ISSN 1342-2

Research Reports on Mathematical and Computing Sciences

Intuitionistic fragment of the $\lambda\mu$-calculus

Naosuke Matsuda

March 2015, C–282

Department of
Mathematical and
Computing Sciences
Tokyo Institute of Technology

SERIES C: Computer Science

INTUITIONISTIC FRAGMENT OF THE $\lambda\mu$-CALCULUS

NAOSUKE MATSUDA

Abstract. Parigot [8] gave an elegant proof system "the $\lambda\mu$-calculus" for classical logic, and gave a correspondence between classical logic and some kind of program structures which the λ-calculus cannot capture. Although the correspondence is due to the power of inference rules of the $\lambda\mu$-calculus, some of those rules are admissible in some intuitionistic proof systems. It therefore must be natural to ask whether it is possible to extend the correspondence between intuitionistic logic and program structures. In this paper, we give an answer to this question by giving a natural subsystem of the $\lambda\mu$-calculus which corresponds to intuitionistic logic and investigating it.

1. Introduction

It is well-known that Gentzen's proof system **NJ** of intuitionistic logic has a close connection called "*Curry-Howard correspondence*" with the λ-*calculus* [1] (see [10]). Because λ-terms can be viewed as a kind of programs (see [9]), this correspondence is also viewed as a correspondence between **NJ** and programs. An extension of this correspondence was given by Griffin [2]. He discovered that the double negation elimination law corresponds to some kind of program structures which the λ-calculus cannot capture. His idea was refined by Parigot [8]. Parigot introduced a natural deduction style proof system called "the $\lambda\mu$-*calculus*" for classical logic and gave a proofs-as-programs correspondence specifically. His system has more expressive power than the λ-calculus actually. In [10], for example, the $\lambda\mu$-terms "**catch** a **in** M" and "**throw** N **to** a" satisfying the following property are given (see [10] in detail).

$$\textbf{catch } a \textbf{ in } C[\textbf{throw } N \textbf{ to } a]$$
is reduced to N by Parigot's reduction $\rhd_c$.

With these terms, the $\lambda\mu$-calculus can capture the catch-throw program.

Then, why can the $\lambda\mu$-calculus capture more various program structures? It is an answer from proof theoretic aspect that the $\lambda\mu$-calculus admits more various unnatural derivations than the λ-calculus (**NJ**). Because a calculation process is given by derivation reductions from an unnatural derivation into a natural derivation, flexible [2] proof systems can capture more various program structures. For example, the behaviour of the term **catch** a **in** C[**throw** N **to** a] written above is

[1] In this paper, "the λ-calculus" means the system which consists of: (1) The set of all λ-terms, (2) Its (simply) typing system, and (3) β-reduction.

[2] Here, "the system **L** is flexible" means "**L** admits various unnatural derivations". This property may be a disadvantage if we view this system as a proof system. However, it becomes a great advantage if we view this system as a calculation model.

due to the following derivation reductions.

$$\dfrac{\dfrac{\dfrac{\neg\alpha\vdash\neg\alpha \quad \begin{array}{c}\vdots\\ \Gamma\vdash\Delta,\alpha\end{array}}{\neg\alpha,\Gamma\vdash\Delta,\bot}}{\neg\alpha,\Gamma\vdash\Delta,\beta\to\gamma} \quad \begin{array}{c}\vdots\\ \Pi\vdash\Theta,\beta\end{array}}{\begin{array}{c}\neg\alpha,\Gamma,\Pi\vdash\Delta,\Theta,\gamma\\ \vdots\end{array}} \quad \rhd \quad \dfrac{\dfrac{\neg\alpha\vdash\neg\alpha \quad \begin{array}{c}\vdots\\ \Gamma\vdash\Delta,\alpha\end{array}}{\neg\alpha,\Gamma\vdash\Delta,\bot}}{\begin{array}{c}\neg\alpha,\Gamma\vdash\Delta,\gamma\\ \vdots\end{array}}$$

$$\dfrac{\dfrac{\neg\alpha\vdash\neg\alpha \quad \begin{array}{c}\vdots\\ \Gamma\vdash\Delta,\alpha\end{array}}{\neg\alpha,\Gamma\vdash\Delta,\bot}}{\begin{array}{c}\Gamma\vdash\Delta,\alpha\\ \vdots\end{array}} \quad \rhd \quad \begin{array}{c}\vdots\\ \Gamma\vdash\Delta,\alpha\\ \vdots\end{array}$$

We cannot construct these derivations in **NJ** actually.

On the other hand, there also exist proof systems for intuitionistic logic which admit the above derivations and derivation reductions. For example, the system **LJ′** [3] has **LK**-like inference rules and admits the above figures. Therefore it must be natural to ask whether it is possible to extend the correspondence between intuitionistic logic and program structures. Then two questions arise:

(♯1) Is there a natural subset of the set Λ_μ of all $\lambda\mu$-terms, which corresponds to intuitionistic logic and can capture more program structures than the λ-calculus?

(♯2) If such subsystem exists, what kind of program structures can it capture (or what kind of program structures can't it capture)?

In section 2, we give an answer to the former question by giving an intuitionistic fragments $\Lambda_\mu^{\mathrm{Int}}$ of Λ_μ which has many good properties such as:

- $\Lambda_\mu^{\mathrm{Int}}$ is closed under Parigot's reduction.
- $\Lambda_\mu^{\mathrm{Int}}$ can capture some program structures which λ-terms cannot capture. In particular, the terms **catch** a **in** M and **throw** N **to** a are both included in this subset in some sense.

In the remainder of this section, we describe some basic concepts we are going to use.

