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A New lattice construction for partial key exposure attack for RSA

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Abstract

In this paper we present a new lattice construction for a lattice based partial key exposure attack for the RSA cryptography. We consider the situation that the RSA secret key d is small and a sufficient amount of the LSBs (least significant bits) of d are known by the attacker. We show that our lattice construction is theoretically more efficient than known attacks proposed in [1, 6].

Keywords: RSA, cryptanalysis, partial key exposure attack, lattice basis reduction, Coppersmith's method

1 Introduction

In this paper we present a new lattice construction for a lattice based partial key exposure attack for the RSA cryptography in the situation that the secret key d is small and its LSBs (least significant bits) are exposed.

Boneh and Durfee [1] proposed the lattice based attack for the RSA cryptography. Its basic idea is to reduce the RSA key finding problem to problems of finding small roots of modular equations such as $f(x_1, \ldots, x_n) \equiv 0 \pmod{W}$, which are solved by the Coppersmith technique [5], the technique that solves a given modular equation by converting it to an ordinary equation by using a lattice basis reduction algorithm such as the LLL algorithm [9]. Boneh and Durfee [1] showed that the secret key d can be obtained from a public key pair e and N in polynomial time in $\log N$ when $d < N^{0.292}$.

Since Boneh and Durfee's work, many of its variants have been proposed [3, 6]. Blömer and May [3] extended the technique for a partial key exposure attack, i.e., a problem of computing d from e, N and some partial information on d. This approach has been further extended by Ernst et al. [6] for several partial key exposure situations. In this paper we consider one of those situations where the secret key d is small and some LSBs of d are given (besides e and N), and we show some improvement over the algorithm by Ernst et al., thereby solving an open problem raised in [6].

In order to state our improvement we need some notations; see the next section for the precise definition. Let (e, N) be an RSA public key pair and let d be its corresponding secret key. Here as usual we use $\ell_N =$ (the bit-length of N) as a security parameter. We consider the situation that $\ell_d =$ (the bit length of d) is relatively small compared with ℓ_N and some ℓ_0 least significant bits of d are known. Let $\beta = \ell_d/\ell_N$ and $\delta = (\ell_d - \ell_0)/\ell_N$; that is, they are respectively the ratios of the bit-length of d and its unknown part. Now the asymptotic performance of the algorithms in [1, 6] can be summarized in Figure 1. (This figure is a rough image, not accurate.)

The algorithm of [1] works asymptotically when the parameters take values in the left of a vertical line labelled "$\beta = 0.292$". That is, it obtains the secret key for

$$\beta < 1 - \frac{1}{\sqrt{2}} = 0.292... \text{ and any } \delta. \tag{1}$$

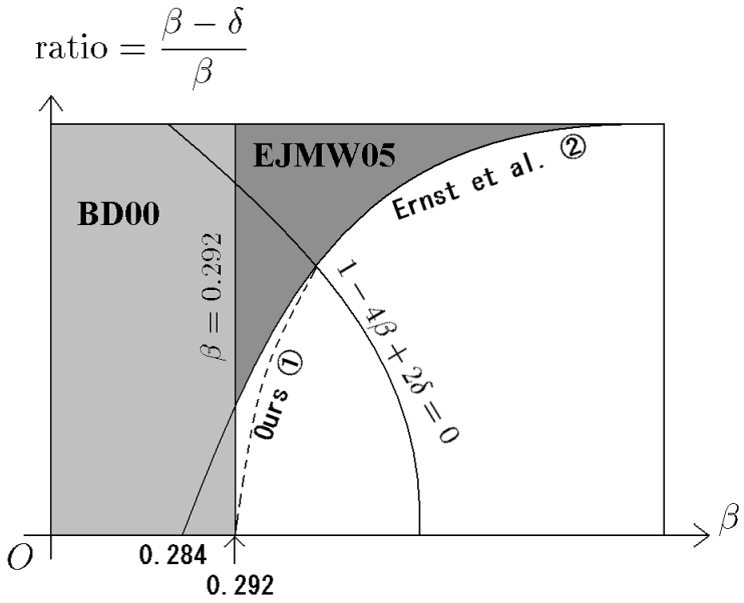


Figure 1: Our recoverable range

The limit of Boneh and Durfee (1) and that of Ernst et al. (2) are corresponding to the left/above of the line $\beta = 0.292$ and the line `Ernst et al.` ② respectively. Our new improvement area (3) is the left side of the dashed line `Ours`① in the area left of line $1 - 4\beta + 2\delta = 0$.

The algorithm of [6] works when

$$\delta < \frac{5}{6} - \frac{1}{3}\sqrt{1+6\beta}. \tag{2}$$

That is it works when β and δ take values in the left/above of a curve labelled `Ernst et al.` ②. As shown in the Figure 1, the algorithm of [6] improves the solvable parameter range when δ is small; it has been however left open [6] to develop an algorithm that has a better solvable parameter range than both [1] and [6].

In this paper we propose an algorithm that can work asymptotically when the parameters take values in the left/above of the dashed line of Figure 1. More precisely, it works when

$$1 - 4\beta + 2\delta > 0 \text{ if } 2\sqrt{2(1-2\beta)(\beta-\delta)}(\delta-\beta) - 2\beta^2 + 3\beta + \delta - 1 < 0 \tag{3}$$

and

$$1 - 4\beta + 2\delta \leq 0 \text{ if } \delta < \frac{5}{6} - \frac{1}{3}\sqrt{1+6\beta}. \tag{4}$$

Note that the range (3) is our new improvement which is the left/above of the dashed line `Ours`① in Figure 1, while the range (4) is already given by [6]. Also as shown in Section 5, compared with the construction of [6], our lattice construction produces a much smaller instance for a lattice basis reduction algorithm, which improves the total running time significantly.

This paper is organized as follows. In Section 2, we introduce some notations and lemmas about the lattice based partial key exposure attack. Section 3 provides the overview of the lattice based partial key exposure attack. In Section 4 we describe the construction and the performance of our lattice. Section 5 provides the results of our computer experiments. The analysis of Section 4 is explained in the Appendix section.

2 Preliminaries

We introduce some notations and state some known facts used in the following discussions. Then we review some key technical lemmas used in the lattice based attack.

We use standard RSA notations throughout this paper. A given RSA instance is defined by p, q, e, and d, where p and q are large primes, e is a public key, and d is a secret key. Let $N = p \times q$, and let $\varphi(N)$ be the Euler's function; here we may simplify assume that $\varphi(N) = (p-1)(q-1)$. The key relation is

$$ed \equiv 1 \pmod{\varphi(N)}. \tag{5}$$

The partial key exposure attack is to compute the secret key d from partial information on d, and the public key (e, N). In this paper, we consider the situation that some LSBs of d are exposed, that is, recovering d from LSBs of d (together with e and N). We use d_0 to denote the exposed part and $\tilde{d}$ to denote the non-exposed part. That is, we assume that

$$d = \tilde{d} \cdot M + d_0, \tag{6}$$

where $M = 2^k$ and $k = \lg(d_0)$ [1]. We will use M for denoting this number throughout this paper. Define $\beta = \log_N d$ and $\delta = \log_N \tilde{d}$. That is, β and δ are the rough ratios of the bit-length of d and $\tilde{d}$ relative to that of N respectively.

In our algorithm, we need to solve some modular equations such as $f(x, y) \equiv 0 \pmod{W}$ for some polynomials $f(x, y)$. Furthermore, we want to obtain solutions in a certain range. In general, this task is not easy. However there are some cases where we may be able to use the standard numerical method for solving these modular equations. The Howgrave-Graham lemma provides us with one of such cases.

In order to state the Howgrave-Graham Lemma, we introduce the following norm defined by any given non-negative integers X and Y, which we call in this paper "*XY-norm.*"

Definition 1. XY-norm Let X and Y be natural numbers and $f(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ be a polynomial with integral coefficient.

We define the *XY-norm* of $f(x, y)$ by

$$||f(x, y)||_{XY} \stackrel{\text{def}}{=} \sqrt{\sum_{i,j} a_{i,j}^2 X^{2i} Y^{2j}}.$$

Lemma 1. (Howgrave-Graham [7]) For any positive integers X, Y and W, let $f(x, y)$ be a bivariate polynomial consisting w terms with integral coefficient such that the following holds

$$||f(x, y)||_{XY} < \frac{W}{\sqrt{w}}.$$

Then we have

$$f(x, y) \equiv 0 \pmod{W} \Leftrightarrow f(x, y) = 0$$

within the range of $|x| < X$ and $|y| < Y$.

Note that $f(x, y) = 0$ clearly implies $f(x, y) \equiv 0 \pmod{W}$. What is important is its converse. This lemma guarantees that the solution of $f(x, y) \equiv 0 \pmod{W}$ in the target range can be found (if exists) from the solutions of $f(x, y) = 0$, which can be obtained by the standard numerical method.

[1] We use $\lg(x)$ to denote the length of the binary representation of x.

In order to use the above lemma, we need to obtain polynomials with a small XY-norm. The key idea of the lattice based attack is to formulate this task as the shortest vector problem and use an approximate solution computed by a polynomial time lattice basis reduction algorithm for the shortest vector problem.

We introduce some definitions and some lemmas about the lattice. Consider linearly independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^{\tilde{n}}$, then the lattice with basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ is defined by

$$L(\mathbf{b}_1, \ldots, \mathbf{b}_n) = \left\{ \sum_{i=1}^{n} a_i \mathbf{b}_i \,\middle|\, a_i \in \mathbb{Z} \text{ for } i = 1, \ldots, n \right\}. \tag{7}$$

That is, the lattice is an integral linear combination of its basis vectors. We denote by n a number of vectors, which is usually called *lattice dimension*, and denote by $\tilde{n}$ a number of component of vector in basis, which we call *lattice component size*. Note that the lattice is a additive subgroup of $\mathbb{R}^{\tilde{n}}$.

The shortest vector problem, for given basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$, is to find a vector $\mathbf{v}$ such that

- $\mathbf{v} \in L(\mathbf{b}_1, \ldots, \mathbf{b}_n) \setminus \{\mathbf{0}\}$ and
- $|\mathbf{v}| \leq |\mathbf{v}'|$ for $^{\forall}\mathbf{v}' \in L(\mathbf{b}_1, \ldots, \mathbf{b}_n) \setminus \{\mathbf{0}\}$.

