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title: Hausdorff Dimension and the Stochastic Traveling Salesman Problem
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Abstract

The traveling salesman problem is a problem of finding the shortest tour through given points. In 1959, Beardwood, Halton, and Hammersley studied asymptotic length of the shortest tour through points on Euclidean space. In particular they described the optimal tour length with Euclidean dimension for the case that points are distributed with respect to Lebesgue absolutely continuous measure. In this paper we reinterpret and generalize their results in terms of fractal geometry. We give the asymptotic order of the optimal tour length in terms of Hausdorff dimension.
key words: TSP, BHH theorem, Hausdorff dimension, fractal geometry.

1 Introduction

The traveling salesman problem (TSP) is a problem of finding the shortest tour through given points. We study asymptotic length of the shortest tour through points on Euclidean space.

Though TSP is an NP-hard problem, Karp [5] showed that if points $X_1, \cdots, X_n$ are uniformly distributed on the unit square then there is a polynomial time algorithm that generate a tour of length $L(X_1, \cdots, X_n)$ such that

$$\lim_{n\to\infty} L(X_1, \cdots, X_n)/L_{opt}(X_1, \cdots, X_n) = 1, a.s.,$$

where L_{opt} is the length of the shortest tour. Karp's algorithm is based on the following theorem by Beardwood, Halton, and Hammersley (BHH theorem):

Theorem 1.1 (BHH[2]) *If points $X_1, \cdots, X_n$ are i.i.d. random variables with respect to distribution μ on $[0,1]^d$ then*

$$\lim_{n\to\infty} L_{opt}(X_1, \cdots, X_n)/n^{1-\frac{1}{d}} = \beta(d) \int_{[0,1]^d} f(x)^{1-\frac{1}{d}} dx, \ \mu - a.s.,$$

where $\beta(d)$ is a constant that depend on the dimension d, and $f(x)$ is the density of μ with respect to Lebesgue measure.

We show that an analogous result holds for singular distributions. To state the result we introduce some notations and results shown in [3]. Let $x \in [0,1]^d$. Let $B_r(x)$ be the d-dimensional ball with center x and radius r. Let μ_h be a probability distribution on $[0,1]^d$ such that

$$\lim_{r \to 0} \log \mu_h(B_r(x) \cap [0,1]^d)/\log r = h, \ \ \mu_h - a.s. \tag{1}$$

Let $H(\mu_h)$ be the support set of μ_h, i.e.,

$$H(\mu_h) = \{x | \lim_{r \to 0} \log \mu_h(B_r(x) \cap [0,1]^d)/\log r = h\}. \tag{2}$$

Then it is known that

$$\dim H(\mu_h) = h, \tag{3}$$

where $\dim H$ is the Hausdorff dimension of H. For a proof of (3) see [3]. Note that many of sets including fractal sets are described by such a manner [3].

We prove that:

Theorem 1.2 (Main result) *If points $X_1, \cdots, X_n$ are i.i.d. random variables with respect to μ_h, then under conditions on μ_h, there exist two constants c_1 and c_2 $(0 < c_1 \leq c_2 < \infty)$ such that for $h > 1$*

$$c_1 \leq \liminf_n L_{opt}(X_1, \cdots, X_n)/n^{1-\frac{1}{h}} \leq \limsup_n L_{opt}(X_1, \cdots, X_n)/n^{1-\frac{1}{h}} \leq c_2, \ \ \mu_h - a.s.,$$

and for $0 < h \leq 1$, $L_{opt}(X_1, \cdots, X_n) = O(\sqrt{\log n})$, $\mu_h - a.s.$

Note that if $h < d$, the measure μ_h is singular with respect to Lebesgue measure on $[0,1]^d$; and therefore BHH theorem cannot be applied to the measure μ_h since the density of the absolutely continuous part is 0.