1.1. **Preliminary: The $\lambda\mu$-calculus.** Suppose that a countable set P of atomic types (propositional variables) is given. Then the set Fml of all types (or formulas) based on P and the set $\mathrm{Fml}_\to$ of all function types (implicational formulas) based on P are defined by the following grammar.

$$\begin{aligned}&\alpha,\beta\in\mathrm{Fml} ::= p \mid \bot \mid (\alpha\to\beta)\\ &\alpha,\beta\in\mathrm{Fml}_\to ::= p \mid (\alpha\to\beta)\\ &p\in\mathrm{P}\end{aligned}$$

[3]**LJ′** is an intuitionistic proof system obtained by restricting Gentzen's **LK** as follows: the right-implication rule is allowed only when the principal formulas are the only formulas in the succedents of the lower sequents. See [11, p. 52] in detail.

Parentheses are omitted in such a way that $\alpha \to \beta \to \gamma$ denotes $(\alpha \to (\beta \to \gamma))$.

We use metavariables $\varphi, \psi, \alpha, \beta, \ldots$ to stand for arbitrary types, and $p, q, r, \ldots$ for arbitrary atomic types. We write $\alpha \equiv \beta$ if α is syntactically equal to β.

Suppose that a countable set V_λ of λ-variables and a countable set V_μ of μ-variables are given. Then, the set Λ of all λ-*terms* based on V_λ and the set Λ_μ of all $\lambda\mu$-*terms* based on $\mathrm{V}_\lambda \cup \mathrm{V}_\mu$ are defined by the following grammar.

$$\begin{aligned} &M, N \in \Lambda ::= x \mid (MN) \mid (\lambda x.M) \\ &M, N \in \Lambda_\mu ::= x \mid (MN) \mid (\lambda x.M) \mid (aM) \mid (\mu a.M) \\ &x \in \mathrm{V}_\lambda,\ a \in \mathrm{V}_\mu \end{aligned}$$

Parentheses are omitted in such a way that $MNPQ$ denotes $(((MN)P)Q)$ and $\lambda x.MN$ denotes $(\lambda x.(MN))$. We use the abbreviation such as

$$\lambda x_1 \ldots x_n.M \equiv (\lambda x_1.(\lambda x_2.(\ldots(\lambda x_n.M)\ldots))).$$

We use metavariables $M, N, P, Q, \ldots$ to stand for arbitrary terms, $x, y, z, \ldots$ for arbitrary λ-variables, and $a, b, c, \ldots$ for arbitrary μ-variables. We also use notations such as $[N/x]M$ (the substitution of N for free occurrences of x in M), $\mathrm{V}_{\lambda(\mu)}(M)$ (the set of all $\lambda(\mu)$-variables in M), $\mathrm{FV}_{\lambda(\mu)}(M)$ (the set of all free $\lambda(\mu)$-variables in M), $\mathrm{BV}_{\lambda(\mu)}(M)$ (the set of all bound $\lambda(\mu)$-variables in M), $\mathrm{Sub}(M)$ (the set of all subterms of M), $M \equiv N$ (M is syntactically equal to N).

A typing judgement is an expression of the form $\Gamma \vdash M : \alpha,\ \Delta$ where:

- Γ is a set of pairs of λ-variables and types written $x : \beta$.
- M is a $\lambda\mu$-term.
- α is a type.
- Δ is a set of pairs of μ-variables and types written $a : \gamma$.

We also assume the following properties.

- If $x : \alpha \in \Gamma$ and $\alpha \not\equiv \beta$, then $x : \beta \notin \Gamma$.
- If $a : \alpha \in \Delta$ and $\alpha \not\equiv \beta$, then $a : \beta \notin \Delta$.

We use abbreviations such as $\Gamma, \Delta = \Gamma \cup \Delta$ and $x : \alpha = \{x : \alpha\}$. We also write $\vdash M : \alpha,\ \Delta \ \equiv\ \emptyset \vdash M : \alpha,\ \Delta$ and $\Gamma \vdash M : \alpha \ \equiv\ \Gamma \vdash M : \alpha,\ \emptyset$.

Definition 1.1. ($\mathbf{TA}_{\lambda\mu}$ [8, Subsection 3.3 (Typed $\lambda\mu$-calculus)]) The typing system $\mathbf{TA}_{\lambda\mu}$ for Λ_μ consists of the following rules.
[axiom] (Var) $x : \alpha \vdash x : \alpha$
[Inference rule]

$$\frac{\Gamma_1 \vdash M : \alpha \to \beta,\ \Delta_1 \qquad \Gamma_2 \vdash\ N : \alpha,\ \Delta_2}{\Gamma_1, \Gamma_2 \vdash MN : \beta,\ \Delta_1, \Delta_2}\ (\mathrm{App})$$

$$\frac{\Gamma \vdash M : \beta,\ \Delta}{\Gamma \setminus \{x : \alpha\} \vdash \lambda x.M : \alpha \to \beta,\ \Delta}\ (\mathrm{Abs})$$

$$\frac{\Gamma \vdash M : \alpha,\ \Delta}{\Gamma \vdash aM : \bot,\ \Delta \cup \{a : \alpha\}}\ (\text{App-}\mu) \qquad \frac{\Gamma \vdash M : \bot,\ \Delta}{\Gamma \vdash \mu a.M : \alpha,\ \Delta \setminus \{a : \alpha\}}\ (\text{Abs-}\mu)$$

Furthermore, the typing system $\mathbf{TA}_\lambda$ for Λ is defined by the rules (Var), (App), (Abs).

Remark 1.2. This formulation is a little different from Parigot's original one ([8]). However this formulation has the following nice property.

$$x_1 : \alpha_1, \ldots, x_n : \alpha_n \vdash M : \varphi, a_1 : \beta_1, \ldots, a_m : \beta_m \text{ is provable in } \mathbf{TA}_{\lambda\mu}$$
$$\Longrightarrow \ \mathrm{FV}_\lambda(M) = \{x_1, \ldots, x_n\} \text{ and } \mathrm{FV}_\mu(M) = \{a_1, \ldots, a_m\}.$$

1.2. **Preliminary: Tree sequent calculus.** There are many proof systems for intuitionistic logic which are more flexible than **NJ**. Kashima's tree sequent calculus **TLJ** [4] is one of the most flexible proof systems. We are going to extract a natural subset of Λ_μ by use of this system.