That is, this problem is to find a non-zero vector having the minimum length in $L(\mathbf{b}_1, \ldots, \mathbf{b}_n)$. In order to obtain polynomials with small XY-norms, we need to compute short vectors as an approximate solution of this problem. We will use a polynomial time algorithm, named LLL, proposed in [9]. Some improvements have been proposed [11, 12], but as shown later, these improvements are not essential for our application.

The approximation ratio of the LLL algorithm is exponential, it is however enough for our propose. The following theorem guarantees an upper bound of the length of a vector computed by the LLL algorithm. The LLL algorithm computes a special basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$, named reduced basis, from given basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Our interest is short vectors in the reduced basis in the following theorem.

Theorem 1. [1, Fact 3.3] Let $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be a given linearly independent basis. Then we can find linearly independent lattice vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that

$$\begin{aligned} |\mathbf{v}_1| &\leq 2^{(n-1)/4} |\det(L)|^{1/n}, \text{ and} \\ |\mathbf{v}_2| &\leq 2^{n/2} |\det(L)|^{1/(n-1)}. \end{aligned} \tag{8}$$

Here, L is the lattice with basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$, and $\det(L)$ is the determinant of the lattice defined by

$$\det(L) = \prod_{i=1}^{n} |\mathbf{b}_i^*|, \text{ where} \tag{9}$$

$\mathbf{b}_1^*, \ldots, \mathbf{b}_n^*$ is the Gram-Schmidt orthogonal basis of given basis.

We will use (9) to evaluate the determinant of our lattice in the later section.

Note that the shortest vector problem is defined on vectors, while our targets are polynomials. Thus we consider some way to map polynomials to vectors. We consider some natural and simple mapping; For example, a polynomial $f(x, y) = -3x^3 + 4x^2y - 2xy^2 + 7xy^3$ is mapped to a vector $(-3X^3, 4X^2Y, -2XY^2, 7XY^3)$ by some natural numbers X and Y. To state this correspondence formally, we first need to fix some linear ordering on pairs (i, j) of nonnegative integers. With respect to this ordering let $(i(t),j(t))$ denote the t-th pair. Then our correspondence between polynomials and vectors is define as follows.

Definition 2. Polynomials ↔ vectors Let J be a sequence of pairs of nonnegative integers, where we assume some linear order on J, let it be fixed, and let $\tilde{n}$ denote $|J|$, the length of the sequence. We also fix some positive integers X and Y. W.r.t. these X and Y, for any $f(x,y) = \sum_{1\le t\le \tilde{n}} a_{i(t),j(t)} x^{i(t)} y^{j(t)}$, the following vector $\mathbf{b}$ is the vectorisation of $f(x,y)$ with parameter X and Y, and it is denoted by $\mathcal{V}_J(f;X,Y)$.

On the other hand, for any $\mathbf{b}$ of size $\tilde{n}$, a polynomial $f(x,y)$ defined from $\mathbf{b}$ by interpreting it as below is called the functionalisation of $\mathbf{b}$ and it is denoted by $\mathcal{F}_J(\mathbf{b};X,Y)$.

$$\begin{array}{lcccccccccc} f(x,y) & = & a_{i(1),j(1)}x^{i(1)}y^{j(1)} & + & a_{i(2),j(2)}x^{i(2)}y^{j(2)} & + & \cdots & + & a_{i(|J|),j(|J|)}x^{i(|J|)}y^{j(|J|)} & & \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ \mathbf{b} & = (& a_{i(1),j(1)}X^{i(1)}Y^{j(1)} & , & a_{i(2),j(2)}X^{i(2)}Y^{j(2)} & , & \cdots & , & a_{i(|J|),j(|J|)}X^{i(|J|)}Y^{j(|J|)} &). \end{array}$$

Remark When J is clear from the context, we often omit J and write as $\mathcal{V}(f;X,Y)$ and $\mathcal{F}(\mathbf{b};X,Y)$. Then from the definition, the following relationships are immediate.

$$\begin{array}{l} ||f(x,y)||_{XY} = |\mathcal{V}(f;X,Y)|, \text{ and} \\ ||\mathcal{F}(\mathbf{b};X,Y)||_{XY} = |\mathbf{b}|. \end{array} \tag{10}$$

3 Overview of the Partial key Exposure Attack

We give an overview of the lattice based partial key exposure attack in the situation that LSBs of d are exposed. The goal of the attack is to compute a secret key d from d_0, least significant bits of d, and the public key pair. The lattice based attack achieves this goal by using a lattice reduction algorithm and Howgrave-Graham's lemma. It is said [6] (and some papers) that this attack is effective if

(i) d, and unknown part of d are short,
(ii) e and N are of similar bit length, and
(iii) p and q are of similar bit length.

In order to be precede, we consider in this paper, the following conditions.

(a) $\delta = \log_N \tilde{d}$ is smaller than 0.5
(b) $\lg(e) = \lg(N)$, and
(c) $\lg(p) = \lg(q)$

In the following, we assume all parameters satisfy these conditions. More precisely, we will use the following inequalities in the later.

$$e < \varphi(N) \text{ and } p + q < 3\sqrt{N}. \tag{11}$$

Our objective is to compute d from a public key pair (e,N) and d_0. As explained in Introduction, the key relation is the modular equation (5), from which it is easy to derive

$$ed = 1 - x\varphi(N) = 1 - x(y+N)$$

for some $x, y \in \mathbb{Z}$. Also by using (6), we can deduce from the above that $e(\tilde{d}\cdot M + d_0) = 1 - x(y+N)$ and hence we have

$$x(N+y) + (ed_0 - 1) \equiv 0 \pmod{eM}. \tag{12}$$

We show here that it is relatively easy to enumerate all solutions (x, y) of (12). First note that a solution (x, y) exists if for integer y,

$$\gcd\left(\frac{N+y}{g}, \frac{eM}{g}\right) = 1 \text{ where } g = \gcd(N+y, eM, ed_0 - 1).$$

In fact in this case, we can compute x by

$$x = \left(\frac{1-ed_0}{g}\right) \cdot \left(\left(\frac{N+y}{g}\right)^{-1} \bmod \frac{eM}{g}\right)$$

But clearly what we need is some specific solution of (12). Among solutions (x, y) of (12), we say that (x_0, y_0) is *useful* if it indeed satisfies the following equation, from which we can recover the secret key.

$$d = \frac{1 - x_0(N + y_0)}{e} \tag{13}$$

Thus, our task is not computing *some* solutions (x, y), but computing this useful solution among (x, y) satisfying (12). Below we use (x_0, y_0) to denote this useful solution. Let us consider a size of the useful solution (x_0, y_0). We have the following upper bounds. Here, we use (11) and the fact that $\varphi(N) = N + y_0$ if (x_0, y_0) is the useful solution.

$$\begin{aligned} |x_0| &= \left|\frac{ed-1}{N+y_0}\right| < \frac{ed}{\varphi(N)} < d = N^{\beta}, \text{ and} \\ |y_0| &= |N - \varphi(N)| = p + q - 1 < 3N^{0.5}. \end{aligned} \tag{14}$$

Now let $X = \lceil N^{\beta} \rceil$ and $Y = \lceil 3N^{0.5} \rceil$. Then, the useful solution (x_0, y_0) is a solution of (12) satisfying $|x_0| < X$ and $|y_0| < Y$.

Conversely, we consider some heuristic condition on δ for a solution satisfying $|x| < X$ and $|y| < Y$ is useful. We assume that solutions of (12) are random numbers on $\{0, \ldots, eM-1\}^2$. Since the number of solution pairs of (12) is smaller than eM, we expect the number of solutions satisfy $|x| < X$ and $|y| < Y$ is smaller than

$$eM \cdot \frac{4XY}{(eM)^2} = \frac{4XY}{eM} \approx \frac{4 \cdot N^{\beta} \cdot 3N^{0.5}}{N \cdot N^{\beta-\delta}} \approx N^{\delta-0.5}.$$

Thus, if this value is smaller than 1, we may expect a solution within the range $|X| < N^{\beta}$ and $|y| < N^{0.5}$ is only one, which is the useful solution guaranteed by (14). From this observation we propose a condition $\delta < 0.5$ and the following heuristic assumption.

Heuristic Assumption Consider the case $\delta < 0.5$. Then, there is only useful solution (x_0, y_0) within the range of $|x_0| < N^{\beta}, |y_0| < 3N^{0.5}$ of the following equation.

$$x(N + y) + (ed_0 - 1) \equiv 0 \pmod{eM}$$

Furthermore we can recover the secret key d by (13)[2]. □

Remark. This assumption shows we can obtain the secret key by the exhaustive search when $\delta < 0.5$.

[2]We can compute the factoring of N by $\varphi(N) = N + y_0$.

1. Based on $f_{\mathrm{main}}(x,y)$ (and $f_{\mathrm{M}}(x,y)$), define a certain family of polynomials $h_1(x,y),\ldots,h_n(x,y)$ such that

$$f_{\mathrm{main}}(x,y) \equiv 0 \pmod{eM} \Rightarrow h_c(x,y) \equiv 0 \pmod{(eM)^m} \text{ for } c = 1,\ldots,n.$$

   (Here m and n are some algorithm parameter defined later.)
2. Set $X = \lceil N^{\beta} \rceil$ and $Y = \lceil 3N^{0.5} \rceil$. Consider vectors by $\mathbf{b}_c = \mathcal{V}_J(h_c; X, Y)$ for $c = 1,\ldots,n$. Here, a sequence J is a set of appropriately ordered integer pairs (i,j) such that a monomial $x^i y^j$ appears in $h_c(x,y)$.
3. For $\mathbf{b}_1,\ldots,\mathbf{b}_n$, compute reduced basis by a lattice basis reduction algorithm. We denote by $\mathbf{v}_1,\ldots,\mathbf{v}_n$ this reduced basis.
4. Define $g_1(x,y)$ and $g_2(x,y)$ by $g_a(x,y) = \mathcal{F}_J(\mathbf{v}_a; X, Y)$ respectively. Obtain solutions of $g_1(x,y) = g_2(x,y) = 0$ numerically. Then from these solutions, compute d as given by (13).

