ISSN 1342-2812

Research Reports on Mathematical and Computing Sciences

A tight bound of the largest eigenvalue of sparse random graph

Tomonori Ando

March 2009, C–262

Department of
Mathematical and
Computing Sciences
Tokyo Institute of Technology

SERIES C: Computer Science

A tight bound of the largest eigenvalue of sparse random graph

Tomonori Ando
Reserch Report C-262

Abstract

We analyze the largest eigenvalue and eigenvector for the adjacency matrices of sparse random graph. Let λ_1 be the largest eigenvalue of an n-vertex graph, and v_1 be its corresponding normalized eigenvector. For graphs of average degree $d \log n$, where d is a large enough constant, we show $\lambda_1 = d \log n + 1 \pm o(1)$ and $\langle \mathbf{1}, v_1 \rangle = \sqrt{n}\left(1 - \Theta\left(\frac{1}{\log n}\right)\right)$. It shows a limitation of the existing method of analyzing spectral algorithms for NP-hard problems.

1 Introduction

$G_{n,p}$ is the random graph model in which there is an n-vertex graph, and every edge is included independently with probability p. For such a graph G, we study the largest eigenvalue of its adjacency matrix and a corresponding eigenvector.

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be eigenvalues of the adjacency matrix of G, and $\lambda = \max[\lambda_2, |\lambda_n|]$. Note that np is the average degree of G. Then it is well known that $\lambda_1 \geq np$ and $\lambda = \Omega(\sqrt{np})$. For $p = \Theta(1)$, it is known that $\lambda_1 = np + 1 - 2p + \epsilon$ with $|\epsilon| = O\left(\frac{1}{\sqrt{n}}\right)$. This was shown by Furedi and Komlos [2]. Let Δ be the maximum degree of G. Krivelevich and Sudakov show that $\lambda_1 = (1 + o(1)) \max\{\sqrt{\Delta}, np\}$ [5]. This result does not depend on size of np. Moreover, for $np = \Omega(\log n)$ and $np = O\left(n^{\frac{1}{3}}/(\log n)^{\frac{5}{3}}\right)$, it is shown that $\lambda = O(\sqrt{np})$ and $\lambda_1 = np + \epsilon$ with $|\epsilon| = O(\sqrt{np})$ by Feige and Ofek [1]. We extend the $\lambda_1 = np + 1 - 2p + \epsilon$ result to lower values of p in the $G_{n,p}$ model. Let v_1 be the corresponding normalized eigenvector of λ_1, and define $v_\phi = \frac{1}{\sqrt{n}}\overrightarrow{\mathbf{1}}$.

Theorem 1 Let d be a sufficiently large constant, and $p = \frac{d \log n}{n}$. There exists $c > 0$ such that

$$|\lambda_1 - (d \log n + 1)| \leq O\left(\frac{1}{\sqrt{d \log n}}\right)$$

with probability at least $1 - \frac{1}{n^c}$.

Theorem 2 Let d be a sufficiently large constant, and $p = \frac{d \log n}{n}$. There exists $c > 0$ such that

$$\left| |\langle v_\phi, v_1 \rangle| - \left(1 - \frac{1}{2d \log n}\right) \right| \leq O\left(\frac{1}{(d \log n)^{\frac{3}{2}}}\right)$$

with probability at least $1 - \frac{1}{n^c}$.

By the method of analyzing spectral algorithms in [3, 4], it is shown that those spectral algorithms output the wrong assignment for at most $O\left(\frac{1}{p}\right)$ variables. The method is based on the similarity between v_ϕ and v_1. Our result $\langle v_\phi, v_1 \rangle = 1 - \Theta\left(\frac{1}{\log n}\right)$ shows that the method in [3, 4] never prove that spectral algorithms output the wrong assignment for at most $o\left(\frac{1}{p}\right)$ variables for $p = \Theta\left(\frac{\log n}{n}\right)$.

1.1 Notation

The number of vertices in a graph is denoted by n, and p denotes the probability of an edge in the $G_{n,p}$ model. Let d be sufficiently large constant. We assume that $p = d \log n / n$. Let G be a random graph taken from $G_{n,p}$ and A be its adjacency matrix of G. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A, and $v_1, \ldots, v_n$ be its corresponding orthonormal eigenvectors. Define $v_\phi = \frac{1}{\sqrt{n}} \overrightarrow{\mathbf{1}}$. For regular graphs $v_1 = v_\phi$. Though G is nearly regular and v_1 is close to v_ϕ, v_1 differs from v_ϕ slightly.