Let $\mathbb{N}^{<\omega}$ be the set of all finite sequences of natural numbers and $*$ be the concatenation function on $\mathbb{N}^{<\omega}$, that is, $\langle n_1, \ldots, n_k\rangle * \langle m_1, \ldots, m_l\rangle = \langle n_1, \ldots, n_k, m_1, \ldots, m_l\rangle$. We write the empty sequence as ϵ. We use the abbreviation such as $\langle n\rangle = n$ if it causes no confusion. We define a partial order $\preceq$ on $\mathbb{N}^{<\omega}$ as follows.

$$\overline{n} \preceq \overline{m} \iff \exists \overline{k} \in \mathbb{N}^{<\omega} \text{ such that } \overline{m} = \overline{n} * \overline{k}$$

We write $\overline{n} \prec \overline{m}$ if both $\overline{n} \preceq \overline{m}$ and $\overline{n} \neq \overline{m}$ hold, and write $\overline{n} \prec_1 \overline{m}$ if there exists a natural number k such that $\overline{m} = \overline{n} * k$. A *tree* $\mathcal{T}$ is a finite subset of $\mathbb{N}^{<\omega}$ which satisfies:

- $\epsilon \in \mathcal{T}$.
- $\overline{n} \in \mathcal{T},\ \overline{m} \preceq \overline{n} \implies \overline{m} \in \mathcal{T}$.

We say $\overline{n}$ is a node of $\mathcal{T}$ if $\overline{n} \in \mathcal{T}$. We also say $\overline{n}$ is a parent-node of $\overline{m}$ (or $\overline{m}$ is a child-node of $\overline{n}$) if $\overline{n} \prec_1 \overline{m}$, and say $\overline{n}$ is an ancestor of $\overline{m}$ (or $\overline{m}$ is a descendant of $\overline{n}$) if $\overline{n} \prec \overline{m}$.

Definition 1.3 (**TLJ**).

(1) A *tree sequent* is an expression of the form $\Gamma \vdash^{\mathcal{T}} \Delta$ where:
   - $\mathcal{T}$ is a tree.
   - Γ and Δ are sets of pairs of nodes of $\mathcal{T}$ and formulas written $\overline{n} : \alpha$.

   We abbreviate $\emptyset \vdash^{\mathcal{T}} \Delta$ to $\vdash^{\mathcal{T}} \Delta$.

   A tree sequent is viewed as a tree in which each node is labelled with a sequent. For example, the tree sequent

$$\epsilon : \alpha_1,\ \epsilon : \alpha_2,\ \langle 1\rangle : \beta_1,\ \langle 2\rangle : \gamma_1,\ \langle 2,1\rangle : \delta_1,\ \langle 2,1\rangle : \delta_2$$
$$\vdash^{\mathcal{T}} \epsilon : \alpha_3, \langle 1\rangle : \beta_2,\ \langle 2\rangle : \gamma_2,\ \langle 2\rangle : \gamma_3,\ \langle 2\rangle : \gamma_4,\ \langle 2,2\rangle : \varepsilon_1,\ \langle 2,2\rangle : \varepsilon_2$$

$$(\mathcal{T} = \{\epsilon, \langle 1\rangle, \langle 1,1\rangle, \langle 2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\})$$

   can be viewed as the tree in figure 1.

(2) The tree sequent calculus **TLJ** is a proof system which derives tree sequents, and consists of the following rules.

   [axiom] (Id) $\overline{n} : \alpha \vdash^{\mathcal{T}} \overline{n} : \alpha$ $\quad$ ($\bot$) $\overline{n} : \bot \vdash^{\mathcal{T}} \overline{n} : \alpha$

   [inference rule]

$$\frac{\Gamma_1 \vdash^{\mathcal{T}} \Gamma_2}{\Delta_1, \Gamma_1 \vdash^{\mathcal{T}} \Gamma_2, \Delta_2} \text{ (Weakening)}$$

$$\frac{\Gamma_1 \vdash^{\mathcal{T}} \Delta_1,\ \overline{n} : \alpha \quad \overline{n} : \alpha,\ \Gamma_2 \vdash^{\mathcal{T}} \Delta_2}{\Gamma_1, \Gamma_2 \vdash^{\mathcal{T}} \Delta_1, \Delta_2} \text{ (Cut)}$$

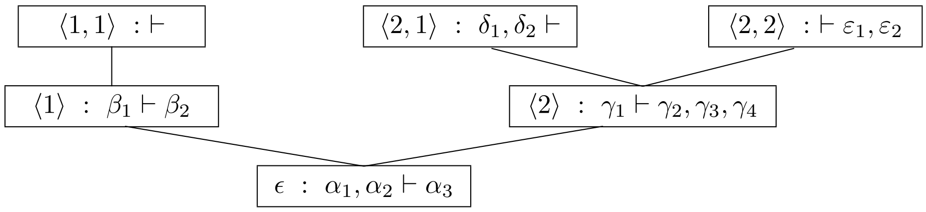

FIGURE 1

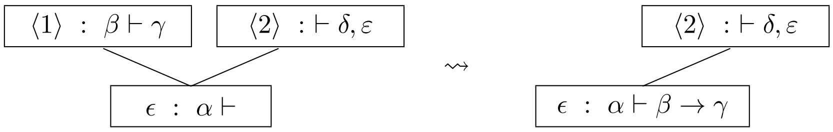

FIGURE 2. $(\vdash\to)$

$$\frac{\overline{n} * k : \alpha,\ \Gamma \vdash^{\mathcal{T}} \Delta}{\overline{n} : \alpha,\ \Gamma \vdash^{\mathcal{T}} \Delta}\ (h \vdash) \quad \frac{\Gamma \vdash^{\mathcal{T}} \Delta,\ \overline{n} : \alpha}{\Gamma \vdash^{\mathcal{T}} \Delta,\ \overline{n} * k : \alpha}\ (\vdash h)$$