Figure 2: Outline of the lattice based attack

For our discussion, let us define the following two functions [3]

$$\begin{aligned} f_{\mathrm{main}}(x,y) &\stackrel{\mathrm{def}}{=} x(N+y) + (ed_0 - 1) \\ &= (ed_0 - 1) + Nx + xy, \text{ and} \end{aligned} \tag{15}$$

$$f_{\mathrm{M}}(x,y) \stackrel{\mathrm{def}}{=} M(-1 + x(N+y)). \tag{16}$$

$f_{\mathrm{main}}(x,y)$ is the left-hand side of equation (12). The motivation of $f_{\mathrm{M}}(x,y)$ will be explained later; here we only point out that $f_{\mathrm{M}}(x_0,y_0) \equiv 0 \pmod{eM}$, where (x_0,y_0) is a useful solution.

Now we summarise the above explanation. Our technical goal is to obtain the useful (x_0,y_0) satisfying (12), in other words, a pair satisfying *both* $f_{\mathrm{main}}(x_0,y_0) \equiv 0 \pmod{eM}$ *and* $|x_0| < N^{\beta}$ and $|y_0| < 3N^{0.5}$. For achieving this technical goal by solving the mod equation, we make use of Howgrave-Graham Lemma, and for this purpose, we modify $f_{\mathrm{main}}(x,y)$ to some family of functions with small XY-norm. The task of defining these polynomials is formulated as the shortest vector problem, and known polynomial time algorithm such as the LLL algorithm is used. This is the rough sketch of the lattice based attack. The outline of our algorithm is stated as Figure 2.

Some remarks may be necessary. Note first that m and n are algorithmic parameters; m is chosen appropriately and n is the number of polynomials $h_a(x,y)$ that is also determined appropriately based on m.

Secondly note that we have the following relation between these polynomials.

$$\begin{aligned} &f_{\mathrm{main}}(x,y) \equiv 0 \pmod{eM} \;\Rightarrow\; h_c(x,y) \equiv 0 \pmod{(eM)^m} \text{ for } c = 1,\ldots,n \\ &\Rightarrow\; g_a(x,y) \equiv 0 \pmod{(eM)^m} \text{ for } a = 1,2 \;\Leftrightarrow\; g_a(x,y) = 0 \text{ for } a = 1,2. \end{aligned} \tag{17}$$

The key point of (17) is the relation $g_a(x,y) \equiv 0 \pmod{(eM)^m} \Leftrightarrow g_a(x,y) = 0$ for $a = 1,2$. This holds when $||g_a(x,y)||_{XY} = |\mathbf{v}_a|$ is smaller than $(eM)^m/\sqrt{w}$ from Howgrave-Graham's lemma. (Here w is the number of terms of each $g_a(x,y)$.) Then we have a relation

$$f_{\mathrm{main}}(x,y) \equiv 0 \pmod{eM} \Rightarrow g_a(x,y) = 0 \text{ for } a = 1,2.$$

[3] Ernst et al. [6] reduced the problem to the problem of finding small solution of an ordinal equation

$$f_{\mathrm{LSB}}(x,y,z) = eMx - Ny + yz + e\tilde{d} - 1 = 0.$$

Hence we can obtain the useful solution from computing all solutions of $g_1(x,y) = g_2(x,y) = 0$ by some numerical method if exists.

By (8) and (10), we have

$$\begin{aligned} ||g_1(x,y)||_{XY} &= |\mathbf{v}_1| \leq 2^{(n-1)/4} \det(L)^{1/n}, \text{ and} \\ ||g_2(x,y)||_{XY} &= |\mathbf{v}_2| \leq 2^{n/2} \det(L)^{1/(n-1)}. \end{aligned}$$

Since the second bound is larger, we obtain the following sufficient condition for using Howgrave-Graham's lemma:

$$2^{n/2} \det(L)^{1/(n-1)} < \frac{(eM)^m}{\sqrt{w}}. \tag{18}$$

We modify (18) to a more simple approximate bound. First we note that the factor $2^{n/2}$ and $\sqrt{w}$ $(=\Theta(\sqrt{n}))$ are negligible. Then we have a condition $\det(L)^{1/(n-1)} < (eM)^m$ instead of (18). Next we neglect the difference between $\det(L)^{1/(n-1)}$ and $\det(L)^{1/n}$. Thus, instead of using the condition (18), we will use the following one.

$$\det(L)^{1/n} < (eM)^m \tag{19}$$

For example, when attacking 1024-bit RSA instances by using lattices having dimension $n \approx 100$, we may have $(eM)^m > 2^{10000}$ whereas $2^{n/2} \cdot \sqrt{w} \approx 2^{50}$, and the ratio $\det(L)^{1/(n-1)} / \det(L)^{1/n} = \det(L)^{1/n(n-1)} \approx 2^{100}$.

Now our goal is to construct a lattice satisfying (19), and we will show in the next section that it is possible if β and δ satisfies the condition (3) given in Introduction.

4 Our Construction

In this section, we explain our construction and a main result. A difference between former algorithm and ours is only lattice construction satisfying (19). We will give its analysis in the Appendix.

Let β and δ be assumed bounds defined above, m be an algorithm parameter introduced in the above outline, τ be a parameter used to optimise the bounds by β and δ. We fix them throughout this section. We introduce index series $I_a(m,\tau,\beta,\delta)$ for constructing our lattice $L(m,\tau,\beta,\delta)$.

Definition 3. We define our sequence $I_1(m,\tau,\beta,\delta), I_2(m,\tau,\beta,\delta)$ and $I_3(m,\tau,\beta,\delta)$ (In short, I_1, I_2 and I_3 respectively). Here we set

$$\begin{aligned} I_1(m,\tau,\beta,\delta) &= \{(i,j) \in \mathbb{Z} \times \mathbb{Z} | 0 \leq i \leq m, 0 \leq j \leq i\}, \\ I_2(m,\tau,\beta,\delta) &= \{(i,j) \in \mathbb{Z} \times \mathbb{Z} | 0 \leq i \leq m, i < j \leq i + \tau m\} \text{ and} \\ I_3(m,\tau,\beta,\delta) &= \{(i,j) \in \mathbb{Z} \times \mathbb{Z} | 0 \leq i \leq m, j \leq 2(1-\beta)i\} \setminus (I_1 \cup I_2). \end{aligned}$$

We consider the order $\prec$ in I_1, I_2 and I_3 by the lexicographic order of (i,j) [4]. Then we define index sequence $I(m,\tau,\beta,\delta)$ by concatenating I_1, I_2 and I_3. That is, order of elements in I is defined as follows for $(i,j) \in I_k$ and $(i',j') \in I_{k'}$,

$$(i,j) \prec (i',j') \Leftrightarrow \begin{cases} k < k' \text{ or} \\ k = k' \text{ and } (i,j) \prec (i',j') \text{ in } I_k. \end{cases}$$

We define our polynomials $f_{i,j}(x,y)$ to construct our lattice (this is $h_c(x,y)$ in the outline).

[4] Notice that a symbol $\prec$ means R.H.S. is exactly larger than L.H.S., not equal. A symbol $\preceq$ means R.H.S. is equal to or larger than L.H.S.

Definition 4.

$$f_{i,j}(x,y) = \begin{cases} (eM)^{m-j}x^{i-j}(f_{\mathrm{main}}(x,y))^j & \text{for } (i,j) \in I_1 \\ (eM)^{m-i}y^{j-i}(f_{\mathrm{main}}(x,y))^i & \text{for } (i,j) \in I_2 \\ e^{m-i}y^{j-i}(f_{\mathrm{M}}(x,y))^i & \text{for } (i,j) \in I_3 \end{cases} \tag{20}$$

Then we define a sequence $J(m,\tau,\beta,\delta) = \{(i',j')|\text{a monomial } x^{i'}y^{j'} \text{ is appeared in some } f_{i,j}(x,y)\}$ where we assume the standard lexicographic order in $J(m,\tau,\beta,\delta)$. We simplify denote this by J.

It is clear that $f_{\mathrm{main}}(x,y) \equiv 0 \pmod{eM} \Rightarrow f_{i,j}(x,y) \equiv 0 \pmod{(eM)^m}$ for $(i,j) \in I$. The number of polynomials $|I|$ is just n in Figure 2. Note also that $|J| = \tilde{n}$, the number of components of each vector $\mathbf{b}_{i,j}$ is $O(|I|)$ since we can rewrite a set J by $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} | 0 \le i \le m, 0 \le j \le i + (1-2\beta)m\}$. Hence $|I|$ and $|J|$ has a same order $\Theta(m^2)$.

By using these polynomials and indecies, we define our lattice $L(m,\tau,\beta,\delta)$ by

$$L(m,\tau,\beta,\delta) = L(\mathbf{b}_{i_1,j_1},\ldots,\mathbf{b}_{i_n,j_n}).$$

Here $(i_1,j_1),\ldots,(i_n,j_n)$ are index sequence in I and $\mathbf{b}_{i_\ell,j_\ell} = \mathcal{V}_J(f_{i_\ell,j_\ell};X,Y)$, a vectorisation of $f_{i_\ell,j_\ell}(x,y)$ with parameters $X = \lceil N^\beta \rceil$ and $Y = \lceil 3N^{0.5} \rceil$. The lattice dimension and lattice component size of $L(m,\tau,\beta,\delta)$ are $|I|$ and $|J|$ respectively.

For evaluating the determinant of L, we will show

$$\begin{aligned} |\mathbf{b}^*_{i,j}| &= (eM)^{m-j}X^iY^j \text{ for } (i,j) \in I_1, \\ |\mathbf{b}^*_{i,j}| &= (eM)^{m-i}X^iY^j \text{ for } (i,j) \in I_2, \text{and} \\ e^{m-j}M^mX^iY^j \le |\mathbf{b}^*_{i,j}| &< 2e^{m-j}M^mX^iY^j \text{ for } (i,j) \in I_3. \end{aligned} \tag{21}$$

Here $\mathbf{b}^*_{i_1,j_1},\ldots,\mathbf{b}^*_{i_n,j_n}$ is a Gram-Schmidt orthogonal basis of given basis $\mathbf{b}_{i_1,j_1},\ldots,\mathbf{b}_{i_n,j_n}$. We give the proof of these bounds in the Appendix. (Lemma 3, Lemma 4 and Lemma 5). Now we assume that these bounds hold, and we introduce an "evaluator" for deciding suitable τ. The evaluator $\mathrm{eval}(i,j)$ for $\mathbf{b}^*_{i,j}$ is defined by

$$\mathrm{eval}(i,j) = \log_N \left(|\mathbf{b}^*_{i,j}|/(eM)^m \right). \tag{22}$$

We define the evaluator for any index sequence K by

$$\mathrm{eval}(K) = \sum_{(i,j)\in K} \mathrm{eval}(i,j). \tag{23}$$

Then we can state the condition (19) in terms of eval.

