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Weighted Random Popular Matchings

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Abstract: For a set A of n applicants and a set I of m items, let us consider the problem of matching applicants to items, where each applicant $x \in A$ provides its *preference list* defined on items. We say that an applicant x prefers an item p than an item q if p is located at higher position than q in its preference list. For any matchings $\mathcal{M}$ and $\mathcal{M}'$ of the matching problem, we say that an applicant x prefers $\mathcal{M}$ over $\mathcal{M}'$ if x prefers $\mathcal{M}(x)$ over $\mathcal{M}'(x)$. For the matching problem, we say that $\mathcal{M}$ is *more popular* than $\mathcal{M}'$ if the number of applicants preferring $\mathcal{M}$ over $\mathcal{M}'$ is larger than the number of applicants preferring $\mathcal{M}'$ over $\mathcal{M}$, and define $\mathcal{M}$ to be a *popular matching* if there are no other matchings that are more popular than $\mathcal{M}$. Assume that A is partitioned into $A_1, A_2, \ldots, A_k$ and each A_i is assigned a weight w_i such that $w_1 > w_2 > \cdots > w_k > 0$. For such a matching problem, we say that $\mathcal{M}$ is *more popular* than $\mathcal{M}'$ if the total weight of applicants preferring $\mathcal{M}$ over $\mathcal{M}'$ is larger than the total weight of applicants preferring $\mathcal{M}'$ over $\mathcal{M}$, and define $\mathcal{M}$ to be a *k-weighted popular matching* if there are no other matchings that are more popular than $\mathcal{M}$. Mahdian showed that if $m > 1.42n$, then a random instance of the matching problem has a popular matching with high probability, but nothing is known for the k-weighted matching problems. In this paper, we analyze the k-weighted matching problems, and we show that for any β such that $m = \beta n$, (lower bound) if $\beta/n^{1/3} = o(1)$, then a random instance of the 2-weighted matching problems does not have a 2-weighted popular matching with probability $1 - o(1)$; (upper bound) if $n^{1/3}/\beta = o(1)$, then a random instance of the 2-weighted matching problems has a 2-weighted popular matching with probability $1 - o(1)$.

1 Introduction

For a set A of n applicants and a set I of m items, let us consider the problem of matching applicants to items, where each applicant $x \in A$ provides its *preference list* defined on a subset $J_x \subseteq I$. A preference list $\vec{p}_x$ of each applicant x may contain ties among the items and it ranks a subset of J_x, i.e., J_x is partitioned into $J_x^1, J_x^2, \ldots, J_x^d$, where J_x^h is a set of the h^{th} preferred items. We say that an applicant x prefers $p \in J_x$ than $q \in J_x$ if $p \in J_x^i$ and $q \in J_x^h$ for $i < h$, and we say that an applicant x has a tie if there exists an h such that $p, q \in J_x^h$. An instance of the matching problem consists of a bipartite graph $B = (A \cup I, D)$ in which there exists an edge $(x, p) \in D$ for each $x \in A$ and each $p \in J_x$. For any matching $\mathcal{M}$ of $B = (A \cup I, D)$, let $\mathcal{M}(x)$ be an item $p \in I$ that is matched to an applicant x by $\mathcal{M}$. For any matchings $\mathcal{M}$ and $\mathcal{M}'$ of $B = (A \cup I, D)$, we say that an applicant x prefers $\mathcal{M}$ over $\mathcal{M}'$ if the applicant x prefers $\mathcal{M}(x)$ over $\mathcal{M}'(x)$. For a bipartite graph $B = (A \cup I, D)$, we say that $\mathcal{M}$ is *more popular* than $\mathcal{M}'$ if the total *number* of applicants preferring $\mathcal{M}$ over $\mathcal{M}'$ is larger than the total *number* of applicants preferring $\mathcal{M}'$ over $\mathcal{M}$, and define $\mathcal{M}$ to be a *popular matching* [6] if there exist no other matchings that are more popular than $\mathcal{M}$. The notion of popular matchings have applications in the real world, e.g., mail-based DVD rental systems such as NetFlix [1].

Assume that the set A of applicants is partitioned into $A_1, A_2, \ldots, A_k$ and each category A_i is assigned a weight $w_i > 0$ such that $w_1 > w_2 > \cdots > w_k$. This setting can be regarded as a case where the applicants in A_1 are platinum members, the applicants in A_2 are gold members, the applicants in A_3 are silver members, the applicants in A_4 are regular members, etc. In a way similar to the above, we can consider k-weighted matching problems and an instance of k-weighted matching problems is also given by a bipartite graph $B_k = (A \cup I, D)$. For a k-weighted matching problem $B_k = (A \cup I, D)$, we say that $\mathcal{M}$ is *more popular* than $\mathcal{M}'$ if the total *weight* of applicants preferring $\mathcal{M}$ over $\mathcal{M}'$ is larger than the total *weight* of applicants preferring $\mathcal{M}'$ over $\mathcal{M}$.

Definition 1.1 [8]**:** *For any k-weighted matching problem $B_k = (A \cup I, D)$, a matching $\mathcal{M}$ of B_k is* k-weighted popular matching *if there exist no other matchings that are more popular than $\mathcal{M}$.*

In this paper, we simply refer to 1-weighted matching problems as matching problems, and also simply refer to 1-weighted popular matchings as popular matchings.

We say that a preference list $\vec{p}_x$ of an applicant x is *complete* if $J_x = I$, i.e., the applicant x represents its preferences on all items, and define a k-weighted matching problem $B_k = (A \cup I, D)$ to be complete if for each applicant $x \in A$, a preference list $\vec{p}_x$ of the applicant x is complete. We also say that an preference list $\vec{p}_x$ of an applicant x is *strict* if $|J_x^h| = 1$ for each h, i.e., the applicant x prefers each item in J_x differently, and we define a k-weighted matching problem $B_k = (A \cup I, D)$ to be strict if for each applicant $x \in A$, a preference list $\vec{p}_x$ of the applicant x is strict.