$$\frac{\Gamma \vdash^{\mathcal{T}} \Delta}{\Gamma \vdash^{\mathcal{T} \cup \{\overline{n}\}} \Delta}\ (\text{Grow}) \quad (\mathcal{T} \cup \{\overline{n}\} \text{ is also a tree})$$

$$\frac{\Gamma_1 \vdash^{\mathcal{T}} \Delta_1,\ \overline{n} : \alpha \qquad \overline{n} : \beta,\ \Gamma_2 \vdash^{\mathcal{T}} \Delta_2}{\overline{n} : \alpha \to \beta,\ \Gamma_1, \Gamma_2 \vdash^{\mathcal{T}} \Delta_1, \Delta_2}\ (\to\vdash)$$

$$\frac{\overline{n} * k : \alpha,\ \Gamma \vdash^{\mathcal{T}} \Delta,\ \overline{n} * k : \beta}{\Gamma \vdash^{\mathcal{T} \setminus \{n*k\}} \Delta,\ \overline{n} : \alpha \to \beta}\ (\vdash\to)$$

In the last figure, because $\Gamma \vdash^{\mathcal{T} \setminus \{n*k\}} \Delta,\ \overline{n} : \alpha \to \beta$ is also a tree sequent, the node $\overline{n} * k$ and its descendants do not occur in the lower sequent (see also figure 2). We say $\overline{n} * k$ is the *eigen-node* of this $(\vdash\to)$-rule.

We write $\vdash_{\mathbf{TLJ}} \varphi$ (φ is provable in **TLJ**) if $\vdash_{\mathbf{TLJ}} \vdash^{\{\epsilon\}} \epsilon : \varphi$.

Theorem 1.4 ([3, 4, 6])**.** φ is provable in **TLJ** if and only if φ is intuitionistically valid.

TLJ can be reformed into a natural deduction like proof system as follows.

Definition 1.5 (**TNJ**)**.** The system **TNJ** consists of the rules (Id), (Weakening), $(h \vdash)$, $(\vdash h)$, (Grow), $(\vdash\to)$ and the following rules.

$$\frac{\Gamma \vdash^{\mathcal{T}} \Delta,\ \overline{n} : \bot}{\Gamma \vdash^{\mathcal{T}} \Delta,\ \overline{n} : \alpha}\ (\text{Absurd}) \quad \frac{\Gamma_1 \vdash^{\mathcal{T}} \Delta_1,\ \overline{n} : \alpha \to \beta \qquad \Gamma_2 \vdash^{\mathcal{T}} \Delta_2,\ \overline{n} : \alpha}{\Gamma_1, \Gamma_2 \vdash^{\mathcal{T}} \Delta_1, \Delta_2,\ \overline{n} : \beta}\ (\mathbf{MP})$$

Then the following theorem can be proved easily.

Theorem 1.6. ([6, Theorem 6.2.]) The following conditions are all equivalent.

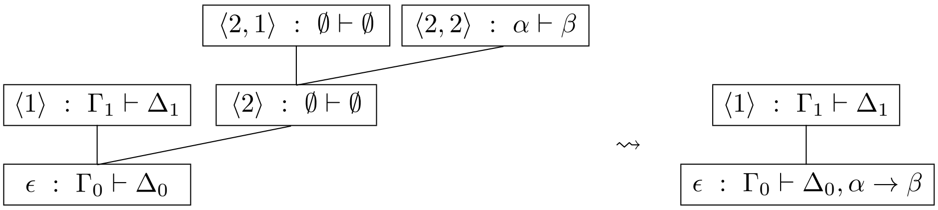

FIGURE 3. $(\vdash\to)^*$

(1) φ is provable in **TLJ**.
(2) φ is provable in **TNJ**.
(3) φ is intuitionistically valid.

Lemma 1.7. The following rule is admissible in **TLJ** (see also figure 3).

$$\frac{\overline{n} * i * \overline{m} : \alpha,\ \Gamma \vdash^{\mathcal{T}} \Delta,\ \overline{n} * i * \overline{m} : \beta}{\Gamma \vdash^{\mathcal{T} \setminus \{\overline{k} \ \mid \ \overline{k} \succeq \overline{n} * i\}} \Delta,\ \overline{n} : \alpha \to \beta} \ (\vdash\to)^*$$

($n * i$ and its descendants do not occur in $\Gamma \cup \Delta$)

Proof. As an example, we give a proof figure which makes up the figure written in figure 3 (and we can make up the other cases similarly).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle 2,2\rangle : \alpha,\ \Gamma_0, \Gamma_1 \vdash^{\mathcal{T}_3} \Delta_0, \Delta_1,\ \langle 2,2\rangle : \beta}{\Gamma_0, \Gamma_1 \vdash^{\mathcal{T}_2} \Delta_0, \Delta_1,\ \langle 2\rangle : \alpha \to \beta}\ (\vdash\to)}{\langle 2,1\rangle : \alpha,\ \Gamma_0, \Gamma_1 \vdash^{\mathcal{T}_2} \Delta_0, \Delta_1,\ \langle 2\rangle : \alpha \to \beta,\ \langle 2,1\rangle : \beta}\ (\text{Weakening})}{\Gamma_0, \Gamma_1 \vdash^{\mathcal{T}_1} \Delta_0, \Delta_1,\ \langle 2\rangle : \alpha \to \beta}\ (\vdash\to)}{\langle 2\rangle : \top,\ \Gamma_0, \Gamma_1 \vdash^{\mathcal{T}_1} \Delta_0, \Delta_1,\ \langle 2\rangle : \alpha \to \beta}\ (\text{Weakening})}{\Gamma_0, \Gamma_1 \vdash^{\mathcal{T}_0} \Delta_0, \Delta_1,\ \epsilon : \top \to \alpha \to \beta}\ (\vdash\to) \quad \dfrac{\langle 2\rangle : p \vdash^{\mathcal{T}_1} \langle 2\rangle : p}{\vdash^{\mathcal{T}_0} \epsilon : \top}\ (\vdash\to)}{\Gamma_0, \Gamma_1 \vdash^{\mathcal{T}_0} \Delta_0, \Delta_1,\ \epsilon : \alpha \to \beta}\ (\mathbf{MP})$$