Lemma 2. For any index sequence I satisfying

$$\mathrm{eval}(I) < 0, \tag{24}$$

the condition (19) holds.

Proof. By (9) and the definition of the evaluator, we have

$$N^{\mathrm{eval}(I)} = N^{\sum_{(i,j)\in I}\mathrm{eval}(i,j)} = \prod_{(i,j)\in I}\left(\frac{|\mathbf{b}_{i,j}|}{(eM)^m}\right) = \det(L)/(eM)^{|I|\cdot m}.$$

Thus, (24) is equivalent to $\det(L) < (eM)^{|I|\cdot m}$. Which is indeed the condition (19).

□

Thus, we use $\mathrm{eval}(I) < 0$ as our (approximate) sufficient condition that the lattice based attack (under our construction of the lattice L) breaks a given RSA instance. Note that this condition is based on our heuristic assumption and it is only an approximate condition because of the approximation of (18) by (19); yet further approximation is used below for estimating eval. This means that the condition for parameters β and δ we will derive below is, strictly speaking, not accurate nevertheless, we will argue by using our approximation for avoiding unnecessary complications. Justification of our approximation analysis and together with our heuristic assumption will be given by computer experiments shown later.

Now by using the bound (21) (see Proposition 1 in Section A.1), we can approximately evaluate $\mathrm{eval}(i, j)$ as follows [5].

$$\mathrm{eval}(i,j) \approx \begin{cases} i\beta - (\beta - \delta + 0.5)j & (i,j) \in I_1 \\ (\delta - 1)i + 0.5j & (i,j) \in I_2 \\ (-1+\beta)i + 0.5j & (i,j) \in I_3. \end{cases}$$

From this we can approximately estimate $\mathrm{eval}(I)$. From some calculation (see Section A.2) we have

$$\mathrm{eval}(I) = \left(\frac{1}{6}(\beta + \delta - 0.5) + \frac{\tau}{4}(2\delta + \tau - 1) - \frac{1}{12}\frac{(1-2\beta-\tau)^3}{1-2\beta} \right) m^3 + o(m^3) \tag{25}$$

for $1 - 2\beta - \tau > 0$, and we have

$$\mathrm{eval}(I) = \left(\frac{1}{6}(\beta + \delta - 0.5) + \frac{\tau}{4}(2\delta + \tau - 1) \right) m^3 + o(m^3) \tag{26}$$

for $1 - 2\beta - \tau \leq 0$. Note that a condition $1 - 2\beta - \tau > 0$ and $1 - 2\beta - \tau \leq 0$ are corresponding to the cases $I_3 \neq \phi$ and $I_3 = \phi$ respectively. Then following the argument of [1, 6], we analyze the bound for *the ideal case* by assuming that m is sufficiently large. (We draw a figure to show the bounds for some concrete values of m, see Figure 4 in Section 5.)

First we consider the case of $1-2\beta-\tau > 0$. Assuming that m is sufficiently large, the condition (24) is approximately equivalent to

$$\frac{1}{6}(\beta + \delta - 0.5) + \frac{\tau}{4}(2\delta + \tau - 1) - \frac{1}{12}\frac{(1-2\beta-\tau)^3}{1-2\beta} < 0, \tag{27}$$

where its left-hand side is minimised when τ takes value $\tau_0 = \sqrt{2(1-2\beta)(\beta-\delta)}$. Hence, substituting this to (27), we have

$$\frac{1}{3}\sqrt{2(1-2\beta)(\beta-\delta)}(\delta - \beta) - \frac{1}{3}\beta^2 + \frac{1}{2}\beta + \frac{1}{6}\delta - \frac{1}{6} < 0. \tag{28}$$

This is equivalent to (3).

Next we consider the case of $1 - 2\beta - \tau \leq 0$. Again assuming m is sufficiently large, we have the following approximate condition (24).

$$\frac{1}{6}(\beta + \delta - 0.5) + \frac{\tau}{4}(2\delta + \tau - 1) < 0. \tag{29}$$

By similar argument, we substitute $\tau_1 = \frac{1-2\delta}{2}$ to τ for minimising (29), and derive the following condition (29).

$$\delta < \frac{5}{6} - \frac{1}{3}\sqrt{1 + 6\beta}. \tag{30}$$

[5] Here a symbol $A \approx B$ means $\frac{A}{B} \to 1$ when N (the RSA bit length) goes to infinity.

Step 1:	(Make sample RSA instance) Randomly choose $\ell/2$-bit primes p and q, and let $N = pq$. (In our program, we choose p and q the Euler-Jacobi pseudoprime to bases 2, 3, 5, 7 and 11.) Randomly choose $\lfloor \beta\ell \rfloor$-bit random odd integer as the secret key d such that $\gcd(d, (p-1)(q-1)) = 1$, and let $M = 2^{\lfloor (\beta-\delta)\ell \rfloor}$ and $d_0 = d \bmod M$. Compute the public key $e \equiv d^{-1} \pmod{(p-1)(q-1)}$, and let $f_{\mathrm{main}}(x, y) = x(N+y)+(ed_0-1)$, and let $y_0 = 1-p-q$ and $x_0 = (1-ed)/((p-1)(q-1))$.
Step 2:	Let $\tau = \sqrt{2(1-2\beta)(\beta-\delta)}$, $X = \lfloor N^{\beta} \rfloor$ and $Y = \lfloor 3N^{0.5} \rfloor$. Then construct our lattice $L(m, \tau, \beta, \delta)$.
Step 3:	Apply the L^2 algorithm for $L(m, \tau, \beta, \delta)$.
Step 4:	Sort the vectors of reduced basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ by these length to $\mathbf{v}'_1, \ldots, \mathbf{v}'_n$. Compute $g_i(x, y) = \mathcal{F}_I(\mathbf{v}'_i, X, Y)$.
Step 5:	Check $g_1(x_0, y_0) = g_2(x_0, y_0) = 0$ holds or not.

Figure 3: Our computer experiment procedure

This is equivalent to the condition proposed by [6]. Therefore, we can recover the secret key of RSA in polynomial time in $\log N$ when β and δ satisfies (3) or (4).

5 Computer Experiments

We carried out our *preliminary* computer experiments to check that our approach works and estimate its efficiency. We conducted our computer experiments on the TSUBAME supercomputer [6] with C++ implementation using the NTL library [10]. We use the L^2 algorithm [11, 12] for lattice basis reduction.

The procedure of our experiments is shown in Figure 3. The algorithm part is essentially the same as the one outlined in Figure 2. (At Step 4, the vectors obtained by the L^2 algorithm are sorted by their length; this is because those vectors are approximate ones and we cannot guarantee that $\mathbf{v}_1$ and $\mathbf{v}_2$ are the smallest in reduced basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$.) Input parameters of this experiments are ℓ, m, β and δ, which are respectively the bit-size of N, the parameter for constructing lattice, the ratio of $\lg(d)$ to $\lg(N)$, and the ratio of $\lg(\tilde{d})$ to $\lg(N)$. (See Figure 2 for the parameters m and n, and see Section 3 for the parameters β and δ.) A word "dim." means the lattice dimension in experiments, i.e., $n = |I|$ in our construction. By "total CPU time" and "L^2 time" we mean the time of Step 2 through Step 5 and that of Step 3 respectively. Recall that the lattice component size is bounded by a constant time of the lattice dimension.

Results are in Table 1 and Table 2. Table 1 shows the qualities of our lattices, that is, whether the experiment is succeeded or not, for these parameters. On the other hand, Table 2 shows the computational time of our experiments for various ℓ and m, and fixed β and δ.

Quality of Lattice

For checking our approach indeed works and estimating the quality of our lattice construction, we carried out our preliminary computer experiments.

Note first that the bounds (3) and (4) are the ideal ones obtained by the asymptotic analysis assuming m is sufficiently large. For each given value of m, we can determine the range of β and δ satisfying $\mathrm{eval}(I(m, \tau_0, \beta, \delta)) < 0$ by numerically analyzing the original expressions (i.e., (42) $\sim$ (44) of Appendix A.2). Figure 4 shows the bounds of β and δ obtained in this way for some m

[6] TSUBAME is a grid type supercomputer at Tokyo Inst. of Tech., whose performance is currently (by Top500, June 2008) the 24th in the World. Note, however, we have not been able to make a parallel version of our algorithm; TSUBAME'S massive parallelism has been used only for reducing the total experiment time.

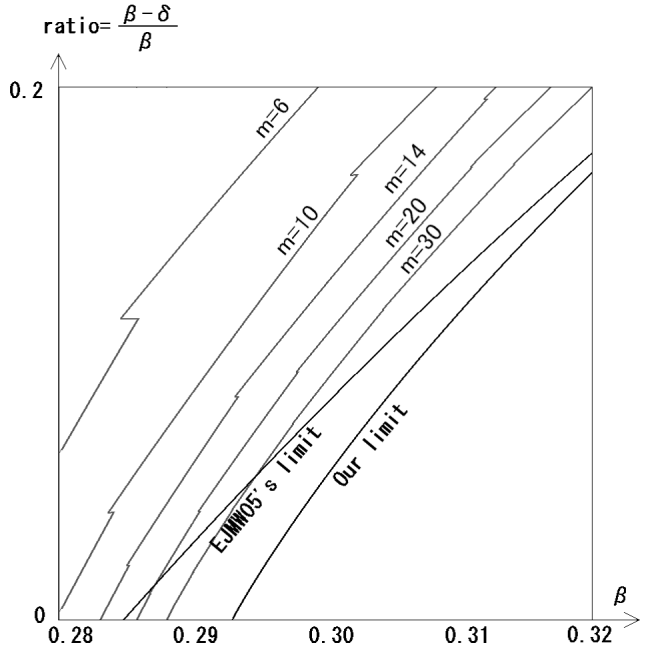


Figure 4: Our recoverable range for many m

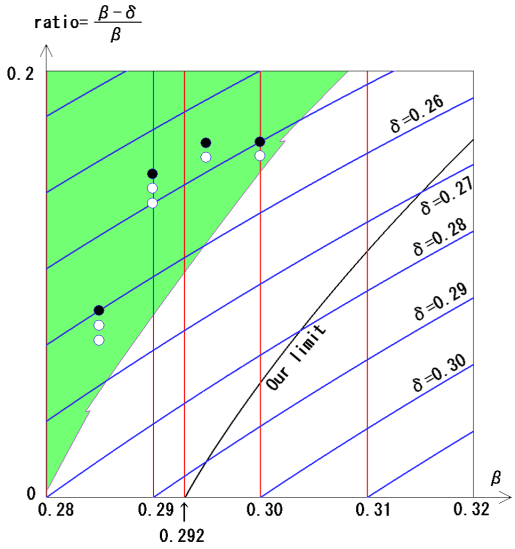
values, and the theoretical limit of our construction (3) and that of [6]'s construction (2). It can be seen that these bounds get close to our ideal bound when m gets large. We focus on the range of $0.28 \leq \beta \leq 0.32$ and $0 \leq \text{ratio} \leq 0.2$, mainly our new improvement area.