1.1 Known Results

For strict matching problems, Abraham, et al. [2, Theorem 3.1] presented a deterministic $O(n+m)$ time algorithm that outputs a popular matching if it exists, and for matching problems with ties, Abraham, et al. [2, Theorem 3.2] also showed a deterministic $O(\sqrt{n}m)$ time algorithm that outputs a popular matching if it exists. To derive these algorithms, Abraham, et al. [2] introduced notions of f-items (the first items) and s-items (the second items), and characterized popular matchings by f-items and s-items. Mestre [8] generalized those results to k-weighted matching problems. For strict k-weighted matching problems, Mestre [8, Theorem 2] showed a deterministic $O(n+m)$ time algorithm that outputs a k-weighted popular matching if it exists, and for k-weighted matching problems with ties, Mestre [8, Theorem 3] showed a deterministic $O(\min(k\sqrt{n}, n)m)$ time algorithm that outputs a k-weighted popular matching if it exists.

In general, the matching problems do not always have popular matchings. Mahdian [7] answered to a question of when the matching problems have popular matchings. In fact, Mahdian presented that if $m > 1.42n$, then a random instance of the matching problems has a popular matching with probability $1 - o(1)$ [7, Theorem 1], and also showed that if $m < 1.42n$, then a random instance of the matching problems does not have a popular matching with probability $1 - o(1)$ [7, §4].

1.2 Main Results

In this paper, we answer to a question of when k-weighted matching problems have k-weighted popular matchings. More precisely, we show the following results:

Theorem 4.1: Let $m = \beta n$. If $\beta/n^{1/3} = o(1)$, then a random instance of the 2-weighted matching problems does not have a 2-weighted popular matching with probability $1 - o(1)$.

Theorem 5.1: Let $m = \beta n$. If $n^{1/3}/\beta = o(1)$, then a random instance of the 2-weighted matching problems has a 2-weighted popular matching with probability $1 - o(1)$.

In the case of the matching problems, it suffices to consider only the set F of f-items and the set S of s-items. In the case of the 2-weighted matching problems, however, we need to separately consider the set F_1 of f_1-items, the set S_1 of s_1-items, the set F_2 of f_2-items, and the set S_2 of s_2-items, and we also need to carefully deal with the case that $S_1 \cap F_2 \neq \emptyset$, which makes the analysis of the 2-weighted matching problems much harder than that of the matching problems.

2 Preliminaries

Let A be the set of n applicants and I be the set of m items. For some $\beta \geq 1$, let $m = \beta n$. We assume that A is partitioned into A_1 and A_2 and we refer to A_1 (resp. A_2) as the first (resp. the second) category. For any constant $0 < \delta < 1$, we also assume that $|A_1| = \delta|A| = \delta n$ and $|A_2| = (1-\delta)|A| = (1-\delta)n$. Let $w_1 > w_2 > 0$ be weights of the first category A_1 and the second category A_2, respectively. In this paper, we only consider 2-weighted matching problems that are complete and/or strict.

We define f-items and s-items [2, 8] as follows: For each applicant $x \in A_1$, let $f_1(x)$ be the most preferred item in its preference list $\vec{p}_x$. We refer to $f_1(x)$ as an f_1-item of the applicants $x \in A_1$ and use F_1 to denote the set of all f_1-items of applicants $x \in A_1$. For each applicant $x \in A_1$, let $s_1(x)$ be the most preferred item in its preference list $\vec{p}_x$ that is not in F_1. We refer to $s_1(x)$ as an s_1-item of the applicant $x \in A_1$ and use S_1 to denote the set of all s_1-items of applicants $x \in A_1$. For each applicant $y \in A_2$, let $f_2(y)$ be the most preferred item in its preference list $\vec{p}_y$ that is not in F_1. We refer to $f_2(y)$ as an f_2-item of the applicants $y \in A_2$ and use F_2 to denote the set of all f_2-items of applicants $y \in A_2$. For each applicant $y \in A_2$, let $s_2(y)$ be the most preferred item in its preference list $\vec{p}_y$ that is not in $F_1 \cup F_2$. We refer to $s_2(y)$ as an s_2-item of the applicant $y \in A_2$ and use S_2 to denote the set of all s_2-items of applicants $y \in A_2$. From the definitions of F_1, S_1, F_2, S_2, we have that $F_1 \cap S_1 = \emptyset$, $F_1 \cap F_2 = \emptyset$, and $F_2 \cap S_2 = \emptyset$, however, we may have that $S_1 \cap F_2 \neq \emptyset$ or $S_1 \cap S_2 \neq \emptyset$.

Mestre [8, Definition 1] defined a notion of *well-formed matchings* for k-weighted matching problems $B_k = (A \cup I, E)$ that is a generalization of well-formed matchings for matching problems due to Abraham, et al. [2]. Then we show the notion of well-formed matchings for strict k-weighted matching problems $B_k = (A \cup I, E)$ for the case where $k = 2$.

Definition 2.1: *For a strict 2-weighted matching problem $B_2 = (A \cup I, D)$, a matching $\mathcal{M}$ is* well-formed *if (1) each $x \in A_1$ is matched to $f_1(x)$ or $s_1(x)$ by $\mathcal{M}$; (2) each each $y \in A_2$ is matched to $f_2(y)$ or $s_2(y)$ by $\mathcal{M}$; (3) each $p \in F_1$ is matched to $x \in A_1$ by $\mathcal{M}$, where $f_1(x) = p$; (4) each $q \in F_2$ is matched to $y \in A_2$ by $\mathcal{M}$, where $f_2(y) = q$.*

For strict 2-weighted matching problems $B_2 = (A \cup I, E)$, Mestre [8] showed the following relations between 2-weighted popular matchings and well-formed matchings.