2 The analysis

To bound λ_1 and $|\langle v_\phi, v_1 \rangle|$, we use the following three properties of a random graph.

Lemma 3

$$\Pr\left[\left|v_\phi^t A v_\phi - d \log n\right| \geq \frac{2d \log n}{n^{\frac{1}{4}}}\right] \leq \exp\left(-2\sqrt{n} d \log n\right).$$

Lemma 4 For every $c > 0$ there exists $k > 0$ such that

$$\Pr\left[{}^\exists i \in \{2, \ldots, n\}, |\lambda_i| > k\sqrt{d \log n}\right] \leq \frac{1}{n^c}.$$

Lemma 5 $f(G)$ denotes $\sum_{v\in V}(deg_v-(n-1)p)^2$. Then we have

$$\Pr\left[\left|f(G)-n^2p\right|>2n^{\frac{2}{3}}\right]\le 3\exp\left(-\frac{n^{\frac{1}{15}}}{512}\right).$$

It is not hard to prove Lemma 3. Since $\frac{n}{2}v_\phi^t A v_\phi$ is equal to the number of edge of G, we can prove by Chernoff bound. Lemma 4 was shown in [1] by Feige and Ofek. They also showed essentially the same Lemma 5.

Since $v_1,\ \dots\ ,v_n$ are orthonormal, there exist $\alpha_1,\ \dots\ ,\alpha_n$ with $\sum_{i=1}^n\alpha_i^2=1$ such that $v_\phi=\sum_{i=1}^n\alpha_i v_i$. Furthermore, there exist $\alpha,\ \beta$ and w with $\alpha^2+\beta^2=1,\ w\perp v_\phi$ and $\|w\|=1$ such that $v_1=\alpha v_\phi+\beta w$. Note that $\alpha=\alpha_1=\langle v_\phi,v_1\rangle$; hence, we have $\sum_{i=2}^n\alpha_i^2=\beta^2$. Our goal is to show that β is reasonably close to 0 with high probability, but it is in fact nonnegligible with high probability. We prove this by Lemma 6 and Lemma 9.

Lemma 6 There exists $c>0$ such that

$$\beta^2\le\frac{1}{d\log n}+O\left(\frac{1}{(d\log n)^{\frac{3}{2}}}\right)$$

with probability at least $1-\frac{1}{n^c}$.

Proof Note that

$$f(G)=n\left\|\left(A-\frac{n-1}{n}d\log nE\right)v_\phi\right\|^2.$$

Thus from Lemma 5, we have the following with high probability

$$\left\|\left(A-\frac{n-1}{n}d\log nE\right)v_\phi\right\|^2\le d\log n+2n^{-\frac{1}{3}}.$$

On the other hand, substituting $v_\phi=\sum_{i=1}^n\alpha_i v_i$ and using Lemma 4, the left hand side of the above is bounded as following by

$$\begin{aligned}\left\|\left(A-\frac{n-1}{n}d\log nE\right)v_\phi\right\|^2 &= \sum_{i=1}^n\alpha_i^2\left(\frac{n-1}{n}d\log n-\lambda_i\right)^2\\ &\ge \sum_{i=2}^n\alpha_i^2\left(\frac{n-1}{n}d\log n-k\sqrt{d\log n}\right)^2.\end{aligned}$$

Hence, we have the following for some constant $c_1 > 0$.

$$\begin{aligned}\sum_{i=2}^{n}\alpha_i^2 &\le \frac{d\log n + 2n^{-\frac{1}{3}}}{(\frac{n-1}{n}d\log n - k\sqrt{d\log n})^2} \\ &\le \frac{d\log n + 2n^{-\frac{1}{3}}}{(d\log n - 2k\sqrt{d\log n})^2} \\ &= \frac{\left(d\log n - 4k\sqrt{d\log n} + 4k^2\right) + 4k\sqrt{d\log n} - 4k^2 + 2n^{-\frac{1}{3}}}{d\log n(\sqrt{d\log n} - 2k)^2} \\ &\le \frac{1}{d\log n} + \frac{c_1 k}{(d\log n)^{\frac{3}{2}}}.\end{aligned}$$