Our computer experiments are summarized in Figure 5, which are for the cases $m = 10$ and 14. The instance size ℓ $(= \lg(N))$ is 1024 for both cases; since these are still preliminary ones, we conducted only one execution for each instance. Note that a shade area in each figure is the area that $\text{eval}(I(m, \tau_0, \beta, \delta)) < 0$ obtained numerically for each m. A black circle (resp., a white circle) indicates the parameter (β, δ) (or the point $(\beta, (\beta - \delta)/\beta)$) that the experiment succeeds (resp., fails). Those results are shown in detail in Table 1. The word "no(1)" in the column "success" in the tables means $g_1(x_0, y_0) = 0$ whereas $g_2(x_0, y_0) \neq 0$ at the Step 5. In this case we may expect to get the correct solution by generating enough number of polynomials g by changing X and Y randomly within the same bit length. On the other hand, the word "no(0)" means that $g_1(x_0, y_0) \neq 0$ and $g_2(x_0, y_0) \neq 0$ at Step 5, which we regarded as a failure.

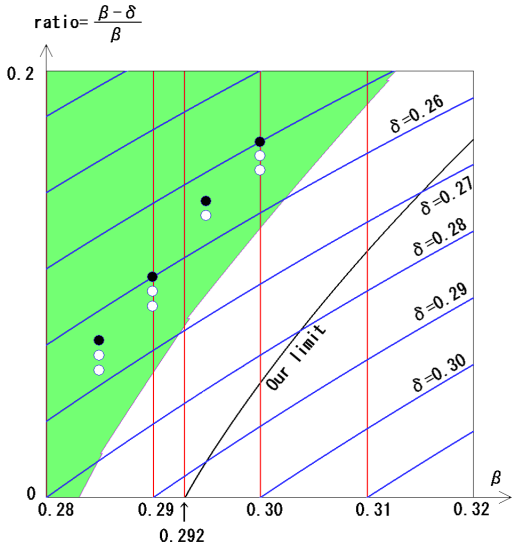
Computational Time

Next we examine the efficiency of our algorithm based on our experiments. As seen by our analysis and experiments, the algorithm shows better performance by using large m. On the other hand, by using larger m, the lattice dimension also get larger. More specifically, the lattice dimension is $\Theta(m^2)$ by our construction. We examine how this indeed effects to the running time of the algorithm. (*Cf.* The lattice dimension of those used in [6] is $\Theta(m^3)$.)

We carried out computer experiments with parameters $(\beta, \delta) = (0.3, 0.225)$ and $(\beta, \delta) = (0.4, 0.12)$, whereas parameters m and ℓ are chosen as $m = 4$, 6, 8, 10 and 12, and $\ell = 512$, 1024, and 2048. Table 2 is experiments for $\ell = 512$, 1024, and 2048. Total CPU time and L^2 time in these tables are the average running time of five executions. Notice that all experiments in Table 2 are success. These results show the total CPU time and L^2 time are close, which shows



$m = 10$



$m = 14$

Figure 5: Summary of our experiments for $\ell = 1024$

Experiment parameters			Results	
β	δ	dim.	L^2 time	Success
0.285	0.260	88	3 hrs. 42 min.	yes
0.285	0.262	88	3 hrs. 46 min.	no(1)
0.285	0.264	88	3 hrs. 53 min.	no(0)
0.290	0.246	87	3 hrs. 15 min.	yes
0.290	0.248	87	3 hrs. 24 min.	no(1)
0.290	0.250	87	3 hrs. 1 min.	no(0)
0.295	0.246	92	4 hrs. 2 min.	yes
0.295	0.248	87	2 hrs. 53 min.	no(0)
0.300	0.250	91	3 hrs. 25 min.	yes
0.300	0.252	85	2 hrs. 16 min.	no(0)

$m = 10$

Experiment parameters			Results	
β	δ	dim.	L^2 time	Success
0.285	0.264	162	2 days 20 hrs.	yes
0.285	0.266	162	2 days 18 hrs.	no(1)
0.285	0.268	162	2 days 9 hrs.	no(0)
0.290	0.260	165	2 days 22 hrs.	yes
0.290	0.262	165	2 days 16 hrs.	no(1)
0.290	0.264	165	2 days 16 hrs.	no(0)
0.295	0.254	164	2 days 19 hrs.	yes
0.295	0.256	164	2 days 17 hrs.	no(0)
0.300	0.250	163	3 days 7 hrs.	yes
0.300	0.252	163	3 days 14 hrs.	no(1)
0.300	0.254	163	3 days 11 hrs.	no(0)

$m = 14$

Table 1: Quality of our lattice for $m = 10$ and $m = 14$

Experiment parameters				Results	
β	δ	m	dim.	L^2time	Total CPU time
0.3	0.225	4	17	1.7 sec.	1.8 sec.
		6	36	1 min. 6 sec.	1 min. 8 sec.
		8	58	10 min. 31 sec.	10 min. 42 sec.
		10	91	1 hour 25 min.	1 hour 26 min
		12	124	6 hrs. 13 min.	6 hrs 15 min.
		14	171	25 hrs. 28 min.	25 hrs. 35 min.
0.4	0.12	4	15	1.5 sec.	1.7 sec.
		6	35	2 min. 4 sec.	2 min. 7 sec.
		8	54	16 min. 32 sec.	16 min. 46 sec.
		10	77	1 hour 10 min	1 hour 11 min.
		12	117	10 hrs. 11 min.	10 hrs. 14 min.
		14	150	29 hrs 56 min.	30 hrs. 3 min.

$\ell = 512$

Experiment parameters				Results	
β	δ	m	dim.	L^2time	Total CPU time
0.3	0.225	4	17	5.6 sec.	6.2 sec.
		6	36	4 min. 34 sec.	4 min. 42 sec.
		8	58	40 min. 5 sec.	40 min. 44 sec.
		10	91	5 hrs. 33 min.	5 hrs. 36 min.
		12	124	21 hrs. 25 min.	21 hrs. 33 min.
		14	171	127 hrs. 0 min.	127 hrs. 10 min.
0.4	0.12	4	15	6.4 sec.	7.1 sec.
		6	35	8 min. 41 sec.	8 min. 52 sec.
		8	54	64 min. 22 sec.	65 min. 10 sec.
		10	77	4 hrs. 56 min.	4 hrs. 59 min.
		12	117	43 hrs. 35 min.	43 hrs. 45 min.
		14	150	159 hrs. 40 min.	160 hrs. 4 min.

$\ell = 1024$

Experiment parameters				Results	
β	δ	m	dim.	L^2time	Total CPU time
0.3	0.225	4	17	27 sec.	29 sec.
		6	36	21 min. 34 sec.	22 min. 3 sec.
		8	58	2 hrs. 46 min.	2 hrs. 48 min.
		10	91	24 hrs. 8 min.	24 hrs. 17 min.
0.4	0.12	4	15	32 sec.	35 sec.
		6	35	42 min. 34 sec.	43 min. 13 sec.
		8	54	4 hrs. 37 min.	4 hrs. 40 min.
		10	77	21 hrs. 19 min.	21 hrs. 29 min.

$\ell = 2048$

Table 2: Total CPU time for ℓ=512, 1024 and 2048

a lattice reduction algorithm is the main part of the lattice based attack. From these tables, we obtain our computational time is approximately

$$\text{Time} \approx 0.35 \times \left(\frac{\ell}{512}\right)^2 \cdot \left(\frac{m}{2}\right)^8 \cdot \log_2 \ell \cdot \log_2 m \text{ sec for } \beta = 0.3 \text{ and } \delta = 0.225, \text{ and}$$
$$\text{Time} \approx 0.6 \times \left(\frac{\ell}{512}\right)^2 \cdot \left(\frac{m}{2}\right)^8 \cdot \log_2 \ell \cdot \log_2 m \text{ sec for } \beta = 0.4 \text{ and } \delta = 0.12.$$

It should be noted that the construction of [6] yields a lattice of dimension $\Theta(m^3)$ and that it took about five hours on average (according to our experiments) to solve the instances obtained by this construction for $m = 6$.

6 Conclusion

We gave a new lattice construction for the lattice based attack for the RSA cryptography in the situation that d is small and LSBs of d is exposed. By this construction, the theoretical recoverable range has been improved as shown in Figure 1, which solves the open problem raised in [6]. Also as shown by our preliminary experimental results, the total efficiency of the lattice based attack has been improved significantly compared with [6]. Some more improvement, however, is necessary for using this technique with large m. One possibility is to make use of parallelization for processing the lattice basis reduction algorithm. From a theoretical view point, it would be interesting if we can understand the limitation of this approach.

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A Appendix: Analysis

This section provides the precise analysis for some results in Section 4. Our objective is to derive our theoretical recoverable limit of our construction

$$\frac{1}{3}\sqrt{2(1-2\beta)(\beta-\delta)}(\delta-\beta)-\frac{1}{3}\beta^2+\frac{1}{2}\beta+\frac{1}{6}\delta-\frac{1}{6}<0 \tag{28}$$

and

$$\delta<\frac{5}{6}-\frac{1}{3}\sqrt{1+6\beta}. \tag{30}$$

In order to obtain these inequalities, we first prove the approximated estimation

$$\mathrm{eval}(i,j)\approx\begin{cases}\beta i-(\beta-\delta+0.5)j & (i,j)\in I_1\\ (\delta-1)i+0.5j & (i,j)\in I_2\\ (-1+\beta)i+0.5j & (i,j)\in I_3\end{cases} \tag{31}$$

in Section A.1. Next in Section A.2 we explain the way to derive our theoretical limit. We fix algorithm parameters and RSA instances throughout this section, and denote $f^*_{i,j}(x,y)$ the functionalisation of $\mathbf{b}^*_{i,j}$. (We recall that $\{\mathbf{b}^*_{i,j}\}_{(i,j)\in I}$ is the Gram-Schmidt orthogonal basis of our basis $\{\mathbf{b}_{i,j}\}_{(i,j)\in I}\stackrel{\mathrm{def}}{=}\{\mathcal{V}(f_{i,j};X,Y)\}_{(i,j)\in I}$.)