Proposition 2.1: *For strict 2-weighted matching problems $B_2 = (A \cup I, E)$ with $A = A_1 \cup A_2$, if $\mathcal{M}$ is a 2-weighted popular matching, then it is a well-formed matching.*

Proposition 2.2: *For strict 2-weighted matching problems $B_2 = (A \cup I, E)$ with $A = A_1 \cup A_2$ and $w_1 \geq 2w_2$, if $\mathcal{M}$ is a well-formed matching, then it is a 2-weighted popular matching.*

For a 2-weighted matching problem $B_2 = (A \cup I, D)$, we define a graph $G = (V, E)$ as follows: Let $V = F_1 \cup S_1 \cup F_2 \cup S_2$. For an applicant $x \in A_1$, connect $f_1(x)$ and $s_1(x)$, and let $e_x = (f_1(x), s_1(x)) \in E_1$. For an applicant $y \in A_2$, connect $f_2(y)$ and $s_2(y)$, and let $e_y = (f_2(y), s_2(y)) \in E_2$. Define $E = E_1 \cup E_2$. Note that the graph $G = (V, E)$ consists of $M = |V| \leq m$ vertices and $n = |A|$ edges. If $e_1 \in E_1$ and $e_2 \in E_2$ are incident to the same vertex $p \in V$, then $p \in S_1 \cap F_2$ or $p \in S_1 \cap S_2$. This property makes the 2-weighted matching problems harder than the matching problems. For any 2-weighted matching problem $B_2 = (A \cup I, D)$, we use the graph $G = (V, E)$ and show a necessary and sufficient condition for $B_2 = (A \cup I, D)$ to have well-formed matchings (see Corollary 3.1).

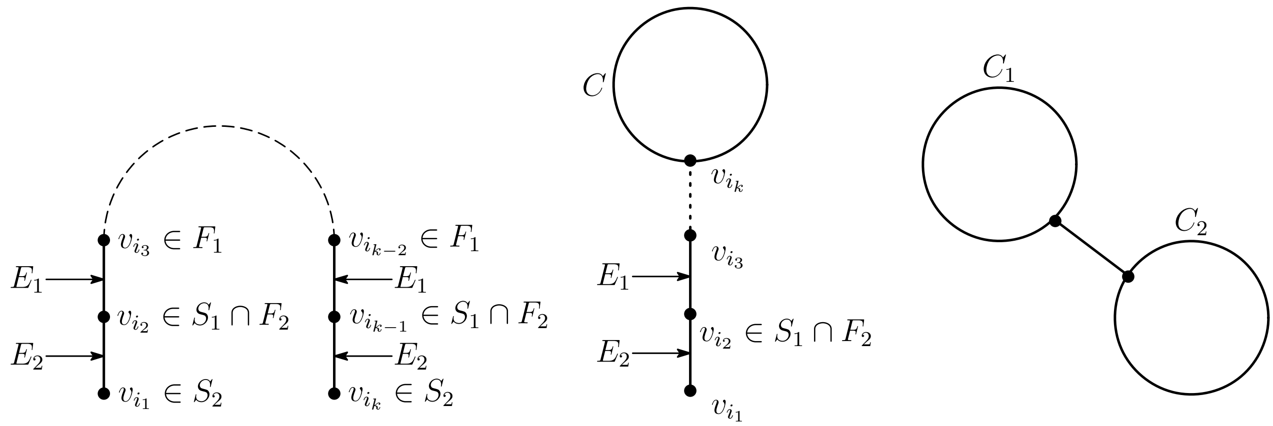
Lemma 2.1: *A strict 2-weighted matching problem $B_2 = (A \cup I, D)$ has a well-formed matching iff the graph $G = (V, E)$ has an orientation $\mathcal{O}$ on edges such that (a) each $p \in V$ has at most one incoming edge; (b) each $p \in F_1$ has one incoming edge in E_1; (c) each $q \in F_2$ has one incoming edge in E_2.*

Proof: Assume that a 2-weighted matching problem $B_2 = (A \cup I, D)$ has a well-formed matching $\mathcal{M}$ and let $A = A_1 \cup A_2$. We define an orientation $\mathcal{O}$ on edges of the graph $G = (V, E)$ as follows: For each applicant $a \in A_i$, orient an edge $e_a = (f_i(a), s_i(a)) \in E_i$ toward $\mathcal{M}(a)$. Since $\mathcal{M}$ is a matching between A and I, we have that each $p \in V$ has at most one incoming edge. From the condition (3) of Definition 2.1, it follows that each $p \in F_1$ has one incoming edge in E_1, and from the condition (4) of Definition 2.1, it follows that each $q \in F_2$ has one incoming edge in E_2. Thus we have that the orientation $\mathcal{O}$ on edges of the graph $G = (V, E)$ satisfies the conditions (a), (b), and (c).

Assume that the graph $G = (V, E)$ has an orientation $\mathcal{O}$ on edges satisfying the conditions (a), (b), and (c). Define a matching $\mathcal{M}$ for the 2-weighted matching problem $B_2 = (A \cup I, D)$ as follows: For each $x \in A_1$, its f_1-item $f_1(x)$ (resp. its s_1-item $s_1(x)$) is matched to x if $\mathcal{O}$ orients the edge $e_x = (f_1(x), s_1(x)) \in E_1$ by $f_1(x) \leftarrow s_1(x)$ (resp. $f_1(x) \rightarrow s_1(x)$), and for each $y \in A_2$, its f_2-item $f_2(y)$ (resp. its s_2-item $s_2(y)$) is matched to y if $\mathcal{O}$ orients the edge $e_y = (f_2(y), s_2(y)) \in E_2$ by $f_y(y) \leftarrow s_2(y)$ (resp. $f_2(y) \rightarrow s_2(y)$). From the condition (a) of the orientation $\mathcal{O}$, it is immediate that $\mathcal{M}$ is a matching for $B_2 = (A \cup I, D)$. From the definition of the graph $G = (V, E)$, we have that $\mathcal{M}$ satisfies the conditions (1) and (2) of Definition 2.1. The condition (b) of the orientation $\mathcal{O}$ implies that each $p \in F_1$ is matched to $x \in A_1$ by $\mathcal{M}$, where $f_1(x) = p$, and the condition (c) of the orientation $\mathcal{O}$ implies that each $q \in F_2$ is matched to $y \in A_2$ by $\mathcal{M}$, where $f_2(y) = q$. Thus the matching $\mathcal{M}$ of the graph $B_2 = (A \cup I, D)$ satisfies the conditions (1), (2), (3), and (4) of Definition 2.1. ■

3 Characterization for 2-Weighted Matching Problems

In this section, we present a necessary and sufficient condition for 2-weighted matching problems to have 2-weighted popular matchings. For a graph $G = (V, E)$, consider the subgraphs in Figure 1.