□

Corollary 7 There exists $c > 0$ such that

$$\begin{aligned}\alpha^2 &\ge 1 - \frac{1}{d\log n} - \frac{c_1 k}{(d\log n)^{\frac{3}{2}}}, \\ |\alpha| &\ge 1 - \frac{1}{2d\log n} - \frac{2c_1 k}{(d\log n)^{\frac{3}{2}}} \quad \text{and} \\ |\beta| &\le \frac{1}{\sqrt{d\log n}} + \frac{c_1 k}{2d\log n}\end{aligned}$$

with probability at least $1 - \frac{1}{n^c}$.

Proof Thus $\alpha^2 + \beta^2 = 1$, α^2 is bounded. If we suppose that the second inequality or the third is not true, the first inequality conflict. □

Lemma 8 There exists $c > 0$ such that

$$\lambda_1 \le d\log n + 1 + O\left(\frac{1}{\sqrt{d\log n}}\right)$$

with probability at least $1 - \frac{1}{n^c}$.

Proof We assume that α is positive. The negative case can be argued similarly. Let $w' = Av_\phi - d\log n v_\phi$. By calculating the inner product of $Av_\phi = d\log n v_\phi + w'$ and v_1, we have

$$\begin{aligned}\alpha\lambda_1 &= \alpha d\log n + \langle v_1, w'\rangle \\ &= \alpha d\log n + \langle \alpha v_\phi, w'\rangle + \langle \beta w, w'\rangle \\ &\le \alpha d\log n + \alpha\langle v_\phi, w'\rangle + |\beta|\|w\|\|w'\|.\end{aligned}$$

By lemma 3, the upper bound of $\langle v_\phi, w'\rangle$ is

$$\langle v_\phi, w'\rangle = v_\phi^t A v_\phi - d\log n \le \frac{d\log n}{n^{\frac{1}{4}}}.$$

Thus by using the bound of $|\beta|$ stated in Corollary 7, we have

$$\alpha\lambda_1 \le \alpha d\log n + \frac{d\log n}{n^{\frac{1}{4}}} + \left(\frac{1}{\sqrt{d\log n}} + \frac{c_1 k}{2d\log n}\right)\|w'\|. \tag{1}$$

Note here that $w' = Av_\phi - d\log n v_\phi$. Thus by using Lemma 5, we have

$$\begin{aligned}
\|w'\| &= \left\| Av_\phi - \frac{n-1}{n} d\log n v_\phi - \frac{1}{n} d\log n v_\phi \right\| \\
&\le \left\| Av_\phi - \frac{n-1}{n} d\log n v_\phi \right\| + \left\| \frac{1}{n} d\log n v_\phi \right\| \\
&\le \sqrt{d\log n} + n^{-\frac{1}{3}} + \frac{d\log n}{n}
\end{aligned}$$

Substituting this to (1), we obtain

$$\begin{aligned}
\alpha\lambda_1 &\le \alpha d\log n + \frac{d\log n}{n^{\frac{1}{4}}} + \left(\frac{1}{\sqrt{d\log n}} + \frac{c_1 k}{2d\log n}\right)\left(\sqrt{d\log n} + n^{-\frac{1}{3}} + \frac{d\log n}{n}\right) \\
&= \alpha d\log n + O\left(\frac{1}{\sqrt{d\log n}}\right) + \left(\frac{1}{\sqrt{d\log n}} + O\left(\frac{1}{d\log n}\right)\right)\left(\sqrt{d\log n} + O(1)\right) \\
&= \alpha d\log n + 1 + O\left(\frac{1}{\sqrt{d\log n}}\right).
\end{aligned}$$

By Corollary 7, the lower bound of $|\alpha|$ is $1 - \frac{1}{d\log n}$. Therefore we have the following for some constant $c_2 > 0$.

$$\lambda_1 \le d\log n + 1 + \frac{c_2 k}{\sqrt{d\log n}}.$$

□

Lemma 9 There exists $c > 0$ such that

$$\beta^2 \ge \frac{1}{d\log n} - O\left(\frac{1}{(d\log n)^{\frac{3}{2}}}\right)$$

with probability at least $1 - \frac{1}{n^c}$.