The theorem above shows that if points are distributed over a set $H(\mu_h)$ of Hausdorff dimension h $(< d)$, then the optimal tour length is much shorter than that of the case for uniform distribution for large number of points. Roughly speaking, this is because if $h < d$, the points $X_1, \cdots, X_n$ are distributed over the d-dimensional volume 0 set and therefore the average distance from a given point $X \in H(\mu_h)$ to the nearest point of $X_1, \cdots, X_n$ is much smaller than that of the case for uniform distribution.

Our results are reinterpretation and generalization of results of [2, 7, 8] in terms of fractal geometry.

2 Average optimal tour length

The proof is almost parallel to those of Stadje [7] and Steel [8]. In this paper we consider the class of distributions that satisfy the following condition:

Condition 1 *Let μ_h be a distribution on $[0,1]^d$ that satisfies the following property: There exist a subset $H(\mu_h)$ of $[0,1]^d$ such that*

$$\mu_h(H(\mu_h)) = 1,$$

and for $x \in H(\mu_h)$

$$\mu_h(B_r(x) \cap [0,1]^d) = f(x) r^{h+g(r,x)}, \tag{4}$$

where

$$h > 0, \ \ f(x) > 0, \ \ \lim_{r \to 0} g(r,x) = 0,$$

and f is the density. Let $\tilde{\mu}$ be the measure defined by $\tilde{\mu}_h(B_r(x)) = r^{h+g(r,x)}$. We assume that $\tilde{\mu}_h([0,1]^d) < \infty$.

Note that μ_h and $H(\mu_h)$ that satisfy the condition above satisfy (1) and that $\dim H(\mu_h) = h > 0$. Conversely if μ_h satisfies (1) and $h > 0$, then there exists $H(\mu_h), g,$ and f that satisfy the condition above such that $\mu_h(H(\mu_h)) = 1$ and $\dim H(\mu_h) = h > 0$.

Let

$$q_n(x) = E(\min_{1 \le i \le n} |X_i - x|). \tag{5}$$

In [7], Stadje showed that if $X_1, \cdots, X_n$ are i.i.d. random variables with respect to an absolutely continuous distribution with respect to Lebesgue measure on $[0,1]^d$ then

$$\lim_{n \to \infty} n^{\frac{1}{d}} q_n(x) = f(x)^{-\frac{1}{d}} d^{-1} \pi^{-\frac{1}{2}} \Gamma(\frac{1}{d}) \Gamma(1 + \frac{d}{2})^{\frac{1}{d}}, \tag{6}$$

where f is the density and $f(x) > 0$.

In the following let $X_1, \cdots, X_n$ be i.i.d. random variables with respect to μ_h in (5). We show that an analogous result of (6) holds for the distribution μ_h.

Lemma 2.1 *Let $h(n)$ be function of n such that $\lim_{n \to \infty} h(n) = h > 0$. For any positive constant a, b, and c, we have*

$$\lim_{n \to \infty} (cn)^{\frac{1}{h(n)}} \int_0^a (1 - c r^{h(n)})^n dr = \lim_{n \to \infty} (cn)^{\frac{1}{h(n)}} \int_0^{n^{-\frac{1}{(1+b)h}}} (1 - c r^{h(n)})^n dr = \frac{\Gamma(\frac{1}{h})}{h}. \tag{7}$$

Proof) We prove (7) by Laplace method. Let $c r^{h(n)} = \frac{1}{n} \tilde{r}^{h(n)}$, i.e., $\tilde{r} = (cn)^{\frac{1}{h(n)}} r$. Then we have

$$\int_0^{n^{-\frac{1}{(1+b)h}}} (1 - c r^{h(n)})^n dr = (cn)^{-\frac{1}{h(n)}} \int_0^\infty I_{[0, c^{\frac{1}{h(n)}} n^{-\frac{1}{(1+b)h} + \frac{1}{h(n)}}]} \exp\{n \log(1 - \frac{1}{n} \tilde{r}^{h(n)})\} d\tilde{r},$$