A.1 Approximating the Evaluator

We first compute the exact value of $|\mathbf{b}^*_{i,j}|$ for $(i,j)\in I_1$ and I_2 in Lemma 3 and Lemma 4 respectively. Next we consider the bound for $|\mathbf{b}^*_{i,j}|$ for $(i,j)\in I_3$ in Lemma 5. Then we estimate the approximated value of $\mathrm{eval}(i,j)\stackrel{\mathrm{def}}{=}\log_N(|\mathbf{b}^*_{i,j}|/(eM)^m)$ in Proposition 1.

Lemma 3. For any $(i,j)\in I_1$, we have

$$|\mathbf{b}^*_{i,j}|=(eM)^{m-j}X^iY^j. \tag{32}$$

Proof. We in fact prove that

$$\mathbf{b}^*_{i,j}=(0,0,\ldots,(eM)^{m-j}X^iY^j,0,\ldots,0)=\mathcal{V}((eM)^{m-j}X^iY^j;X,Y). \tag{33}$$

by mathematical induction on (i,j). That is, $\mathbf{b}^*_{i,j}$ is the vectorisation of $(eM)^{m-j}X^iY^j$.

We first recall that $\mathbf{b}^*_{i,j}$ is defined by

$$\mathbf{b}^*_{i,j}=\mathbf{b}_{i,j}-\sum_{(i',j')\prec(i,j)}\mu_{(i,j),(i',j')}\mathbf{b}^*_{i',j'}\text{ and }\mu_{(i,j),(i',j')}\stackrel{\mathrm{def}}{=}\frac{\langle\mathbf{b}_{i,j},\mathbf{b}^*_{i',j'}\rangle}{\langle\mathbf{b}^*_{i',j'},\mathbf{b}^*_{i',j'}\rangle}.$$

where the order $\prec$ is defined in Section 4.

Consider the base case, i.e. the case $(i,j)=(0,0)$, we have

$$\mathbf{b}^*_{0,0}=\mathbf{b}_{0,0}=\mathcal{V}((eM)^m;X,Y).$$

Now we assume that (33) holds for $(i',j') \prec (i,j)$. Then we have for $(i,j) \in I_1$ with some constants α,

$$f_{i,j}(x,y) = (eM)^{m-j}x^{i-j}((ed_0-1)+Nx+xy)^j = (eM)^{m-j}x^iy^j + \sum_{(i',j')\prec(i,j)} \alpha_{i',j'}x^{i'}y^{j'}. \quad (34)$$

We consider the vectorisation of (34), we have

$$\mathbf{b}_{i,j} = \mathcal{V}((eM)^{m-j}x^iy^j;X,Y) + \sum_{(i',j')\prec(i,j)} \alpha_{i',j'}\mathbf{b}^*_{i',j'}. \quad (35)$$

Here we use the assumption of induction $\mathbf{b}^*_{i',j'} = \mathcal{V}((eM)^{m-j'}x^{i'}y^{j'};X,Y)$.
Then we have for $(i'',j'') \prec (i,j)$,

$$\begin{aligned}\langle \mathbf{b}_{i,j}, \mathbf{b}^*_{i'',j''}\rangle &= \left\langle \mathcal{V}((eM)^{m-j}x^iy^j;X,Y) + \sum_{(i',j')\prec(i,j)} \alpha_{i',j'}\mathbf{b}^*_{i',j'}, \mathbf{b}^*_{i'',j''}\right\rangle \\ &= \langle \alpha_{i'',j''}\mathbf{b}^*_{i'',j''}, \mathbf{b}^*_{i'',j''}\rangle\end{aligned}$$

since $\left\langle \alpha_{i',j'}\mathbf{b}^*_{i',j'}, \mathbf{b}^*_{i'',j''}\right\rangle = 0$ for $(i',j') \neq (i'',j'')$, and $\left\langle \mathcal{V}((eM)^{m-j}x^iy^j;X,Y), \mathbf{b}^*_{i'',j''}\right\rangle = 0$. This means

$$\alpha_{i'',j''} = \frac{\left\langle \mathbf{b}_{i,j}, \mathbf{b}^*_{i'',j''}\right\rangle}{\left\langle \mathbf{b}^*_{i'',j''}, \mathbf{b}^*_{i'',j''}\right\rangle} = \mu_{(i,j),(i'',j'')}. \quad (36)$$

Substituting this to (35) we have

$$\mathcal{V}((eM)^{m-j}x^iy^j;X,Y) = \mathbf{b}_{i,j} - \sum_{(i',j')\prec(i,j)} \mu_{(i,j),(i',j')}\mathbf{b}^*_{i',j'} = \mathbf{b}^*_{i,j}.$$

□
We can prove the next lemma by a similar way to lemma 3.

Lemma 4. For any $(i,j) \in I_2$, we have

$$|\mathbf{b}^*_{i,j}| = (eM)^{m-i}X^iY^j.$$

The evaluation of $|\mathbf{b}^*_{i,j}|$ for $(i,j) \in I_3$ is a bit involved. Our objective is to show the following lemma.

Lemma 5. For any $(i,j) \in I_3$, we have

$$e^{m-i}M^mX^iY^j \leq |\mathbf{b}^*_{i,j}| < 2e^{m-i}M^mX^iY^j.$$

Now we start with the following two simple lemmas.

Lemma 6. Let $\mathbf{b}_1,\ldots,\mathbf{b}_n$ be linearly independent vectors and $\mathbf{b}^*_1,\ldots,\mathbf{b}^*_n$ be orthogonal set generated by the Gram-Schmidt process. Then for any natural number i and any real numbers $\alpha_1,\ldots,\alpha_{i-1},\alpha^*_1,\ldots,\alpha^*_{i-1}$ we have

$$|\mathbf{b}^*_i| \leq \left|\mathbf{b}_i + \sum_{j=1}^{i-1}\alpha_j\mathbf{b}_j + \sum_{j=1}^{i-1}\alpha^*_j\mathbf{b}^*_j\right|.$$

Proof. By definition of the Gram-Schmidt basis, with suitable coefficients β_{ij} we have

$$\mathbf{b}_i^* = \mathbf{b}_i + \sum_{j=1}^{i-1} \beta_{ij} \mathbf{b}_j^*.$$

Then with some real numbers γ_j, we have

$$\left| \mathbf{b}_i + \sum_{j=1}^{i-1} \alpha_j \mathbf{b}_j + \sum_{j=1}^{i-1} \alpha_j^* \mathbf{b}_j^* \right|^2 = \left| \mathbf{b}_i^* - \sum_{j=1}^{i-1} \beta_{ij} \mathbf{b}_j^* + \sum_{j=1}^{i-1} \alpha_j \left(\mathbf{b}_j^* - \sum_{k=1}^{j-1} \beta_{jk} \mathbf{b}_k^* \right) + \sum_{j=1}^{i-1} \alpha_j^* \mathbf{b}_j^* \right|^2$$
$$= \left| \mathbf{b}_i^* + \sum_{j=1}^{i-1} \gamma_j \mathbf{b}_j^* \right|^2 = |\mathbf{b}_i^*|^2 + \left| \sum_{j=1}^{i-1} \gamma_j \mathbf{b}_j^* \right|^2 \geq |\mathbf{b}_i^*|^2.$$

□

Lemma 7. For any positive integers i and j such that $XY > 2i$, we have

$$||(-1+xy)^i y^{j-i}||_{XY} < 2X^i Y^j.$$

Proof. This is derived as follows.

$$\begin{aligned}
&\left(||(-1+xy)^i y^{j-i}||_{XY}\right)^2 \\
&= \left(\left\| y^{j-i} \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} (xy)^k \right\|_{XY} \right)^2 \\
&= Y^{2(j-i)} \sum_{k=0}^{i} \binom{i}{k}^2 (XY)^{2k} < Y^{2(j-i)} \sum_{k=0}^{i} \sum_{k'=0}^{i} \binom{i}{k} \binom{i}{k'} X^{k+k'} Y^{k+k'} \\
&= Y^{2(j-i)} (1+XY)^{2i} < Y^{2(j-i)} \left(\left(1 + \frac{1}{2i} \right) XY \right)^{2i} \\
&< Y^{2(j-i)} \left(4^{1/(2i)} XY \right)^{2i} = 4X^{2i} Y^{2j} \ \square
\end{aligned}$$

Now for our analysis we define $g_{i,j}(x,y) = e^{m-i} y^{j-i} (-1 + Nx + xy)^i$, that is $f_{i,j}(x,y) = M^m g_{i,j}(x,y)$ for $(i,j) \in I_3$. We first show that $g_{i,j}(x,y)$ is expressed as a linear combination of $(-1+xy)^i y^s$, $g_{i',j'}(x,y)$ and $(-1+xy)^{j'} x^{i'-j'}$ for $(i',j') \prec (i,j)$.

Lemma 8. For any i and s, $1 \leq s \leq i$, there exist some constants α's and β's such that the following holds:

$$\begin{aligned}
g_{i,i+s}(x,y) &= e^{m-i} y^s (-1+xy)^i + \sum_{(i',j') \in A(i,i+s)} \alpha_{i',j'} g_{i',j'}(x,y) \\
&\quad + \sum_{(i',j') \in B(i,i+s)} \beta_{i',j'} (-1+xy)^{j'} x^{i'-j'} \qquad (37)
\end{aligned}$$

Here, we denote $A(i,i+s) = \{(i',j') \in \mathbb{Z}^2 | i-s < i' \leq i, i' < j' \leq 2i'+s-i, (i',j') \neq (i,i+s)\}$ and $B(i,i+s) = \{(i',j') \in \mathbb{Z}^2 | i-s \leq i' \leq i, 0 \leq j' \leq i'\}$.