Proof The argument is similar to the proof of Lemma 6. First by Lemma 5, we have

$$\left\| \left(A - \frac{n-1}{n} d \log n E \right) v_\phi \right\|^2 \geq d \log n - 2n^{-\frac{1}{3}}.$$

Then using Lemma 4, the left hand side of the above is bounded as follows.

$$\left\| \left(A - \frac{n-1}{n} d \log n E \right) v_\phi \right\|^2 \leq \alpha_1^2 \left(\frac{n-1}{n} d \log n - \lambda_1 \right)^2 + \sum_{i=2}^{n} \alpha_i^2 \left(d \log n + k\sqrt{d \log n} \right)^2.$$

Now using the bound $\lambda_1 \leq d \log n + 1 + \frac{c_2 k}{\sqrt{d \log n}}$, we have the following for some constant $c_3 > 0$.

$$\begin{aligned}
\sum_{i=2}^{n} \alpha_i^2 \left(d \log n + k\sqrt{d \log n} \right)^2 &\geq d \log n - 2n^{-\frac{1}{3}} - \alpha_1^2 \left(\frac{n-1}{n} d \log n - \lambda_1 \right)^2 \\
\sum_{i=2}^{n} \alpha_i^2 &\geq \frac{d \log n - 2n^{-\frac{1}{3}} - \alpha_1^2 \left(1 + \frac{c_2 k}{\sqrt{d \log n}} + \frac{d \log n}{n} \right)^2}{(d \log n + k\sqrt{d \log n})^2} \\
&= \frac{d \log n - O(1)}{d \log n (\sqrt{d \log n} + k)^2} \\
&= \frac{\left(d \log n + 2k\sqrt{d \log n} + k^2 \right) - 2k\sqrt{d \log n} - k^2 - O(1)}{d \log n (\sqrt{d \log n} + k)^2} \\
&\geq \frac{1}{d \log n} - \frac{c_3 k}{(d \log n)^{\frac{3}{2}}}.
\end{aligned}$$

□

Corollary 10 There exists $c > 0$ such that

$$\begin{aligned}
\alpha^2 &\leq 1 - \frac{1}{d \log n} + \frac{c_3 k}{(d \log n)^{\frac{3}{2}}} \quad , \\
|\alpha| &\leq 1 - \frac{1}{2d \log n} + \frac{c_3 k}{2(d \log n)^{\frac{3}{2}}} \quad \text{and} \\
|\beta| &\geq \frac{1}{\sqrt{d \log n}} - \frac{c_3 k}{d \log n}
\end{aligned}$$

with probability at least $1 - \frac{1}{n^c}$.

Proof The proof of this lemma is similar to Corollary 7. Thus we can use $1 - \alpha^2 = (1 + |\alpha|)(1 - |\alpha|) \leq 2(1 - |\alpha|)$, the second inequality can be proved more easy. □

Lemma 11 There exists $c > 0$ such that

$$\lambda_1 \geq d \log n + 1 - O\left(\frac{1}{\sqrt{d \log n}}\right)$$

with probability at least $1 - \frac{1}{n^c}$.

Proof Let w'' denote $Av_\phi - (v_\phi^t A v_\phi) v_\phi$. w'' and v_ϕ are orthogonal. Thus, by computing $\|w''\|^2$, we derive

$$\|Av_\phi\|^2 = (v_\phi^t A v_\phi)^2 + \|w''\|^2. \tag{2}$$

We estimate the right hand side of the above. $w' = Av_\phi - d \log n v_\phi$, hence,

$$w'' + (v_\phi^t A v_\phi - d \log n) v_\phi = w'.$$

By triangle inequality,

$$\|w''\| + \|(v_\phi^t A v_\phi - d \log n) v_\phi\| \geq \|w'\|.$$

Lemma 5 states $\|w'\| \geq \sqrt{d \log n} - n^{-\frac{1}{3}} - \frac{d \log n}{n}$. On the other hand, since Lemma 3, $\|(v_\phi^t A v_\phi - d \log n) v_\phi\| \leq \frac{2d \log n}{n^{\frac{1}{4}}}$. By Lemma3, we have

$$\begin{aligned} \|w''\| &\geq \sqrt{d \log n} - n^{-\frac{1}{3}} - \frac{d \log n}{n} - \frac{2d \log n}{n^{\frac{1}{4}}} \\ &\geq \sqrt{d \log n} - \frac{3d \log n}{n^{\frac{1}{4}}}. \end{aligned}$$

