_i = \emptyset$ for all $i \geq d(F)+1$. Let t_i, for $0 \leq i \leq d(F)$, be the Boolean function defined by $t_i(x) = 1$ iff $x \in S_i$. Notice from (6) that we have, for each $x \in \{0,1\}^n$,

$$1 = t_0(x) \geq t_1(x) \geq \cdots \geq t_{d(F)}(x) \geq t_{d(F)+1}(x) = 0. \tag{7}$$

Moreover $a \in S_i$ implies $\{b \in \{0,1\}^n \mid b \geq a\} \subseteq S_i$, since each b such that $b \geq a$ is in some chain α with $\alpha^0 = a$, and hence, each t_i is a monotone function.

For $0 \leq i \leq d(F)$ and $1 \leq j \leq m$, let

$$f_j^i(x) = \begin{cases} 0 & \text{if } x \notin S_i, \\ f_j(x) & \text{if } x \in S_i - S_{i+1}, \\ 1 & \text{otherwise (i.e., if } x \in S_{i+1}). \end{cases}$$

Now we show that each f_j^i is monotone, i.e., $d(f_j^i) = 0$, for $0 \leq i \leq d(f)$ and $1 \leq j \leq m$. Suppose $d(f_j^i) > 0$ for some i and j. Then there exists a chain α such that $d(f_j^i, \alpha) \neq \emptyset$. Let $k \in d(f_j^i, \alpha)$, i.e., $1 = f_j^i(\alpha^{k-1}) > f_j^i(\alpha^k) = 0$. It follows that $\alpha^{k-1} \in S_i$ and $\alpha^k \notin S_{i+1}$. Moreover, from $\alpha^{k-1} \leq \alpha^k$ and $\alpha^k \notin S_{i+1}$, we have $\alpha^{k-1} \notin S_{i+1}$ and $\alpha^{k-1} \in S_i - S_{i+1}$. Thus, there exists a chain β with $\beta^0 = \alpha^k$ such that $\#d(F,\beta) = d(F) - i$, and it implies that $\#d(F,\gamma) = d(F) - i + 1$ for a chain γ with $\gamma^0 = \alpha^{k-1}$ and $\gamma^i = \beta^{i-1}$ for $i \geq 1$. However, it contradicts to $\alpha^{k-1} \in S_i - S_{i+1}$.

Now we are ready to show how F is computed. Let $\nu_1, \nu_2, \ldots, \nu_D \in \{\lfloor (d(F)+1)^{1/D} \rfloor, \lfloor (d(F)+1)^{1/D} \rfloor + 1\}$ satisfying $\sum_{\ell=1}^{D} \nu_\ell = \sigma_D(d(F))$. Recall that each t_i for $1 \leq i \leq d(F)$ is a monotone Boolean function. From Lemma 7 with $m = d(F)$ and $h_i = t_i$ for each $1 \leq i \leq d(F)$, the collection of Boolean function $G = \{g_j^\ell \mid 1 \leq \ell \leq D, 1 \leq j \leq \nu_\ell\}$, defined by,

$$g_j^\ell(x) = 1 \quad \text{iff} \quad \left(\sum_{i=1}^{m} t_i(x)\right) \bmod P_\ell < jP_{\ell+1},$$

can be computed by a depth-D circuit which contains $\sigma_D(d(F))$ negation gates. In order to complete the proof, it suffices to show that each f_j is a monotone Boolean function of x and all $g_j^k(x)$'s for $1 \leq k \leq D$ and $1 \leq j \leq \nu_k$, so that, each f_j can be computed by a monotone gate in level $D+1$.

Notice that $f_j(x) = f_j^i(x)$ if $x \in S_i - S_{i+1}$, and $t_i(x) \wedge \neg t_{i+1}(x) = 1$ iff $x \in S_i - S_{i+1}$. Thus,

$$f_j = \bigvee_{i=0}^{d(F)} t_i \wedge \neg t_{i+1} \wedge f_j^i.$$

Notice from $S_0 = \{0,1\}^n$ and $S_{d(F)+1} = \emptyset$ that t_0 and $t_{d(F)+1}$ are constant functions. From (7), we have $\neg t_{i+1}(x) = 1$ iff $\sum_{i'=1}^{d(F)} t_{i'}(x) \leq i$. Then, from (5) with $m = d(F)$ and $h_i = t_i$,

$$\neg t_{i+1}(x) = 1 \quad \text{iff} \quad \sum_{k=1}^{D} P_{k+1} \sum_{j=1}^{\nu_k} g_j^k(x) \geq \sum_{k=1}^{D} \nu_k P_{k+1} - i$$

Thus, $\neg t_{i+1}$ for each $1 \leq i \leq d(F)$ is a monotone function of all $g_j^k(x)$ for $1 \leq k \leq D$ and $1 \leq j \leq \nu_k$. Since each t_i for $1 \leq i \leq d(F)$ and each f_j^i for $0 \leq i \leq d(F)$ and $1 \leq j \leq m$ are monotone Boolean functions of x. Therefore, each f_j, for $1 \leq j \leq m$, is a monotone function of x and each $g_j^k(x)$ for $1 \leq k \leq D$ and $1 \leq j \leq \nu_k$. The proof is completed. ∎

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