Proof. We prove that for any $s \geq 1$, the equation (37) holds for some appropriate α's and β's for all $i \geq 1$ by induction on s. For the base case, i.e., s=1, consider any $i \geq 1$. We have $A(i, i+1) = \phi, B(i, i+1) = \{(i', j') \in \mathbb{Z}^2 | i-1 \leq i' \leq i, 0 \leq j' \leq i'\}$ and

$$\begin{aligned}
g_{i,i+1}(x,y) &= e^{m-i}(-1+xy+Nx)^i y \\
&= e^{m-i}\sum_{j'=0}^{i}\binom{i}{j'}(-1+xy)^{i-j'}(Nx)^{j'}y \\
&= e^{m-i}(-1+xy)^i y + e^{m-i}\sum_{j'=1}^{i}\binom{i}{j'}N^{j'}(-1+xy)^{i-j'}x^{j'}y \\
&= e^{m-i}(-1+xy)^i y + e^{m-i}\sum_{j'=0}^{i-1}\binom{i}{j'+1}N^{j'+1}(-1+xy)^{i-j'-1}x^{j'+1}y \\
&= e^{m-i}(-1+xy)^i y + e^{m-i}\sum_{j'=0}^{i-1}\binom{i}{j'+1}N^{j'+1}(-1+xy)^{i-j'-1}((-1+xy)+1)x^{j'} \\
&= e^{m-i}(-1+xy)^i y + e^{m-i}\sum_{j'=0}^{i-1}\sum_{i'=0}^{1}\binom{i}{j'+1}N^{j'+1}(-1+xy)^{i-i'-j'}x^{j'}.
\end{aligned}$$

Thus, (37) is true for $s = 1$ and any $i \geq 1$ by setting $\beta_{i-i',j'} = e^{m-i}\binom{i}{j'+1}N^{j'+1}$. Note also that the above implies

$$(-1+xy)^i y = \alpha'_i g_{i,i+1}(x,y) + \sum_{j'=0}^{i-1}\sum_{i'=0}^{1}\alpha'_{i,j',i'}(-1+xy)^{i-i'-j'}x^{j'}. \tag{38}$$

for some constants α'. We use this later.

Assume now the lemma holds up to $s-1$ and we prove it for s. First note that by the case for $s-1$,

$$\begin{aligned}
g_{i,i+s}(x,y) &= y g_{i,i+s-1}(x,y) \\
&= e^{m-i}y^s(-1+xy)^i + \sum_{(i',j')\in A(i,i+s-1)}\alpha_{i',j'}y g_{i',j'}(x,y) \\
&\quad + \sum_{(i',j')\in B(i,i+s-1)}\beta_{i',j'}(-1+xy)^{j'}x^{i'-j'}y.
\end{aligned} \tag{39}$$

Thus, it suffices to show that the second and the third terms of (39) can be written in the form of (37). Then we have with some constants α and α'

$$\begin{aligned}
&(\text{The second term of (39)}) = \sum_{(i',j')\in A(i,i+s-1)}\alpha_{i',j'}g_{i',j'+1}(x,y) \\
&= \sum_{(i',j')\in A(i,i+s)}\alpha'_{i',j'}g_{i',j'}(x,y).
\end{aligned}$$

This is because $(i', j'+1) \in A(i, i+s-1) \Rightarrow (i', j') \in A(i, i+s)$. This means the second term of (39) can be written in the form of (37). We divide the third term of (39) for terms with $i' = j'$ and with $i' > j'$.

$$
\begin{aligned}
&\text{(The third term of (39))}\\
&= \sum_{(j',j')\in B(i,i+s-1)} \beta_{j',j'}(-1+xy)^{j'}y + \sum_{\substack{(i',\,j')\,\in\, B(i,\,i\,+\,s\,-\,1)\\ i'\,>\,j'}} \beta_{i',j'}(-1+xy)^{j'}x^{i'-j'}y \qquad (40)
\end{aligned}
$$

We will show these term also can be written in the form of (37). Substituting (38) to the first term of (40). We have for some suitable constant γ,

$$
\begin{aligned}
&\text{(The first term of (40))}= \sum_{j'=i-s+1}^{i} \beta_{j',j'}(-1+xy)^{j'}y\\
&= \sum_{j'=i-s+1}^{i} \beta_{j',j'}\left(\alpha'_{j'}g_{j',j'+1}(x,y) + \sum_{j''=0}^{j'-1}\sum_{i''=0}^{1}\alpha'_{j',j'',i''}(-1+xy)^{j'-i''-j''}x^{j''}\right)\\
&= \sum_{j'=i-s+1}^{i} \gamma_{j'}g_{j',j'+1}(x,y) + \sum_{j'=i-s+1}^{i}\sum_{j''=0}^{j'-1}\sum_{i''=0}^{1}\beta_{j',j'}\alpha'_{j',j'',i''}(-1+xy)^{j'-i''-j''}x^{j''}\\
&= \sum_{j'=i-s+1}^{i} \gamma_{j'}g_{j',j'+1}(x,y) + \sum_{j'=i-s}^{i}\sum_{j''=0}^{j'-1}\gamma_{j',j''}(-1+xy)^{j'-j''}x^{j''}.
\end{aligned}
$$

Then, this is in the form of (39) from

$$
\begin{aligned}
&(j',j'+1)\in A(i,i+s) \text{ for } i-s+1\le j'\le i \text{ and}\\
&(j',j'-j'')\in B(i,i+s) \text{ for } i-s\le j'\le i, 0\le j''\le j'-1.
\end{aligned}
$$

Finally, we show the second term of (40) is also in the form of (39). We have

$$
\begin{aligned}
&\text{(The second term of (40))} = \sum_{\substack{(i',\,j')\,\in\, B(i,\,i\,+\,s\,-\,1)\\ i'\,>\,j'}} \beta_{i',j'}(-1+xy)^{j'}\cdot xy\cdot x^{i'-j'-1}\\
&= \sum_{\substack{(i',\,j')\,\in\, B(i,\,i\,+\,s\,-\,1)\\ i'\,>\,j'}} \left(\beta_{i',j'}(-1+xy)^{j'+1}x^{i'-j'-1} + \beta_{i',j'}(-1+xy)^{j'}x^{i'-j'-1}\right)
\end{aligned}
$$

Thus, the last term is in the form of (39). This is because $(i',j')\in B(i,i+s-1)$ and $i'<j'$ implies $i-s+1\le i'\le i$ and $0\le j'\le i'+1$. This means $(i',j'+1)\in B(i,i+s)$ and $(i'-1,j')\in B(i,i+s)$. □

Corollary 1. For $(i,i+s)\in I_3$, we have

$$
f_{i,i+s}(x,y) = e^{m-i}M^m(-1+xy)^i y^s + \sum_{\substack{(i',\,j')\,\prec\,(i,\,i\,+\,s)\\ (i',\,j')\,\in\, I_3}} \alpha_{i',j'}f_{i',j'}(x,y) + \sum_{\substack{(i',\,j')\,\prec\,(i,\,i\,+\,s)\\ (i',\,j')\,\in\, I_1\,\cup\, I_2}} \beta_{i',j'}f^*_{i',j'}(x,y). \qquad (41)
$$

Here, we recall $f^*_{i',j'}(x,y) = x^{i'}y^{j'}$, which is equal to the functionalisation of $\mathbf{b}^*_{i',j'}$.

Proof. First, we separate the second term of (37) as follows

$$
\sum_{(i',j')\in A(i,i+s)} \alpha_{i',j'}g_{i',j'}(x,y) = \sum_{(i',j')\in A(i,i+s)\cap I_2} \alpha_{i',j'}g_{i',j'}(x,y) + \sum_{(i',j')\in A(i,i+s)\cap I_3} \alpha_{i',j'}g_{i',j'}(x,y).
$$

Recall that all items in $A(i,i+s)$ are smaller than $(i,i+s)$ in the order $\prec$. Thus, we have with some constants for $(i',j') \in A(i,i+s) \cap I_2$

$$\begin{aligned} g_{i',j'}(x,y) &= e^{m-i'}y^{j'-i'}(-1+Nx+xy)^{i'} \\ &= \sum_{\substack{0 \le i'' \le i' \\ j'-i' \le j'' \le j'}} \gamma_{i'',j''}x^{i''}y^{j''} = \sum_{\substack{(i'',j'') \in I_1 \cup I_2 \\ (i'',j'') \preceq (i',j')}} \gamma_{i'',j''}x^{i''}y^{j''} = \sum_{\substack{(i'',j'') \in I_1 \cup I_2 \\ (i'',j'') \prec (i,i+s)}} \gamma_{i'',j''}x^{i''}y^{j''}. \end{aligned}$$

By a similar argument, the third term of (37) is

$$\sum_{(i',j') \in B(i,i+s)} \beta_{i',j'}(-1+xy)^{j'}x^{i'-j'} = \sum_{\substack{0 \le i' \le i \\ 0 \le j' \le i'}} \beta_{i',j'}(-1+xy)^{j'}x^{i'-j'} = \sum_{\substack{0 \le i' \le i \\ 0 \le j' \le i'}} \beta'_{i',j'}x^{i'}y^{j'}$$

Note that all of suffices in β' are in I_1. Therefore, we have with some constants a and b

$$g_{i,i+s}(x,y) = e^{m-i}(-1+xy)^i y^s + \sum_{\substack{(i',j') \prec (i,i+s) \\ (i',j') \in I_3}} a_{i',j'}g_{i',j'}(x,y) + \sum_{\substack{(i',j') \prec (i,i+s) \\ (i',j') \in I_1 \cup I_2}} b_{i',j'}f^*_{i',j'}(x,y),$$

and we get our claim by multiplying M^m.
□

Now we are ready to prove to Lemma 5.

Proof of Lemma 5. Vectorisation of (41) is

$$\mathbf{b}_{i,i+s} = \mathcal{V}(e^{m-i}M^m(-1+xy)^i y^s; X, Y) + \sum_{\substack{(i',j') \prec (i,i+s) \\ (i',j') \in I_3}} \alpha_{i',j'}\mathbf{b}_{i',j'} + \sum_{(i',j') \in I_1 \cup I_2} \beta_{i',j'}\mathbf{b}^*_{i',j'}(x,y).$$

Thus from Lemma 6 and Lemma 7, we have

$$|\mathbf{b}^*_{i,i+s}| \le |\mathcal{V}(e^{m-i}M^m(-1+xy)^i y^s; X, Y)| = ||e^{m-i}M^m(-1+xy)^i y^s||_{XY} < 2e^{m-i}M^m X^i Y^{i+s}.$$

Then substituting $j = i+s$, we showed the upper bound of our claim.