Then,

$$\|w''\|^2 \geq d \log n - \frac{6(d \log n)^2}{n^{\frac{1}{4}}}.$$

On the other hand, again using Lemma 3, we have $v_\phi^t A v_\phi \geq d \log n - \frac{2d \log n}{n^{\frac{1}{4}}}$. Hence we have from (2) that

$$\begin{aligned} \|Av_\phi\|^2 &\geq \left(1 - \frac{4}{n^{\frac{1}{4}}}\right)(d \log n)^2 + d \log n - \frac{6(d \log n)^2}{n^{\frac{1}{4}}} \\ &\geq (d \log n)^2 + d \log n - \frac{10(d \log n)^2}{n^{\frac{1}{4}}}. \end{aligned} \tag{3}$$

Since $v_\phi = \sum_{i=1}^n \alpha_i v_i$, we have $\|Av_\phi\|^2 = \sum_{i=1}^n \alpha_i^2 \lambda_i^2$. By Lemma 4, we have $\lambda_i^2 \leq k^2 d \log n$ for all $i \geq 2$. By lemma 6, we have $\sum_{i=2}^n \alpha_i^2 \leq \frac{1}{d \log n} + \frac{c_1 k}{(d \log n)^{\frac{3}{2}}}$. So, we have

$$\begin{aligned} \|Av_\phi\|^2 &= \sum_{i=1}^n \alpha_i^2 \lambda_i^2 \\ &\leq \alpha_1^2 \lambda_1^2 + k^2 d \log n \left(\frac{1}{d \log n} + \frac{c_1 k}{(d \log n)^{\frac{3}{2}}}\right). \end{aligned} \tag{4}$$

Hence we have from (3) and (4) that

$$\begin{aligned}\alpha^2\lambda_1^2 &\geq (d\log n)^2 + d\log n - \frac{10(d\log n)^2}{n^{\frac{1}{4}}} - k^2 d\log n\left(\frac{1}{d\log n} + \frac{c_1 k}{(d\log n)^{\frac{3}{2}}}\right) \\ &\geq (d\log n)^2 + d\log n - 2k^2.\end{aligned}$$

By Corollary 10, we have

$$\begin{aligned}\lambda_1^2 &\geq \frac{(d\log n)^2 + d\log n - 2k^2}{1 - \frac{1}{d\log n} + \frac{c_3 k}{(d\log n)^{\frac{3}{2}}}} \\ &= \frac{\left((d\log n)^2 - d\log n + c_3 k\sqrt{d\log n}\right) + 2d\log n - c_3 k\sqrt{d\log n} - 2k^2}{1 - \frac{1}{d\log n} + \frac{c_3 k}{(d\log n)^{\frac{3}{2}}}} \\ &= (d\log n)^2 + \frac{\left(2d\log n - 2 + \frac{2c_3 k}{\sqrt{d\log n}}\right) + 2 - \frac{2c_3 k}{\sqrt{d\log n}} - c_3 k\sqrt{d\log n} - 2k^2}{1 - \frac{1}{d\log n} + \frac{c_3 k}{(d\log n)^{\frac{3}{2}}}} \\ &= (d\log n)^2 + 2d\log n - \frac{\Theta(\sqrt{d\log n})}{\Theta(1)}.\end{aligned}$$

For some constant $c_4 > 0$,

$$\lambda_1^2 \geq (d\log n)^2 + 2d\log n - c_4 k\sqrt{d\log n}.$$

Therefore,

$$\lambda_1 \geq d\log n + 1 - \frac{c_4 k}{\sqrt{d\log n}}.$$

□

2.1 Proof of Lemma 5

Let $deg_i = \sum_{j=1}^{n} a_{i,j}$, and $f(G) = \sum_{i=1}^{n}(deg_i - (n-1)p)^2$. In this section, we prove $f(G)$ is close to its expectation. The lemma is proved in Section 5 of [1]. We give a strict proof of the lemma. Let $d_i^* = \sum_{j=1}^{n} a_{j,i+j}$. Let $a'_{j,i+j}$ be

$$a'_{j,i+j} = \begin{cases} a_{j,i+j} & (d_i^* \leq knp) \\ 0 & (d_i^* > knp) \end{cases},$$