(a) Subgraph of Type G_1 (b) Subgraph of Type G_2 (c) Subgraph of Type G_3

Figure 1: (a) a path $P = v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ that has vertices $v_{i_2}, v_{i_{k-1}} \in S_1 \cap F_2$ such that $(v_{i_2}, v_{i_3}) \in E_1$ and $(v_{i_{k-2}}, v_{i_{k-1}}) \in E_1$; (b) a cycle C and a path $P = v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ incident to C at v_{i_k} that has a vertex $v_{i_2} \in S_1 \cap F_2$ such that $(v_{i_2}, v_{i_3}) \in E_1$; (c) a connected component including cycles C_1 and C_2.

Theorem 3.1: *A strict* 2*-weighted matching problem* $B_2 = (A \cup I, D)$ *has a well-formed matching iff the graph* $G = (V, E)$ *includes none of the subgraphs* G_1, G_2, *nor* G_3 *in Figure* 1.

Proof: Assume that the graph $G = (V, E)$ includes one of the subgraphs G_1, G_2, and G_3 in Figure 1. For the case where G includes the subgraph G_1, if the edge $(v_{i_2}, v_{i_3}) \in E_1$ is oriented by $v_{i_2} \leftarrow v_{i_3}$, then the edge $(v_{i_1}, v_{i_2}) \in E_2$ must be oriented by $v_{i_1} \leftarrow v_{i_2}$ to meet the condition (a) of Lemma 2.1. However, this does not meet the condition (c) of Lemma 2.1, since the vertex $v_{i_2} \in S_1 \cap F_2 \subseteq F_2$ has no incoming edges in E_2. Then the edge $(v_{i_2}, v_{i_3}) \in E_1$ is oriented by $v_{i_2} \leftarrow v_{i_3}$. This is also the case for the edge $(v_{i_{k-2}}, v_{i_{k-1}}) \in E_1$, i.e., the edge $(v_{i_{k-2}}, v_{i_{k-1}}) \in E_1$ is oriented by $v_{i_{k-1}} \rightarrow v_{i_{k-2}}$. These results imply that there exists $2 < j < k-1$ such that $v_{i_j} \in V$ has at least two incoming edges, which violates the condition (a) of Lemma 2.1. So if the graph $G = (V, E)$ includes the subgraph G_1, then it does not have any well-formed matching. In a way similar to the case for the subgraph G_2, we can show that if the graph $G = (V, E)$ includes the subgraph G_2, then it does not have any well-formed matching. For the case where the graph $G = (V, E)$ includes the subgraph G_3, we can also show that it does not have any well-formed matching in a way similar to the argument by Mahdian [7].

Assume that the graph $G = (V, E)$ does not include any of the subgraph G_1, G_2, or G_3 and let $\{C_i\}_{i \geq 1}$ be the set of cycles in G. We first orient cycles $\{C_i\}_{i \geq 1}$. Since the graph G does not include the subgraph G_1, we can orient each cycle C_i in one of the clockwise and counterclockwise orientations to meet the conditions (a), (b), and (c) of Lemma 2.1. From the assumption that the graph G does not include the subgraph G_3, the remaining edges can be categorized as follows: Let $E_{\text{tree}}^{\text{cyc}}$ be the set of edges in subtrees of G that are incident to some cycle $C \in \{C_i\}_{i \geq 1}$, and E_{tree} be the set of edges in subtrees of G that are not incident to any cycle $C \in \{C_i\}_{i \geq 1}$. Since the graph G does not include the subgraphs G_1 and G_2, we can orient the edges in $E_{\text{tree}}^{\text{cyc}}$ away from the cycles to meet the conditions (a), (b), and (c) of Lemma 2.1. We notice that the edges in E_{tree} consist of the set of subtrees $\{T_j\}_{j \geq 1}$ of G. For each $T \in \{T_j\}_{j \geq 1}$, let E_T^2 be the set of edges (v, u) that is assigned to some applicant in A_2 and $u \in S_1 \cap F_2$. For each edge $e = (v, u) \in E_T^2$, we first orient the edge e by $v \rightarrow u$ and then the remaining edges in E_T^2 are oriented away from each $u \in S_1 \cap F_2$. By the assumption that the graph G does not include the subgraph G_1, such an orientation meets the conditions (a), (b), and (c) of Lemma 2.1 for each vertex $v \in T$, and this completes the proof. ■

From Propositions 2.1 and 2.2 and Theorem 3.1, we immediately have the following corollary:

Corollary 3.1: *A strict* 2*-weighted matching problem* $B_2 = (A \cup I, D)$ *with* $A = A_1 \cup A_2$ *and* $w_1 \geq 2w_2$ *has a* 2*-weighted popular matching iff the graph* $G = (V, E)$ *includes none of the subgraphs* G_1, G_2*, nor* G_3 *in Figure* 1.

Let us consider a random instance of the 2-weighted matching problem $B_2 = (A \cup I, D)$ that is complete and strict, i.e., each applicant $x \in A$ is assigned a random preference list $\vec{p}_x$, which is a uniformly chosen permutation on the set I of all items. In the subsequent sections, we analyze the probability that a random instance of the complete and strict 2-weighted matching problems has (or does not have) a 2-weighted popular matching. To this end, we define the following process for a random choice of an instance of the graphs $G = (V, E)$:

(1) Each applicant $x \in A_1$ is assigned a uniformly chosen $p \in I$ as an f_1-item $f_1(x)$ and let F_1 be the set of all f_1-items assigned to applicants $x \in A_1$.

(2) Each applicant $x \in A_1$ is assigned a uniformly chosen $q \in I - F_1$ as an s_1-item $s_1(x)$ and let S_1 be the set of all s_1-items assigned to applicants $x \in A_1$.

(3) For each applicant $x \in A_1$, connect $f_1(x)$ and $s_1(x)$ and let $(f_1(x), s_1(x)) \in E_1$.