where I_A is the characteristic function of a set A. Since $c^{\frac{1}{h(n)}} n^{-\frac{1}{(1+b)h} + \frac{1}{h(n)}} \to \infty$ as $n \to \infty$, we have for sufficiently large n,
$I_{[0, c^{\frac{1}{h(n)}} n^{-\frac{1}{(1+b)h} + \frac{1}{h(n)}}]} \exp\{n \log(1 - \frac{1}{n} \tilde{r}^{h(n)})\} \le \exp\{-\tilde{r}^{\frac{h}{2}}\}$, $\int_0^\infty \exp\{-\tilde{r}^{\frac{h}{2}}\} d\tilde{r} < \infty$, and

$\lim_{n\to\infty} I_{[0,c^{\frac{1}{h(n)}}n^{-\frac{1}{(1+b)h}+\frac{1}{h(n)}}]} \exp\{n\log(1-\frac{1}{n}\tilde{r}^{h(n)})\} = \exp\{-\tilde{r}^h\}$ for $\tilde{r} > 0$; and therefore by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int I_{[0,c^{\frac{1}{h(n)}}n^{-\frac{1}{(1+b)h}+\frac{1}{h(n)}}]} \exp\{n\log(1-\frac{1}{n}\tilde{r}^{h(n)})\}d\tilde{r} = \int_0^\infty \exp\{-\tilde{r}^h\}d\tilde{r} = \frac{\Gamma(\frac{1}{h})}{h},$$

which proves the second equality of (7).

For the first equality, observe that

$$\int_{n^{-\frac{1}{(1+b)h}}}^a (1-cr^{h(n)})^n dr \le a(1-cn^{-\frac{h(n)}{(1+b)h}})^n \le a\exp(-cn^{1-\frac{h(n)}{(1+b)h}}). \tag{8}$$

Since $1-\frac{h(n)}{(1+b)h} > 0$ for sufficiently large n, by (8), and the second equality of (7), we have the first equality of (7). ■

In the following, let b be a positive constant, and let

$$\delta(n,x) = \sup_{0\le r\le n^{-\frac{1}{(1+b)h}}} |g(r,x)|, \tag{9}$$

and

$$\delta(n) = \sup_{x\in H(\mu_h)} \delta(n,x).$$

Lemma 2.2 *Let μ_h and $H(\mu_h)$ be a distribution on $[0,1]^d$ and its support set that satisfy Condition 1. Let $C_1^h(x) = f(x)^{-\frac{1}{h}}\frac{\Gamma(\frac{1}{h})}{h}$. Then for $x \in H(\mu_h)$, we have*

$$\limsup_n q_n(x)n^{\frac{1}{h+\delta(n,x)}} \le C_1^h(x) \le \liminf_n q_n(x)n^{\frac{1}{h-\delta(n,x)}}. \tag{10}$$

In particular if $\delta(n,x) = o((\log n)^{-1})$, we have for $x \in H(\mu_h)$,

$$\lim_{n\to\infty} q_n(x)n^{\frac{1}{h}} = C_1^h(x). \tag{11}$$

Proof) Let $x \in H(\mu_h)$. We have

$$\mu_h(\min_{1\le i\le n}|X_i - x| \ge r) = (1-\mu_h(B_r(x)\cap[0,1]^d))^n,$$

and hence

$$\begin{aligned} q_n(x) = E(\min_{1\le i\le m}|X_i - x|) &= \int_0^{\sqrt{d}} \mu_h(\min_{1\le i\le m}|X_i - x| \ge r)dr \\ &= \int_0^{\sqrt{d}} (1-\mu_h(B_r(x)\cap[0,1]^d))^n dr \\ &= \int_0^{a(n)} A_n(r)dr + \int_{a(n)}^{\sqrt{d}} A_n(r)dr \end{aligned} \tag{12}$$

where $A_n(r) = (1-\mu_h(B_r(x)\cap[0,1]^d))^n$, $a(n) = n^{-\frac{1}{(1+b)h}}$, and b is a positive constant.