and $deg'_i = \sum_{j=1}^{n} a'_{i,j}$. Let k be a fixed value . We assume $k = \Omega(1)$. Let D_i be

$$D_i = \begin{cases} (deg'_i - (n-1)p)^2 & (deg'_i \leq k(n-1)p) \\ (k(n-1)p - (n-1)p)^2 & (deg'_i > k(n-1)p) \end{cases},$$

and $D = \sum_{i=1}^{n} D_i$. Note that if $deg_i \le k(n-1)p$ and $d_i^* \le knp$ for any i, D is equal to $f(G)$. Therefore, it suffices to show D is close to its expectation and $D = f(G)$ with high probability. Let $X_i = \mathrm{E}\left[D | d_1^*, \ \ldots \ , d_i^*\right]$ for $i = 0, \ \ldots \ , \lceil \frac{n-1}{2} \rceil$. $X_0, \ \ldots \ , X_{\lceil \frac{n-1}{2} \rceil}$ are the Doob martingale. To use Azuma-Hoeffding inequality, we calculate upper bound of $|X_i - X_{i+1}|$. Let $x_{j,i+1} = \left| \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_i^*\right] - \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_{i+1}^*\right] \right|$. Note that $|X_i - X_{i+1}| \le \sum_{j=1}^{n} x_{j,i+1}$. If d_{i+1}^* is decided, D_j is changed by only $a_{j,j+i+1}, a_{j-i-1,j}$. We divide the analysis of $x_{j,i+1}$ into the three cases, and calculate upper bound of $x_{j,i+1}$.

- Case 1: $a_{j,j+i+1} + a_{j-i-1,j} = 2$ by d_{i+1}^*
  Note that

$$
\begin{aligned}
& \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_i^*\right] \\
= & \sum_{d_{i+1}^*} \Pr\left[d_{i+1}^* | d_1^*, \ \ldots \ , d_i^*\right] \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_i^*, d_{i+1}^*\right] \\
= & \sum_{x \in \{0,1\}} \sum_{y \in \{0,1\}} \Pr\left[a_{j,j+i+1} = x, a_{j-i-1,j} = y | d_1^*, \ \ldots \ , d_i^*\right] \\
& \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = x, a_{j-i-1,j} = y\right] \\
= & \sum_{x \in \{0,1\}} \sum_{y \in \{0,1\}} \sum_{z=1}^{n} \Pr\left[a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg_j' = x + y + z | d_1^*, \ \ldots \ , d_i^*\right] \\
& \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg_j' = x + y + z\right].
\end{aligned}
$$

  In this case, $a_{j,j+i+1} + a_{j-i-1,j} = 2$ by d_{i+1}^*. Moreover, d_k^* and d_l^* are independent for any $k \neq l$. Thus, we have for any integer z

$$
\begin{aligned}
& \sum_{x \in \{0,1\}} \sum_{y \in \{0,1\}} \Pr\left[a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg_j' = x + y + z | d_1^*, \ \ldots \ , d_i^*\right] \\
= & \Pr\left[a_{j,j+i+1} = 1, a_{j-i-1,j} = 1, deg_j' = z + 2 | d_1^*, \ \ldots \ , d_{i+1}^*\right].
\end{aligned}
$$

  Note that for any integer x, y, z

$$
\begin{aligned}
& \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg_j' = x + y + z\right] \\
= & \begin{cases} (x + y + z - (n-1)p)^2 & (x + y + z \le k(n-1)p) \\ (k(n-1)p - (n-1)p)^2 & (x + y + z > k(n-1)p) \end{cases}.
\end{aligned}
$$

  Hence we have

$$
\begin{aligned}
& \Big| \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg_j' = x + y + z\right] \\
& - \mathrm{E}\left[D_j | d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = 1, a_{j-i-1,j} = 1, deg_j' = z + 2\right] \Big| \\
\le & (k(n-1)p - (n-1)p)^2 - (k(n-1)p - 2 - (n-1)p)^2 \le 4knp.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
& \mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_i^*\right] - \mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_{i+1}^*\right] \\
= & \sum_{x\in\{0,1\}} \sum_{y\in\{0,1\}} \sum_{z=1}^{n} \Pr\left[a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg'_j = x+y+z | d_1^*, \ \ldots \ , d_i^*\right] \\
& \Big(\mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg'_j = x+y+z\right] \\
& -\mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = 1, a_{j-i-1,j} = 1, deg'_j = z+2\right]\Big) \\
\leq & \ 4knp.
\end{aligned}$$

- Case 2: $a_{j,j+i+1} + a_{j-i-1,j} = 1$ by d_{i+1}^*
  $x_{j,i+1} \leq 4knp$ is proved in a similar way to Case 1.