Here,

$$\top \equiv p \to p, \quad \mathcal{T}_0 = \{\epsilon, \langle 1\rangle\}, \quad \mathcal{T}_1 = \{\epsilon, \langle 1\rangle, \langle 2\rangle\},$$
$$\mathcal{T}_2 = \{\epsilon, \langle 1\rangle, \langle 2\rangle, \langle 2,1\rangle\}, \quad \mathcal{T}_3 = \{\epsilon, \langle 1\rangle, \langle 2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}.$$

□

2. INTUITIONISTIC FRAGMENT OF THE $\lambda\mu$-CALCULUS

$\mathbf{TA}_{\lambda\mu}$ and **TNJ** are very similar systems. We do not mind saying that the sole difference between them is the label condition of $(\vdash\to)$-rule:

$$\frac{\overline{n} * k : \alpha,\ \Gamma \vdash^{\mathcal{T}} \Delta,\ \overline{n} * k : \beta}{\Gamma \vdash^{\mathcal{T} \setminus \{n * k\}} \Delta,\ \overline{n} : \alpha \to \beta} \ (\vdash\to)$$

($\overline{n} * k$ and its descendant do not occur in the lower sequent)

With this observation, we obtain the following subset $\Lambda_\mu^{\mathrm{Int}}$ of Λ_μ, which corresponds to intuitionistic logic.

Definition 2.1. (1) We say $x \in \mathrm{FV}_\lambda(M)$ occurs *classically* in M if there exist $a \in \mathrm{FV}_\mu(M)$ and $aN \in \mathrm{Sub}(M)$ such that $x \in \mathrm{FV}_\lambda(N)$. We say x occurs *intuitionistically* in M if x does not occurs classically in M [4] . We define $\mathrm{V}_\lambda^{\mathrm{Int}}(M)$ as the set $\{x \mid x \text{ occurs intuitionistically in } M\}$.

(2) $\Lambda_\mu^{\mathrm{Int}}$ is defined as follows.

$$
\begin{aligned}
&(\Lambda_\mu^{\mathrm{Int}}0)\ \mathrm{V}_\lambda \subseteq \Lambda_\mu^{\mathrm{Int}}\\
&(\Lambda_\mu^{\mathrm{Int}}1)\ M, N \in \Lambda_\mu^{\mathrm{Int}} \implies MN \in \Lambda_\mu^{\mathrm{Int}}\\
&(\Lambda_\mu^{\mathrm{Int}}2)\ M \in \Lambda_\mu^{\mathrm{Int}},\ x \in \mathrm{V}_\lambda^{\mathrm{Int}}(M) \implies \lambda x.M \in \Lambda_\mu^{\mathrm{Int}}\\
&(\Lambda_\mu^{\mathrm{Int}}3)\ M \in \Lambda_\mu^{\mathrm{Int}}, a \in \mathrm{V}_\mu \implies aM \in \Lambda_\mu^{\mathrm{Int}}\\
&(\Lambda_\mu^{\mathrm{Int}}4)\ M \in \Lambda_\mu^{\mathrm{Int}}, a \in \mathrm{V}_\mu \implies \mu a.M \in \Lambda_\mu^{\mathrm{Int}}
\end{aligned}
$$

We call a term in $\Lambda_\mu^{\mathrm{Int}}$ a $\lambda\mu^{\mathrm{Int}}$-term.

Intuitively speaking, a closed $\lambda\mu$-term M is in $\Lambda_\mu^{\mathrm{Int}}$ if M does not have subterms of the form $\mu a.(\ldots \lambda x.(\ldots a(\ldots x \ldots)))$.

Example 2.2.

(1) $\lambda x.\rho a.(ax)(by) \in \Lambda_\mu^{\mathrm{Int}}$.

(2) $\lambda x.\mu a.a(x(\lambda y.\mu b.ay))$, whose principal type is $((\alpha \to \beta) \to \alpha) \to \alpha$ (Peirce's law), is not in $\Lambda_\mu^{\mathrm{Int}}$ because y occurs classically in $\mu b.ay$.

Theorem 2.3 (Main theorem)**.** Typed $\lambda\mu^{\mathrm{Int}}$-terms correspond to $[\to, \bot]$-fragment of intuitionistic logic, that is:

(1) If $\varphi \in \mathrm{Fml}$ is intuitionistically valid, then there is a closed $\lambda\mu^{\mathrm{Int}}$-term M_φ such that $\vdash M_\varphi : \varphi$ is provable in $\mathbf{TA}_{\lambda\mu}$.

(2) Let M be a closed $\lambda\mu^{\mathrm{Int}}$-term. If $\vdash M : \varphi$ is provable in $\mathbf{TA}_{\lambda\mu}$, then φ is intuitionistically valid.

Proof of theorem 2.3 - (1). The set

$$\{\alpha \mid \text{ there is a closed } \lambda\mu^{\mathrm{Int}}\text{-term } M \text{ such that } \vdash M : \alpha \text{ is provable in } \mathbf{TA}_{\lambda\mu}\}$$

is closed under modus ponens. In addition, we can see that $\vdash \lambda xy.x : \alpha \to \beta \to \alpha$, $\vdash \lambda xyz.xz(yz) : (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$ and $\vdash \lambda x.\mu a.ax : \bot \to \alpha$ are all provable in $\mathbf{TA}_{\lambda\mu}$. □

To show theorem 2.3-(2), we prepare some notions.