We have

$$\begin{aligned}\int_0^{a(n)} A_n(r)dr &= \int_0^{a(n)} (1-f(x)r^{h+g(r,x)})^n dr \\ &\leq \int_0^{a(n)} (1-f(x)r^{h+\delta(n,x)})^n dr \qquad (13)\\ &= (f(x)n)^{-\frac{1}{h+\delta(n,x)}} \frac{\Gamma(\frac{1}{h})}{h}(1+o(1)), \qquad (14)\end{aligned}$$

where the first equality and the first inequality follow from (4) and (9); for the last equality observe that $\lim_{n\to\infty} \delta(n,x) = 0$, and hence (14) follows from Lemma 2.1.

Since $A_n(r)$ is decreasing as r grows, we have

$$\begin{aligned}\int_{a(n)}^{\sqrt{d}} A_n(r)dr &\leq \sqrt{d}A_n(a(n)) \\ &= \sqrt{d}(1-f(x)a(n)^{h+g(a(n),x)})^n \\ &\leq \sqrt{d}\exp(-f(x)n^{1-\frac{h+g(a(n),x)}{(1+b)h}}). \qquad (15)\end{aligned}$$

Since $\lim_{n\to\infty} g(a(n),x) = 0$, we see $\int_{a(n)}^{\sqrt{d}} A_n(r)dr = o(n^{-\frac{1}{h+\delta(n,x)}})$; hence we have the first inequality of (10). In a similar way, we can prove the other inequality of (10). If $\delta(n,x) = o((\log n)^{-1})$, we have (11). ■

Remark 2.1 *If μ_d is an absolutely continuous distribution with respect to Lebesgue measure on $[0,1]^d$ and if x is a interior point of $[0,1]^d$, we see $\mu_d(B_r(x)) = f(x)c_d r^{d+g(r,x)}$, where c_d $(= \pi^{d/2}/\Gamma((d+2)/2))$ is the volume of the d-dimensional unit ball, $f(x)$ is the density, and $g(r,x) = o((\log r)^{-1})$. By applying Lemma 2.2 to $\mu_d(B_r(x))$, we have (6).*

Lemma 2.3 *Let μ_h be a distribution that satisfy Condition 1.*
Let $C_2^h = E(C_1^h(x)) = E(f(x)^{-\frac{1}{h}})\frac{\Gamma(\frac{1}{h})}{h} \leq \infty$. We have

$$\limsup_n E(q_n(x)n^{\frac{1}{h+\delta(n,x)}}) \leq C_2^h \leq \liminf_n E(q_n(x)n^{\frac{1}{h-\delta(n,x)}}). \qquad (16)$$

In particular if $\delta(n,x) = o((\log n)^{-1})$, we have

$$\lim_{n\to\infty} E(q_n(x))n^{\frac{1}{h}} = C_2^h. \qquad (17)$$

Proof) First we show the lemma when $C_2^h < \infty$. Since $C_2^h = E(C_1^h(x)) < \infty$ and $\mu_h(H_\mu) = 1$, by Fatou lemma and (10), we have (16). If $\delta(n,x) = o((\log n)^{-1})$, we have (17).

Note that by Fatou lemma, $\liminf_n E(q_n(x)n^{\frac{1}{h-\delta(n,x)}}) \geq E(\liminf_n q_n(x)n^{\frac{1}{h-\delta(n,x)}})$ holds without assuming that $q_n(x)n^{\frac{1}{h-\delta(n,x)}}$ is bounded by integrable function; hence the lemma holds for $C_2^h = \infty$. ■

Remark 2.2 *If $h \geq 1$, $E(f(x)^{-\frac{1}{h}})$ always exists and have a finite value, because by Jensen's inequality we have $E((\frac{1}{f(x)})^{\frac{1}{h}}) \leq E(1/f(x))^{\frac{1}{h}} = (\int_{H(\mu_h)} d\tilde{\mu}_h)^{\frac{1}{h}} < \infty$ where $\tilde{\mu}_h$ is the finite measure defined by $\tilde{\mu}_h(B_r(x)) = r^{h+g(r,x)}$.*

In the following for simplicity, L denote L_{opt}. Then it is known that

$$nE(q_{n-1}(X)) \leq E(L(X_1, \cdots, X_n)) \leq 2\sum_{i=1}^{n} E(q_i(X)). \tag{18}$$

For a proof, see [7, 8].