(4) Each applicant $y \in A_2$ is assigned a uniformly chosen $r \in I - F_1$ as an f_2-item $f_2(y)$ and let F_2 be the set of all f_2-items assigned to applicants $y \in A_2$.

(5) Each applicant $y \in A_2$ is assigned a uniformly chosen $s \in I - (F_1 \cup F_2)$ as an s_2-item $s_2(y)$ and let S_2 be the set of all s_2-items assigned to applicants $y \in A_2$.

(6) For each applicant $y \in A_2$, connect $f_2(y)$ and $s_2(y)$ and let $(f_2(y), s_2(y)) \in E_2$.

From Corollary 3.1, it is immediate that a random choice of an instance of the complete and strict 2-weighted matching problems $B_2 = (A \cup I, D)$ is equivalent to a random choice of an instance of the graphs $G = (V, E)$. So in the rest of this paper, we consider a random instance of the graphs $G = (V, E)$ instead of a random instance of the 2-weighted matching problems $B_2 = (A \cup I, D)$.

4 Lower Bounds for 2-Weighted Matching Problems

Let n be the total number of applicants and m be the total number of items. Let $m = \beta n$, where β could be a function of n, and assume that β is large enough so that $m - n = \geq m/c$ for some constant $c > 1$, i.e., $\beta \geq c/(c-1)$. For any constant $0 < \delta < 1$, let $n_1 = \delta n$ be the total number of applicants in A_1 and let $n_2 = (1-\delta)n$ be the total number of applicants in A_2. In this section, we show a lower bound for β of the 2-weighted matching problems not to have a 2-weighted popular matching, i.e.,

Theorem 4.1: *Let $m = \beta n$. If $\beta / n^{1/3} = o(1)$, then a random instance of the 2-weighted matching problems does not have a 2-weighted popular matching with probability $1 - o(1)$.*

Proof: Let F_1, F_2 be the set of the first items for applicants in A_1, A_2, respectively, and let S_1, S_2 be the set of the second items for applicants in A_1, A_2, respectively. By the definitions of F_1, F_2, S_1, S_2, we have that $F_1 \cap S_1 = \emptyset$; $F_1 \cap F_2 = \emptyset$; $F_1 \cap S_2 = \emptyset$; $F_2 \cap S_2 = \emptyset$, but we could have that $S_1 \cap F_2 \neq \emptyset$; $S_1 \cap S_2 \neq \emptyset$. Let $R_1 = I - F_1$ and let $R_2 = R_1 - F_2 = I - (F_1 \cup F_2)$. It is obvious that $1 \leq |F_1| \leq \delta n$; $1 \leq |F_2| \leq (1-\delta)n$, which implies that $m - \delta n \leq |R_1| \leq m$; $m - n \leq |R_2| \leq m$. From Corollary 3.1, we have that the graph $G = (V, E)$ does not have any popular matching iff the graph G includes one of the bad subgraphs of types G_1, G_2, and G_3. To show the theorem, it suffices to consider the case where a random instance of the 2-weighted matching problems $G = (V, E)$ includes the simplest bad subgraphs of type G_1 as shown in Figure 2.

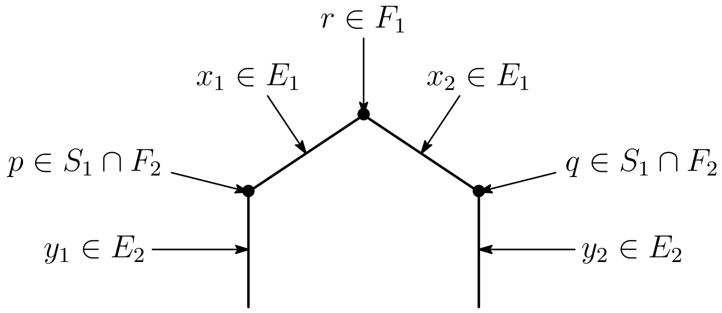


Figure 2: The Simplest "Bad" Subgraphs of Type G_1

For any $x_1, x_2 \in A_1$ and any $y_1, y_2 \in A_2$ such that $x_1 < x_2$ and $y_1 < y_2$, we define a random variable Z_{x_1,x_2,y_1,y_2} to be $Z_{x_1,x_2,y_1,y_2} = 1$ if the vertices x_1, x_2, y_1, and y_2 form the bad subgraph of type G_1 in Figure 2 and $Z_{x_1,x_2,y_1,y_2} = 0$ otherwise. For notational simplicity, we use $\vec{v}$ to denote (x_1, x_2, y_1, y_2) such that $x_1 < x_2$ and $y_1 < y_2$ for any $x_1, x_2 \in A_1$ and $y_1, y_2 \in A_2$. Let T be the set of all such $\vec{v}$'s and $N = |T|$. Since $n_1 = \delta n = |A_1|$ and $n_2 = (1 - \delta)n = |A_2|$, we have that

$$N = \binom{n_1}{2}\binom{n_2}{2} \geq \frac{\delta^2 n^2}{3} \cdot \frac{(1-\delta)^2 n^2}{3} = \frac{\delta^2(1-\delta)^2}{9} n^4. \tag{1}$$

Let $Z = \sum_{\vec{v} \in V} X_{\vec{v}}$. Then it follows from Chebyshev's Inequality [9] that

$$\Pr[Z = 0] \leq \Pr[|Z - \mathbf{E}[Z]| \geq \mathbf{E}[Z]]$$

$$= \Pr\left[|Z - \mathbf{E}[Z]| \geq \frac{\mathbf{E}[Z]}{\sigma_Z}\sigma_Z\right] \leq \frac{\sigma_Z^2}{\mathbf{E}^2[Z]} = \frac{\mathbf{Var}[Z]}{\mathbf{E}^2[Z]}. \tag{2}$$