Definition 2.4. Let M be a closed $\lambda\mu^{\mathrm{Int}}$-term following the Barendregt's convention [5]. It is well-known that $\mathrm{BV}_\lambda(M) \cup \{\epsilon\}$ can also be viewed as a tree-structure by giving the following partial order $\prec^M$ (see also figure 4).

- $x \prec^M y$ if λy occurs in the scope of λx.
- $\epsilon \prec^M x$.

[4]Note that we say x occurs intuitionistically in M even if x does not occur in M.

[5]M follows the Barendregt's convention if all bound variables in M are all different from each other.

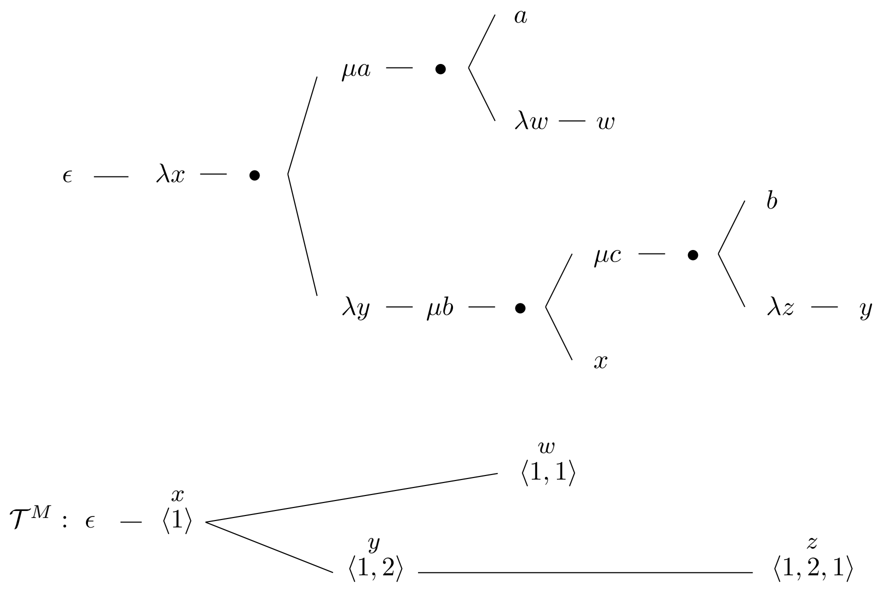

FIGURE 4. $\lambda x.(\mu a.a(\lambda w.w))(\lambda y.\mu b.(\mu c.b(\lambda z.y))x)$

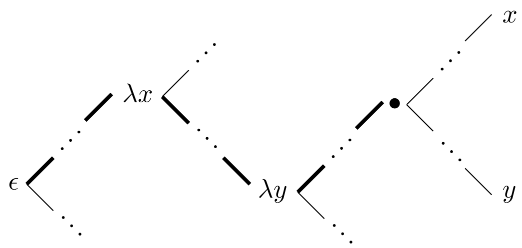

FIGURE 5

We write this tree as $\mathcal{T}^M$. $\prec^M$ is simply written as $\prec$ if it causes no confusion. The node which corresponds to x is simply written as x.

Lemma 2.5. Let M be a closed $\lambda\mu^{\mathrm{Int}}$-term following the Barendregt's convention. Then the following properties hold in $\mathcal{T}^M$.

(1) For each subterm N of M, $\prec$ is a total order on $\mathrm{FV}_\lambda(N)$.
(2) If $x \in \mathrm{FV}_\lambda(N),\ y \in \mathrm{BV}_\lambda(N)$ for some subterm N of M, then $x \prec y$.
(3) Let $\mathrm{FV}^a_\lambda(N) = \{x \in \mathrm{FV}_\lambda(N) \mid \exists aP \in \mathrm{Sub}(N) \text{ such that } x \text{ occurs in } P\}$. Then $\prec$ is a total order on $\mathrm{FV}^a_\lambda(N)$.

Proof. First, we note that "$x \prec y$" means "λx occurs in the path from the root-node ϵ to λy when we view M as a tree" (see figure 5). Then the lemma is obvious because M is a closed term and follows the Barendregt's convention. □

Definition 2.6. For each $N \in \mathrm{Sub}(M)$ and $a \in \mathrm{FV}_\mu(N)$, we assign the node $[N]$ of $\mathcal{T}^M$ and assign the node $[a]^N$ as follows (see also figure 6).

$$[N] = \text{ the greatest }^6\text{element of } \mathrm{FV}_\lambda(N) \cup \{\epsilon\}.$$
$$[a]^N = \text{ the greatest element of } \mathrm{FV}^a_\lambda(N) \cup \{\epsilon\}.$$

From lemma 2.5, we can assign such node, for each $N \in \mathrm{Sub}(M)$ and $a \in \mathrm{FV}_\mu(N)$.

Then we return to the proof of the main theorem.

Proof of theorem 2.3 - (2). Let M be a closed $\lambda\mu^{\mathrm{Int}}$-term such that there is a $\mathbf{TA}_{\lambda\mu}$-derivation Σ of $\vdash M : \varphi$. We can assume that M follows the Barendregt's convention. For each $N \in \mathrm{Sub}(M)$, we define the subtree $T^M(N)$ of $\mathcal{T}^M$ as

$$T^M(N) = \langle \{x \mid \exists y \in \mathrm{FV}_\lambda(N) \text{ such that } x \preceq^M y\}, \prec^M \rangle$$

(see also figure 6). Then we can show, by induction on the size of derivation, the following statement.

$$\Gamma \vdash N : \psi,\ \Delta \text{ occurs in } \Sigma$$
$$\Longrightarrow\ [\Gamma] \vdash^{T^M(N)} [N] : \psi,\ [\Delta]^N \text{ is provable in } \mathbf{TNJ}$$

Here,

$$[\Gamma] = \{[x] : \alpha \mid x : \alpha \in \Gamma\}\ ,\ [\Delta]^N = \{[a]^N : \alpha \mid a : \alpha \in \Delta\}.$$

(1) Suppose $\Gamma \vdash N : \psi,\ \Delta$ is an axiom, that is,

$$\Gamma = \{x : \psi\},\ N \equiv x,\ \Delta = \emptyset$$

for some x. Then we obtain $[x] : \psi \vdash^{T^M(x)} [x] : \psi$ by (Id).