Next, we show the lower bound. As the above argument, we fix $(i,j) \in I_3$. Let $b_{(i,j),(i',j')}$ the $(i',j') - th$ element of $\mathbf{b}_{i,j}$. Then it is clear that $b_{(i,j),(i',j')} = 0$ for $(i,j) \prec (i',j')$ and $b_{(i,j),(i,j)} = e^{m-i}M^m X^i Y^j$. We look at $b^*_{(i,j),(i,j)}$, the (i,j)-th element of $\mathbf{b}^*_{i,j}$. By definition of the Gram-Schmidt basis, we have

$$b^*_{(i,j),(i,j)} = b_{(i,j),(i,j)} - \sum_{(i',j') \prec (i,j)} \mu_{(i,j),(i',j')}b^*_{(i',j'),(i,j)} = b_{(i,j),(i,j)} = e^{m-i}M^m X^i Y^j, \text{ and}$$

$$b^*_{(i,j),(i',j')} = 0 \text{ for } (i,j) \prec (i',j').$$

Thus we have

$$\mathbf{b}^*_{i,j} = (*, *, \ldots, *, e^{m-i}M^m X^i Y^j, 0, \ldots, 0).$$

Therefore,

$$|\mathbf{b}^*_{i,j}| \ge |(0, \ldots, 0, e^{m-i}M^m X^i Y^j, 0, \ldots, 0)| = e^{m-i}M^m X^i Y^j.$$

□

Now we obtain the approximated bounds for evaluators for each (i,j).

Proposition 1.

$$\mathrm{eval}(i,j) \approx \begin{cases} \beta i - (\beta - \delta + 0.5)j & (i,j) \in I_1 \\ (\delta - 1)i + 0.5j & (i,j) \in I_2 \\ (-1+\beta)i + 0.5j & (i,j) \in I_3 \end{cases}$$

Proof. As we explained at the overview, we assumed that

$$X \approx N^{\beta},\ Y \approx N^{0.5},\ \text{and}\ e \approx N,$$

from which by using (6) we derive

$$M = \frac{d - d_0}{\tilde{d}} \approx \frac{N^{\beta}}{N^{\delta}} = N^{\beta - \delta}.$$

Substituting these for each bound of $\mathbf{b}^*_{i,j}$, we have

$$\begin{aligned} \mathrm{eval}(i,j) &\stackrel{\mathrm{def}}{=} \log_N \left(\frac{|\mathbf{b}^*_{i,j}|}{(eM)^m} \right) = \log_N (eM)^{-j} X^i Y^j \approx \beta i - (\beta - \delta + 0.5)j && \text{for } (i,j) \in I_1, \\ \mathrm{eval}(i,j) &\stackrel{\mathrm{def}}{=} \log_N \left(\frac{|\mathbf{b}^*_{i,j}|}{(eM)^m} \right) = \log_N (eM)^{-i} X^i Y^j \approx (\delta - 1)i + 0.5j && \text{for } (i,j) \in I_2, \text{ and} \\ \mathrm{eval}(i,j) &\stackrel{\mathrm{def}}{=} \log_N \left(\frac{|\mathbf{b}^*_{i,j}|}{(eM)^m} \right) \approx \log_N e^{-j} X^i Y^j \approx (-1+\beta)i + 0.5j && \text{for } (i,j) \in I_3. \end{aligned}$$

□

A.2 Some Calculations on eval(I)

This section shows how to get our conditions (28) and (30) in detail.

First we state the expressions for $\mathrm{eval}(I_1)$, $\mathrm{eval}(I_2)$, and $\mathrm{eval}(I_3)$ derived from (23) and Proposition 1.

$$\mathrm{eval}(I_1) = \sum_{i=0}^{m} \sum_{j=0}^{i} \mathrm{eval}(i,j) = \sum_{i=0}^{m} \sum_{j=0}^{i} (i\beta - (\beta - \delta + 0.5)j), \tag{42}$$

$$\mathrm{eval}(I_2) = \sum_{i=0}^{m} \sum_{j=i+1}^{i+\lfloor \tau m \rfloor} \mathrm{eval}(i,j) = \sum_{i=0}^{m} \sum_{j=i+1}^{i+\lfloor \tau m \rfloor} ((\delta - 1)i + 0.5j), \tag{43}$$

$$\mathrm{eval}(I_3) = \sum_{i=\lfloor Bm \rfloor}^{m} \sum_{j=i+\lceil \tau m \rceil}^{\lfloor 2(1-\beta)i \rfloor} \mathrm{eval}(i,j) = \sum_{i=\lfloor Bm \rfloor}^{m} \sum_{j=i+\lceil \tau m \rceil}^{\lfloor 2(1-\beta)i \rfloor} ((-1+\beta) + 0.5j). \tag{44}$$

Here we use B to denote $\frac{\tau}{1-2\beta}$. Figure 6 shows an area of I_1, I_2 and I_3. By definition, these are sets of integral points in each I_k. The cross point of the line $j = 2(1-\beta)$ and $j = i + \tau m$, i.e., $(\frac{\tau}{1-2\beta}m, \frac{2(1-\beta)\tau}{1-2\beta}m)$, exists below the vertex point $(m, 2(1-\beta)m)$ when $1 - 2\beta - \tau > 0$. Thus, we separate our discussion into the case of $1 - 2\beta - \tau > 0$ and that of $1 - 2\beta - \tau \leq 0$.

First we consider the case of $1 - 2\beta - \tau > 0$. This case corresponds $I_3 \neq \phi$ and the following

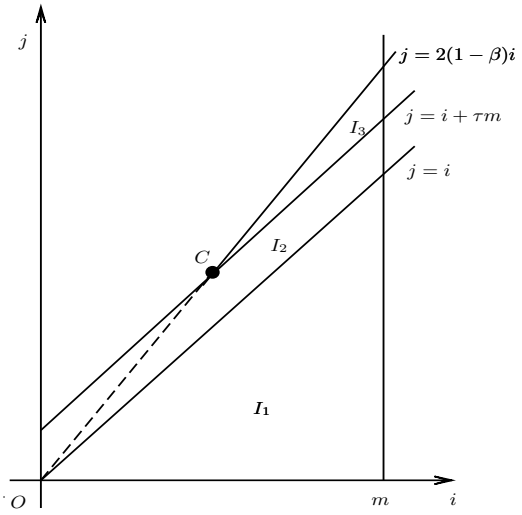

Figure 6: I_1, I_2 and I_3

holds.

$$\begin{aligned}
\mathrm{eval}(I_1) &= \sum_{i=0}^{m}\sum_{j=0}^{i}\mathrm{eval}(i,j) = \frac{1}{6}(\beta+\delta-0.5)m^3+o(m^3),\\
\mathrm{eval}(I_2) &= \sum_{i=0}^{m}\sum_{j=i+1}^{i+\lfloor\tau m\rfloor}\mathrm{eval}(i,j) = \frac{\tau}{4}(2\delta+\tau-1)m^3+o(m^3),\\
\mathrm{eval}(I_3) &= \sum_{i=\lfloor Bm\rfloor}^{m}\sum_{j=i+\lceil\tau m\rceil}^{\lfloor 2(1-\beta)i\rfloor}\mathrm{eval}(i,j) = -\frac{1}{12}\frac{(1-2\beta-\tau)^3}{1-2\beta}m^3+o(m^3).
\end{aligned}$$

From these we derive the following (25) for $\mathrm{eval}(I) \stackrel{\mathrm{def}}{=} \mathrm{eval}(I_1)+\mathrm{eval}(I_2)+\mathrm{eval}(I_3)$.

$$\mathrm{eval}(I) = \left(\frac{1}{6}(\beta+\delta-0.5)+\frac{\tau}{4}(2\delta+\tau-1)-\frac{1}{12}\frac{(1-2\beta-\tau)^3}{1-2\beta}\right)m^3+o(m^3) \tag{25}$$

Now we focus on its major term assuming that m is sufficiently large (as, e.g., [1, 6]). Then we may consider that the condition $\mathrm{eval}(I)<0$ is equivalent to

$$f(\tau) \stackrel{\mathrm{def}}{=} \frac{1}{6}(\beta+\delta-0.5)+\frac{\tau}{4}(2\delta+\tau-1)-\frac{1}{12}\frac{(1-2\beta-\tau)^3}{1-2\beta} < 0.$$

Here $f(\tau)$ is minimised when τ is $\tau_0 = \sqrt{2(1-2\beta)(\beta-\delta)}$ within the range of $0 \le \tau < 1-2\beta$.

Substituting this we have

$$f(\tau_0) = \frac{1}{3}\sqrt{2(1-2\beta)(\beta-\delta)}(\delta-\beta) - \frac{1}{3}\beta^2 + \frac{1}{2}\beta + \frac{1}{6}\delta - \frac{1}{6}.$$

Then the condition $0 \leq \tau_0 < 1-2\beta$ is equivalent to $1-4\beta+2\delta > 0$ and $\beta \geq \delta \geq 0$. Hence we can construct the lattice from indecies having $\mathrm{eval}(I) < 0$ when $f(\tau_0) < 0$ and m is sufficiently large. Rearranging these conditions we have (3).

We next consider the case of $1-2\beta-\tau \leq 0$. This is a case of $I_3 = \phi$, thus

$$\mathrm{eval}(I) = \mathrm{eval}(I_1) + \mathrm{eval}(I_2) = \left(\frac{1}{6}(\beta+\delta-0.5) + \frac{\tau}{4}(2\delta+\tau-1)\right) m^3 + o(m^3).$$

By similar argument, the condition $\mathrm{eval}(I) < 0$ is approximately equivalent to the following condition when m is sufficiently large.

$$\frac{1}{6}(\beta+\delta-0.5) + \frac{\tau}{4}(2\delta+\tau-1) < 0.$$

Note that this left-hand side is minimized when $\tau = \frac{1-2\delta}{2}$. Hence by substituting this, we have

$$\delta < \frac{5}{6} - \frac{1}{3}\sqrt{1+6\beta}.$$

This is equivalent to the range given by [6].