- Case 3: $a_{j,j+i+1} + a_{j-i-1,j} = 0$ by d_{i+1}^*
  Note that

$$\begin{aligned}
& \mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_i^*\right] - \mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_{i+1}^*\right] \\
= & \sum_{(x,y)\neq(0,0)} \sum_{z=1}^{n} \Pr\left[a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg'_j = x+y+z | d_1^*, \ \ldots \ , d_i^*\right] \\
& \Big(\mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = x, a_{j-i-1,j} = y, deg'_j = x+y+z\right] \\
& -\mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_i^*, a_{j,j+i+1} = 0, a_{j-i-1,j} = 0, deg'_j = z\right]\Big).
\end{aligned}$$

Since $\sum_{(x,y)\neq(0,0)} \Pr\left[a_{j,j+i+1} = x, a_{j-i-1,j} = y\right]$ is at most $2p$, we have $x_{v,i+1} \leq 8knp^2$. Case 1 and Case 2 cause at most $2knp$ times for each i. Hence, we have

$$\begin{aligned}
|X_i - X_{i+1}| & \leq \sum_{j=1}^{n} \left|\mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_i^*\right] - \mathrm{E}\left[D_j|d_1^*, \ \ldots \ , d_{i+1}^*\right]\right| \\
& \leq 2knp \cdot 4knp + n \cdot 8knp^2 \leq 16k^2n^2p^2.
\end{aligned}$$

By Azuma-Hoeffding inequality, we have

$$\Pr[|X_{\lceil \frac{n-1}{2} \rceil} - X_0| > \lambda] \leq 2\exp\left(-\frac{\lambda^2}{2 \cdot \lceil \frac{n-1}{2} \rceil \cdot 256(kd\log n)^4}\right).$$

Setting $\lambda = n^{\frac{2}{3}}$, $k = \frac{n^{\frac{1}{15}}}{d\log n}$, the value of X_0 is

$$\begin{aligned}
X_0 = \mathrm{E}[D] & \leq \mathrm{E}[f(G)] \leq nd\log n \\
X_0 = \mathrm{E}[D] & \geq \mathrm{E}[f(G)] - n^3 \Pr[D \neq f(G)] \leq nd\log n - 2(d\log n)^2.
\end{aligned}$$

$\Pr[D \neq f(G)]$ is at most $2 \cdot 2^{-\frac{n-1}{n} n^{\frac{1}{15}}}$. Thus,

$$\begin{aligned} \Pr[|D - nd\log n| > 2n^{\frac{2}{3}}] &\leq \Pr[|X_n - X_0| > n^{\frac{2}{3}}] \\ &\leq 2\exp\left(-\frac{n^{\frac{1}{15}}}{512}\right). \end{aligned}$$

Since $\Pr\left[D \neq f(G)\right] \leq 2 \cdot 2^{-\frac{n-1}{n} n^{\frac{1}{15}}}$, we have $\Pr\left[|f(G) - n^2p| > 2n^{\frac{2}{3}}\right] \leq 3\exp\left(-\frac{n^{\frac{1}{15}}}{512}\right)$.

References

[1] U. Feige and E. Ofek, Spectral Techniques Applied to Sparse Random Graph, Random Structures and Algorithms, 27(2), (2005), 251-275

[2] Z. Furedi and J. Komlos, The eigenvalues of random symmetric matrices, Combinatorica 1(3) (1981), 233-241.

[3] N. Alon and N. Kahale. A Spectral Technique for Coloring Random 3-Colorable Graphs. DIMACS TR-94-35, 1994.

[4] A. Flaxman. A spectral technique for random satisfiable 3CNF formulas. In Proc. 14th ACM-SIAM Symposium on Discrete Algorithms, pp. 357-363, 2003.

[5] M. Krivelevich and B. Sudakov, The largest eigenvalue of sparse random graphs, Combin Probab Comput 12(2003),61-72.