From (18) and Lemma 2.3, we have:

Theorem 2.1 *Assume that $C_2^h < \infty$ and $\delta(n) = o((\log n)^{-1})$. Under Condition 1, for $1 < h$*

$$c_1 \leq \liminf_n E(L(X_1, \cdots, X_n))/n^{1-\frac{1}{h}} \leq \limsup_n E(L(X_1, \cdots, X_n))/n^{1-\frac{1}{h}} \leq c_2, \tag{19}$$

and for $0 < h \leq 1$, $\sup_n E(X_1, \cdots, X_n) < \infty$, where c_1 and c_2 are constants dependent on h such that $0 < c_1 \leq c_2 < \infty$.

3 Concentration

Let F_i be a σ-algebra generated by $X_1, \cdots, X_n$ for $1 \leq i \leq n$, and F_0 be a trivial one. Let f be a measurable function with respect to F_n. Let $d_i = E(f|F_i) - E(f|F_{i-1})$. We see $f - E(f) = \sum_{i=1}^n d_i$, and $\{\sum_{k=1}^i d_k\}_i$ is a martingale sequence with respect to F_i, $1 \leq i \leq n$. For a random variable X, let $\operatorname{ess\,sup}_X f(X) = \inf\{a \,|P(f(X) > a) = 0\}$, and $\operatorname{ess\,inf}_X f(X) = \sup\{a \,|P(f(X) < a) = 0\}$. Let $\tilde{d_i} = \operatorname{ess\,sup} d_i - \operatorname{ess\,inf} d_i$. Then the following Azuma-Hoeffding inequality holds.

Theorem 3.1 (Azuma-Hoeffding[1, 4]) *For any $t > 0$,*
$P(|f - E(f)| \geq t) \leq 2\exp(-2t^2/\sum_{i=1}^n \tilde{d}_i^2)$.

For some applications of the theorem to combinatorics, see [6, 8] and for Markov processes see [9]. In this section we apply Azuma-Hoeffding inequality to our model.

In Theorem 3.1, let $f = L(X_1, \cdots, X_n)$. In order to obtain $\tilde{d_i}$, observe that [7, 8]

$$L(X_1, \cdots, \hat{X}_i, \cdots, X_n) \leq L(X_1, \cdots, X_n) \leq L(X_1, \cdots, \hat{X}_i, \cdots, X_n) + 2\min_{1 \leq j \leq n, j \neq i} |X_i - X_j|,$$

where $(X_1, \cdots, \hat{X}_i, \cdots, X_n)$ is the random vector obtained by deleting X_i from $(X_1, \cdots, X_n)$. Thus we have

$$\begin{aligned} \tilde{d_i} &\leq 2\operatorname*{ess\,sup}_{X_1,\cdots,X_i} E(\min_{1 \leq j \leq n, j \neq i} |X_i - X_j| \,|F_i) \\ &\leq 2\operatorname*{ess\,sup}_{X_1,\cdots,X_i} E(\min_{i < j \leq n} |X_i - X_j| \,|F_i) \\ &= 2\operatorname*{ess\,sup}_{X_i} E(\min_{i < j \leq n} |X_i - X_j| \,|X_i) = 2\operatorname*{ess\,sup}_{X_i} q_{n-i}(X_i), \end{aligned} \tag{20}$$

where the first equality follows from that $X_1, \cdots, X_n$ are i.i.d. random variables.