(2) Suppose $\Gamma \vdash N : \psi,\ \Delta$ is derived by (App).

$$\frac{\begin{array}{cc}\vdots & \vdots \\ \Gamma_1 \vdash P : \alpha \to \psi,\ \Delta_1 & \Gamma_2 \vdash Q : \alpha,\ \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2,\ PQ : \psi}\ (\mathrm{App})$$

$$(\Gamma = \Gamma_1 \cup \Gamma_2,\ N \equiv PQ,\ \Delta = \Delta_1 \cup \Delta_2)$$

By induction hypothesis,

$$[\Gamma_1] \vdash^{T^M(P)} [P] : \alpha \to \psi,\ [\Delta_1]^P$$

and

$$[\Gamma_2] \vdash^{T^M(Q)} [Q] : \alpha,\ [\Delta_2]^Q$$

are both provable in **TNJ**. From lemma 2.5 and the definition of $[\cdot]$, we obtain:

- $T^M(P), T^M(Q) \subseteq T^M(PQ)$.
- $[P], [Q] \preceq [PQ]$.
- $a \in \mathrm{FV}_\mu(PQ) \implies [a]^P, [a]^Q \preceq [a]^{PQ}$.

[6] In the following argument, the sentence "the greatest element of A" means "the greatest element of A with respect to $\prec^M$".

Then we can construct the following derivation.

$$\dfrac{\dfrac{\dfrac{\vdots}{[\Gamma_1] \vdash^{T^M(P)} [P] : \alpha \to \psi,\ [\Delta_1]^P}}{[\Gamma_1] \vdash^{T^M(PQ)} [P] : \alpha \to \psi,\ [\Delta_1]^P}\,(\text{Grow})}{[\Gamma_1] \vdash^{T^M(PQ)} [PQ] : \alpha \to \psi,\ [\Delta_1]^{PQ}}\,(\vdash h) \quad \dfrac{\dfrac{\dfrac{\vdots}{[\Gamma_2] \vdash^{T^M(Q)} [Q] : \alpha,\ [\Delta_2]^Q}}{[\Gamma_2] \vdash^{T^M(PQ)} [Q] : \alpha,\ [\Delta_2]^Q}\,(\text{Grow})}{[\Gamma_2] \vdash^{T^M(PQ)} [PQ] : \alpha,\ [\Delta_2]^{PQ}}\,(\vdash h)$$
$$\overline{[\Gamma_1, \Gamma_2] \vdash^{T^M(PQ)} [PQ] : \psi,\ [\Delta_1, \Delta_2]^{PQ}}\ (\mathbf{MP})$$

(3) Suppose $\Gamma \vdash N : \psi,\ \Delta$ is derived by (Abs).

$$\dfrac{\dfrac{\vdots}{\Gamma' \vdash P : \beta,\ \Delta}}{\Gamma' \setminus \{x : \alpha\} \vdash \lambda x.P : \alpha \to \beta,\ \Delta}\ (\text{Abs})$$

$$(\Gamma = \Gamma' \setminus \{x : \alpha\},\ N \equiv \lambda x.P,\ \psi \equiv \alpha \to \beta)$$

By induction hypothesis, $[\Gamma'] \vdash^{T^M(P)} [P] : \beta,\ [\Delta]^P$ is provable in **TNJ**. From lemma 2.5 and the definition of $[\cdot]$, we obtain:

- $T^M(\lambda x.P) \subset T^M(P)$.
- $[x] = [P]$.
- $[\Delta]^{\lambda x.P} = [\Delta]^P$.
- $y \in \mathrm{FV}_\lambda(\lambda x.P) \implies [y] \preceq [\lambda x.P] \prec [x]$.
- $a \in \mathrm{FV}_\mu(P) \implies [a]^P = [a]^{\lambda x.P} \preceq [\lambda x.P] \prec [x]$.

Then we can obtain the following derivation.

$$\dfrac{\dfrac{\dfrac{\vdots}{[\Gamma'] \vdash^{T^M(P)} [P] : \beta,\ [\Delta]^{\lambda x.P}}}{[x] : \alpha,\ [\Gamma' \setminus \{x : \alpha\}] \vdash^{T^M(P)} [P] : \beta,\ [\Delta]^{\lambda x.P}}\,(\text{Weakening})}{[\Gamma' \setminus \{x : \alpha\}] \vdash^{T^M(\lambda x.P)} [\lambda x.P] : \alpha \to \beta,\ [\Delta]^{\lambda x.P}}\ (\vdash\to)^*$$

(4) Suppose $\Gamma \vdash N : \psi,\ \Delta$ is derived by (App-μ).