To derive the lower bound for $\Pr[Z > 0]$, we estimate the upper bound for $\mathbf{Var}[Z]/\mathbf{E}^2[Z]$. We first consider $\mathbf{E}[Z]$. For each $\vec{v} \in V$, it is easy to see that

$$\begin{aligned} \Pr[Z_{\vec{v}} = 1] &\geq \frac{1}{m} \cdot \left(\frac{1}{m}\right)^2 = \frac{1}{m^3}; \\ \Pr[Z_{\vec{v}} = 1] &\leq \frac{1}{m} \cdot \left(\frac{1}{m - n_1}\right)^2 \leq \frac{1}{m} \cdot \left(\frac{1}{m - n}\right)^2 = \frac{c^2}{m^3}, \end{aligned} \tag{3}$$

where Inequality (3) follows from the assumption that $m - n_1 \geq m - n \geq m/c$ for some constant $c > 1$. Thus from the estimations for $\Pr[Z_{\vec{v}} = 1]$, it follows that

$$\mathbf{E}[Z] = \mathbf{E}\left[\sum_{\vec{v} \in V} Z_{\vec{v}}\right] = \sum_{\vec{v} \in V} \mathbf{E}[Z_{\vec{v}}] = \sum_{\vec{v} \in V} \Pr[Z_{\vec{v}} = 1] \geq \frac{N}{m^3}; \tag{4}$$

$$\mathbf{E}[Z] = \mathbf{E}\left[\sum_{\vec{v} \in V} Z_{\vec{v}}\right] = \sum_{\vec{v} \in V} \mathbf{E}[Z_{\vec{v}}] = \sum_{\vec{v} \in V} \Pr[Z_{\vec{v}} = 1] \leq \frac{c^2 N}{m^3}. \tag{5}$$

We then consider $\mathbf{Var}[Z]$. From the definition of $\mathbf{Var}[Z]$, it follows that

$$\begin{aligned} \mathbf{Var}[Z] &= \mathbf{E}\left[\left(\sum_{\vec{v} \in V} X_{\vec{v}}\right)^2\right] - \left(\mathbf{E}\left[\sum_{\vec{v} \in V} X_{\vec{v}}\right]\right)^2 \\ &= \mathbf{E}\left[\sum_{\vec{v} \in V} X_{\vec{v}}^2 + \sum_{\vec{v} \in V} \sum_{\vec{w} \in V - \{\vec{v}\}} X_{\vec{v}} X_{\vec{w}}\right] - \left(\mathbf{E}\left[\sum_{\vec{v} \in V} X_{\vec{v}}\right]\right)^2 \\ &= \mathbf{E}\left[\sum_{\vec{v} \in V} X_{\vec{v}}\right] - \left(\mathbf{E}\left[\sum_{\vec{v} \in V} X_{\vec{v}}\right]\right)^2 + \sum_{\vec{v} \in V} \sum_{\vec{w} \in V - \{\vec{v}\}} \mathbf{E}[X_{\vec{v}} X_{\vec{w}}] \\ &\leq \mathbf{E}[Z] - \mathbf{E}^2[Z] + \sum_{\vec{v} \in V} \sum_{\vec{w} \in V - \{\vec{v}\}} \mathbf{E}[X_{\vec{v}} X_{\vec{w}}]. \end{aligned} \tag{6}$$

In the following, we estimate the last term of Inequality (6). For each $\vec{v} = (x_1, x_2, y_1, y_2) \in V$, we say that $\vec{w} = (x_1', x_2', y_1', y_2') \in V$ is 2-common to $\vec{v}$ if $x_1 = x_1', x_2 = x_2'$; 1-common to $\vec{v}$ if $x_1 = x_1', x_2 \neq x_2'$ or $x_1 \neq x_1', x_2 = x_2'$; 0-common to $\vec{v}$ if $x_1 \neq x_1', x_2 \neq x_2'$. We note that $x_1 = x_2', x_2 = x_1'$ never occurs because of the assumption that $x_1 < x_2$ and $x_1' < x_2'$. We also notice that $x_1 = x_2', x_2 \neq x_1'$ or $x_1 \neq x_2', x_2 = x_1'$ never occurs, because if $x_1 = x_2', x_2 \neq x_1'$ or $x_1 \neq x_2', x_2 = x_1'$, then $F_1 \cap S_1 \neq \emptyset$ and this contradicts the fact that $F_1 \cap S_1 = \emptyset$. For each $\vec{v} \in V$, we use $V_2(\vec{v})$ to denote the set of $\vec{w} \in V - \{\vec{v}\}$ that is 2-common to $\vec{v}$; $V_1(\vec{v})$ to denote the set of $\vec{w} \in V - \{\vec{v}\}$ that is 1-common to $\vec{v}$; $V_0(\vec{v})$ to denote the set of $\vec{w} \in V - \{\vec{v}\}$ that is 0-common to $\vec{v}$. Then we have that

$$\begin{aligned} \sum_{\vec{v} \in V} \sum_{\vec{w} \in V_2(\vec{v})} \mathbf{E}[X_{\vec{v}} X_{\vec{w}}] &\leq \sum_{\vec{v} \in V} \frac{1}{m} \left(\frac{1}{m - n_1}\right)^4 \binom{n_2}{2} = \frac{1}{m} \left(\frac{1}{m - n_1}\right)^4 \binom{n_2}{2} N \\ &\leq \frac{1}{m} \left(\frac{1}{m - n}\right)^4 \binom{n_2}{2} N \leq \frac{c^4 (1 - \delta)^2 n^2 N}{2m^5}; \end{aligned} \tag{7}$$

$$\sum_{\vec{v} \in V} \sum_{\vec{w} \in V_1(\vec{v})} \mathbf{E}[X_{\vec{v}} X_{\vec{w}}] \leq \sum_{\vec{v} \in V} \frac{1}{m^2} \left(\frac{1}{m - n_1}\right)^4 2 \binom{n_1}{1} \binom{n_2}{2} = \frac{1}{m^2} \left(\frac{1}{m - n_1}\right)^4 2 \binom{n_1}{1} \binom{n_2}{2} N$$

$$
\begin{aligned}
&\leq \frac{1}{m^2}\left(\frac{1}{m-n}\right)^4 2\binom{n_1}{1}\binom{n_2}{2}N \leq \frac{c^4\delta(1-\delta)^2 n^3 N}{m^6}; && (8)\\
\sum_{\vec{v}\in V}\sum_{\vec{w}\in V_0(\vec{v})} \mathbf{E}\left[X_{\vec{v}}X_{\vec{w}}\right] &= \sum_{\vec{v}\in V}\sum_{\vec{w}\in V_0(\vec{v})} \Pr\left[X_{\vec{v}} = \wedge X_{\vec{w}} = 1\right] \\
&= \sum_{\vec{v}\in V} \Pr\left[X_{\vec{v}} = 1\right] \sum_{\vec{w}\in V_0(\vec{v})} \Pr\left[X_{\vec{w}} = 1 : X_{\vec{v}} = 1\right] \\
&= \sum_{\vec{v}\in V} \Pr\left[X_{\vec{v}} = 1\right] \sum_{\vec{w}\in V_0(\vec{v})} \Pr\left[X_{\vec{w}} = 1\right] \leq \mathbf{E}^2[Z]. && (9)
\end{aligned}
$$