To prove the following theorem we need a condition.

Condition 2 *Assume that there exists a positive constant* m *such that* $\inf_{x\in H(\mu_h)} f(x) > m > 0$. *Assume that* $\lim_{n\to\infty}\delta(n) = 0$.

Lemma 3.1 *Under Condition 1 and 2, there exists a constant* M *such that*

$$\sup_{x\in H(\mu_h)} q_n(x) \le Mn^{-\frac{1}{h+\delta(n)}}. \tag{21}$$

Proof) Let $A_n(r)$ and $a(n)$ be the same as in the proof of Lemma 2.2. From (13), Condition 2, and Lemma 2.1, we have for sufficiently large n,

$$\begin{aligned}\int_0^{a(n)} A_n(r)dr &\le \int_0^{a(n)}(1-f(x)r^{h+\delta(n,x)})^n dr \\ &\le \int_0^{a(n)}(1-mr^{h+\delta(n)})^n dr \\ &\le mn^{-\frac{1}{h+\delta(n)}},\end{aligned} \tag{22}$$

where m is a constant. Note that $a(n)\to 0$ as $n\to\infty$.

From (15), we have

$$\int_{a(n)}^{\sqrt{d}} A_n(r)dr \le \sqrt{d}\exp(-f(a(n),x)n^{1-\frac{h+g(a(n),x)}{(1+b)h}}) \le \sqrt{d}\exp(-mn^{1-\frac{h+\delta(n)}{(1+b)h}}). \tag{23}$$

Since $\lim_{n\to\infty}\delta(n) = 0$ (Condition 2), from (22), (23), and (12), we have (21). ■

Theorem 3.2 *Under Condition 1, and 2, if* $\delta(n) = o((\log n)^{-1})$, *there exist constants* M_1, M_2, *and* M_3 *such that*

$$\sum_{i=1}^n \tilde{d}_i^2 \le \begin{cases} M_1, & \text{if } h<2, \\ M_2\log n, & \text{if } h=2, \\ M_3 n^{1-\frac{2}{h}}, & \text{if } h>2, \end{cases}$$

and for any $t>0$,

$$\mu_h(|f-E(f)|\ge t) \le 2\exp(-2t^2/\sum_{i=1}^n \tilde{d}_i^2),$$

where $f = L(X_1,\cdots,X_n)$.

Proof) Since $\mu_h(H(\mu_h)) = 1$, by (20) and Lemma 3.1, we have

$$\tilde{d}_i \le M(n-i)^{-\frac{1}{h}},$$

where M is a positive constant. Theorem 3.2 follows from Theorem 3.1. ■

Theorem 3.3 *Assume that* $\delta(n) = o((\log n)^{-1})$. *Under Condition 1, and 2, for* $1 < h$,

$$c_1 \le \liminf_n L(X_1, \cdots, X_n)/n^{1-\frac{1}{h}} \le \limsup_n L(X_1, \cdots, X_n)/n^{1-\frac{1}{h}} \le c_2,\ \mu_h - a.e., \quad (24)$$

where c_1 *and* c_2 *are constants that depend on* h. *For* $0 < h \le 1$, *we have* $L(X_1, \cdots, X_n) = O(\sqrt{\log n}),\ \mu_h - a.s.$

Proof) By Borel-Cantelli's lemma and Theorem 3.2, we have

$$\limsup_n \frac{|f - E(f)|}{g(n)} \le 1,\ \mu_h - a.s.,$$

where $f = L(X_1, \cdots, X_n)$, and

$$g(n) = \begin{cases} O(\sqrt{\log n}), & \text{if } h < 2, \\ O(\log n), & \text{if } h = 2, \\ O(n^{\frac{1}{2}-\frac{1}{h}}\sqrt{\log n}), & \text{if } h > 2. \end{cases}$$

By Theorem 2.1, we have the theorem. ∎

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