Thus from Inequalities (5), (6), (7), (8), and (9), it follows that

$$
\begin{aligned}
\mathbf{Var}[Z] &\leq \mathbf{E}[Z] - \mathbf{E}^2[Z] + \sum_{\vec{v}\in V}\sum_{\vec{w}\in V-\{\vec{v}\}} \mathbf{E}\left[X_{\vec{v}}X_{\vec{w}}\right] \\
&= \mathbf{E}[Z] - \mathbf{E}^2[Z] + \sum_{\vec{v}\in V}\sum_{\vec{w}\in V_2(\vec{v})} \mathbf{E}\left[X_{\vec{v}}X_{\vec{w}}\right] + \sum_{\vec{v}\in V}\sum_{\vec{w}\in V_1(\vec{v})} \mathbf{E}\left[X_{\vec{v}}X_{\vec{w}}\right] + \sum_{\vec{v}\in V}\sum_{\vec{w}\in V_0(\vec{v})} \mathbf{E}\left[X_{\vec{v}}X_{\vec{w}}\right] \\
&\leq \mathbf{E}[Z] + \frac{c^4(1-\delta)^2 n^2 N}{2m^5} + \frac{c^4\delta(1-\delta)^2 n^3 N}{m^6} \\
&\leq \frac{c^2 N}{m^3} + \frac{c^4(1-\delta)^2 n^2 N}{2m^5} + \frac{c^4\delta(1-\delta)^2 n^3 N}{m^6} \\
&= \frac{c^2 N}{m^3}\left\{1 + \frac{c^2(1-\delta)^2 n^2}{2m^2} + \frac{c^2\delta(1-\delta)^2 n^3}{m^3}\right\} \\
&= \frac{c^2 N}{m^3}\left\{1 + \frac{c^2(1-\delta)^2}{2\beta^2} + \frac{c^2\delta(1-\delta)^2}{\beta^3}\right\} \leq \frac{dN}{m^3}, && (10)
\end{aligned}
$$

where d in Inequality (10) is a constant determined by the constants $0 < \delta < 1$ and $c > 1$. Then from Inequalities (1), (2), (4), and (10), we finally have that

$$
\Pr\left[Z = 0\right] \leq \frac{\mathbf{Var}[Z]}{\mathbf{E}^2[Z]} \leq \frac{dN}{m^3} \cdot \frac{m^6}{N^2} = \frac{dm^3}{N} \leq \frac{9d\beta^3 n^3}{\delta^2(1-\delta)^2 n^4} = \frac{9d}{\delta^2(1-\delta)^2} \cdot \frac{\beta^3}{n},
$$

which implies that $\Pr[Z = 0] = o(1)$ for any β such that $\beta/n^{1/3} = o(1)$. So it follows that if $\beta/n^{1/3} = o(1)$, then $\Pr[Z > 0] = 1 - o(1)$, i.e., a random instance of the 2-weighted matching problems does not have a 2-weighted popular matching with probability $1 - o(1)$. ■

5 Upper Bounds for 2-Weighted Matching Problems

As we have shown in Theorem 4.1, a random instance of the 2-weighted matching problems have no constant threshold of β to admit 2-weighted popular matchings with high probability. In this section, we derive upper bounds for the threshold of β such that a random instance of the 2-weighted matching problems have 2-weighted popular matching with high probability. The following lemma plays a crucial role to derive upper bounds for the threshold of β.

Lemma 5.1: *For any $\beta = 1/o(1)$, a random instance of the 2-weighted matching problems includes a cycles as a subgraph with probability $o(1)$.*

Proof: For any $\ell \geq 2$, we use C_ℓ to denote a cycle with ℓ vertices and ℓ edges and let E_ℓ^{cyc} be the event that G includes C_ℓ. Then from the assumption that $m = \beta n$, it follows that

$$
\Pr\left[G \text{ includes a cycle}\right] = \Pr\left[\bigcup_{\ell\geq 2} E_\ell\right] \leq \sum_{\ell\geq 2} \Pr[E_\ell]
$$

$$
\begin{aligned}
&\leq \sum_{\ell \geq 2} \left\{ \frac{1}{2\ell} \cdot \ell! \cdot \binom{m}{\ell} \cdot \ell! \cdot \binom{n}{\ell} \cdot \left(\frac{1}{m-n} \right)^{2\ell} \right\} \\
&\leq \sum_{\ell \geq 2} \left\{ \frac{1}{2\ell} \cdot m^{\ell} \cdot n^{\ell} \cdot \left(\frac{1}{\beta - 1} \right)^{2\ell} \cdot \frac{1}{n^{2\ell}} \right\} \\
&= \sum_{\ell \geq 2} \frac{1}{2\ell} \cdot \left\{ \frac{\beta}{(\beta-1)^2} \right\}^{\ell} \leq \sum_{\ell \geq 2} \left\{ \frac{\beta}{(\beta-1)^2} \right\}^{\ell}.
\end{aligned}
$$

For any $\beta \geq 4$, it is obvious that $\beta/(\beta-1)^2 \leq 2/\beta$ and $1/(\beta-2) \leq 2/\beta$. Then we have that

$$
\Pr\left[G \text{ includes cycles}\right] \leq \sum_{\ell \geq 2} \left\{ \frac{\beta}{(\beta-1)^2} \right\}^{\ell} \leq \sum_{\ell \geq 2} \left(\frac{2}{\beta} \right)^{\ell} = \frac{4}{\beta(\beta-2)} \leq \frac{8}{\beta^2}.
$$

From the assumption that $\beta = 1/o(1)$, it follows that a random instance of the graphs $G = (V, E)$ include cycles as subgraphs with probability $o(1)$. ■

Theorem 5.1: *Let $m = \beta n$. If $n^{1/3}/\beta = o(1)$, then a random instance of the 2-weighted matching problems has a 2-weighted popular matching with probability $1 - o(1)$.*

Proof: From Lemma 5.1 and the assumption that $n^{1/3}/\beta = o(1)$, we can assume that a random instance of 2-weighted matching problems $G = (V, E)$ includes bad subgraphs of type G_2 or G_3 in Figure 1 with probability $o(1)$. In the rest of the proof, we estimate the probability that a random instance of the graphs $G = (V, E)$ includes a bad subgraph of type G_1 in Figure 1.

For any $\ell \geq 4$, we use P_ℓ to denote a path of type G_1 with $\ell+1$ vertices and ℓ edges and let E_ℓ^{path} be the event that G includes P_ℓ. Then from the assumption that $m = \beta n$, it follows that

$$
\begin{aligned}
&\Pr\left[G \text{ includes a bad subgraph of type } G_1\right] = \Pr\left[\bigcup_{\ell \geq 4} E_\ell^{\text{path}} \right] \\
&\leq \sum_{\ell \geq 4} \Pr\left[E_\ell^{\text{path}}\right] \leq \sum_{\ell \geq 4} \left\{ \frac{1}{(m-n)^{2\ell}} \cdot (\ell+1)! \cdot \binom{m}{\ell+1} \cdot \ell! \cdot \binom{n}{\ell} \right\} \\
&\leq \sum_{\ell \geq 4} \frac{\beta^{\ell+1}}{(\beta-1)^{2\ell}} \cdot n = \frac{\beta^5}{(\beta-1)^8} \cdot n \cdot \sum_{h \geq 0} \left\{ \frac{\beta}{(\beta-1)^2} \right\}^{h}.
\end{aligned}
$$

For any $\beta \geq 4$, it is obvious that $\beta/(\beta-1)^2 \leq 2/\beta$ and $\beta/(\beta-2) \leq 2$. Then we have that

$$
\begin{aligned}
&\Pr\left[G \text{ includes a bad subgraph of type } G_1\right] \\
&\leq \frac{\beta^5}{(\beta-1)^8} \cdot n \cdot \sum_{h \geq 0} \left\{ \frac{\beta}{(\beta-1)^2} \right\}^{h} \leq n \cdot \beta \cdot \left(\frac{2}{\beta} \right)^4 \cdot \sum_{h \geq 0} \left(\frac{2}{\beta} \right)^h \\
&\leq n \cdot \frac{16}{\beta^3} \cdot \frac{\beta}{\beta-2} \leq n \cdot \frac{32}{\beta^3}.
\end{aligned}
$$

From the assumption that $n^{1/3}/\beta = o(1)$, it follows that a random instance of the graphs $G = (V, E)$ includes a bad subgraph of type G_1 with probability $o(1)$. Thus from Lemma 5.1 and Corollary 3.1, we have that if $n^{1/3}/\beta = o(1)$, then a random instance of the 2-weighted matching problems has a 2-weighted popular matching with probability $1 - o(1)$. ■

6 Concluding Remarks

In this paper, we have analyzed the 2-weighted matching problems, and have shown that for any β such that $m = \beta n$, (Theorem 4.1) if $\beta/n^{1/3} = o(1)$, then a random instance of the 2-weighted matching problems does not have a 2-weighted popular matching with probability $1-o(1)$; and (Theorem 5.1) if $n^{1/3}/\beta = o(1)$, then a random instance of the 2-weighted matching problems has a 2-weighted popular matching with probability $1-o(1)$. These results imply that there exists a threshold $\beta = O(n^{1/3})$ to admit 2-weighted popular matchings with high probability, which is quite different to the case for the (1-weighted) matching problems shown by Mahdian [7]. In a way similar to the proof of Theorem 4.1, we can show the following lower bounds of β for any integer $k \geq 2$, i.e.,

Theorem 6.1: *Let $m = \beta n$. If $\beta/n^{1/3} = o(1)$, then a random instance of the k-weighted matching problems does not have a k-weighted popular matching with probability $1-o(1)$.*

Then one of the interesting problems would be the upper bounds of β for any integer $k \geq 2$, i.e.,

- For any integer $k \geq 2$, show upper bounds of β for which a random instance of the k-weighted matching problems has a k-weighted popular matching with probability $1-o(1)$.

References

[1] D.J. Abraham, N. Chen, V. Kumar, and V. Mirrokni. Assignment Problems in Rental Markets. In *Proc. Internet and Network Economics*, Lecture Notes in Computer Science 4286, pp.198-213, 2006.

[2] D.J. Abraham, R.W. Irving, T. Kavitha, and K. Mehlhorn. Popular Matchings. In *Proc. of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms*, pp.424-432, 2005.

[3] D.J. Abraham and T. Kavitha. Dynamic Matching Markets and Voting Paths. In *Proc. of the 10th Scandinavian Workshop on Algorithm Theory*, Lecture Notes in Computer Science 4059, pp.65-76, 2006.

[4] N. Alon and J. Spencer. *The Probabilistic Method*. John Wiley & Sons, 2000.

[5] B. Bollobás. *Random Graphs*. Cambridge University Press, 2001.

[6] P. Gardenfors. Match Making: Assignment Based on Bilateral Preferences. *Behaviourial Sciences*, 20:166-173, 1975.

[7] M. Mahdian. Random Popular Matchings. In *Proc. of the 7th ACM Conference on Electronic Commerce*, pp. 238-242, 2006.

[8] J. Mestre. Weighted Popular Matchings. In *Proc. of the 33rd International Colloquium on Automata, Languages, and Programming (Part I)*, Lecture Notes in Computer Science 4051, pp.715-726, 2006.

[9] R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge University Press, 1995.