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash P : \alpha,\ \Delta'}}{\Gamma \vdash aP : \bot,\ \Delta' \cup \{a : \alpha\}}\ (\text{App-}\mu)$$

$$(N \equiv aP,\ \psi \equiv \bot,\ \Delta = \Delta' \cup \{a : \alpha\})$$

By induction hypothesis, $[\Gamma] \vdash^{T^M(P)} [P] : \alpha,\ [\Delta']^P$ is provable in **TNJ**. From lemma 2.5 and the definition of $[\cdot]$, we obtain:

- $T^M(aP) = T^M(P)$.
- $[a]^P \preceq [a]^{aP} = [aP] = [P]$.
- $a \not\equiv b \in \mathrm{FV}_\mu(P) \implies [b]^P = [b]^{aP}$

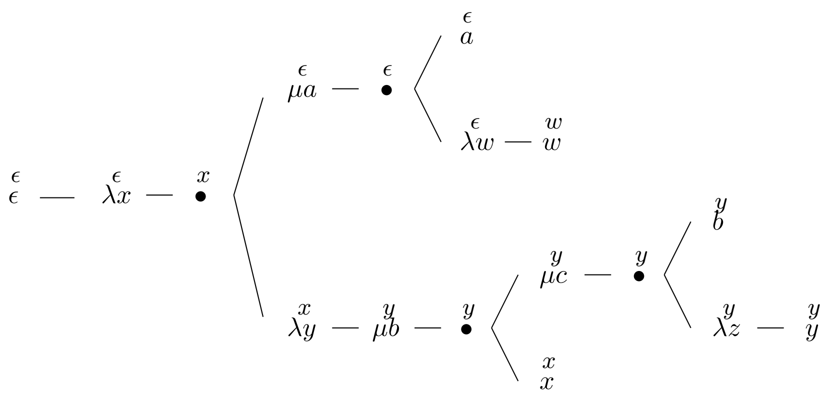

FIGURE 6. $\lambda x.(\mu a.a(\lambda w.w))(\lambda y.\mu b.(\mu c.b(\lambda z.y))x)$.

Then we can construct the following derivation.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ [\Gamma] \vdash^{T^M(aP)} [P] : \alpha,\ [\Delta']^P\end{array}}{[\Gamma] \vdash^{T^M(aP)} [P] : \alpha,\ [a]^P : \alpha,\ [\Delta' \setminus \{a : \alpha\}]^{aP}}\ (\text{Weakening})}{[\Gamma] \vdash^{T^M(aP)} [\Delta' \cup \{a : \alpha\}]^{aP}}\ (\vdash h)}{[\Gamma] \vdash^{T^M(aP)} [aP] : \bot,\ [\Delta' \cup \{a : \alpha\}]^{aP}}\ (\text{Weakening})$$

(5) Suppose $\Gamma \vdash N : \psi,\ \Delta$ is derived by (Abs-μ).

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash P : \bot,\ \Delta'\end{array}}{\Gamma \vdash \mu a.P : \psi,\ \Delta' \setminus \{a : \psi\}}\ (\text{Abs-}\mu)$$

$$(N \equiv \mu a.P,\ \Delta = \Delta' \setminus \{a : \psi\})$$

By induction hypothesis, $[\Gamma] \vdash^{T^M(P)} [P] : \bot,\ [\Delta']^P$ is provable in **TNJ**. From lemma 2.5 and the definition of $[\cdot]$, we obtain:

- $T^M(P) = T^M(\mu a.P)$.
- $[a]^P \preceq [P] = [\mu a.P]$.
- $b \in \mathrm{FV}_\mu(\mu a.P) \implies [b]^{\mu a.P} = [b]^P$.

Then we can construct the following derivation.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ [\Gamma] \vdash^{T^M(\mu a.P)} [\mu a.P] : \bot,\ \Delta'^P\end{array}}{[\Gamma] \vdash^{T^M(\mu a.P)} [\mu a.P] : \bot,\ [a]^P : \bot,\ [\Delta' \setminus \{a : \bot\}]^{\mu a.P}}\ (\text{Weakening})}{[\Gamma] \vdash^{T^M(\mu a.P)} [\mu a.P] : \bot,\ [\Delta' \setminus \{a : \bot\}]^{\mu a.P}}\ (\vdash h)}{[\Gamma] \vdash^{T^M(\mu a.P)} [\mu a.P] : \psi,\ [\Delta' \setminus \{a : \bot\}]^{\mu a.P}}\ (\text{Absurd})$$

□

In addition, we can obtain the following properties (see [1] in detail).

Proposition 2.7. $\Lambda_\mu^{\mathrm{Int}}$ is closed under Parigot's reduction $\triangleright_c$ defined in [8], that is:

$$M \in \Lambda_\mu^{\mathrm{Int}},\ M \triangleright_c N \implies N \in \Lambda_\mu^{\mathrm{Int}}.$$

Proposition 2.8. The $\lambda\mu$-terms

$$\begin{aligned}&\mathbf{catch}\ a\ \mathbf{in}\ M \equiv \mu a.aM,\\ &\mathbf{throw}\ M\ \mathbf{to}\ a \equiv \mu b.aM\end{aligned}$$

are both in $\Lambda_\mu^{\mathrm{Int}}$, if $M \in \Lambda_\mu^{\mathrm{Int}}$.

3. On the $\lambda\rho$-calculus

In [5], Komori gave a natural deduction style proof system called the $\lambda\rho$-calculus for implicational fragment of classical logic (when we treat the $\lambda\rho$-calculus, we treat only implicational formulas). This system give us a more simple logic system than the $\lambda\mu$-calculus.

Definition 3.1. Suppose that a countable set V_λ of λ-variables, and a countable set V_ρ of ρ-variables are given. Then,the set Λ_ρ of all $\lambda\rho$-terms based on $\mathrm{V}_\lambda \cup \mathrm{V}_\mu$ is defined by the following grammar.

$$\begin{aligned}&M, N \in \Lambda_\rho ::= x \mid (MN) \mid (\lambda x.M) \mid (aM) \mid (\rho a.M)\\ &x \in \mathrm{V}_\lambda,\ a \in \mathrm{V}_\rho\end{aligned}$$

Definition 3.2. ($\mathbf{TA}_{\lambda\rho}$ [5, Definition 2.2 (Typed $\lambda\rho$-terms)]) The typing system $\mathbf{TA}_{\lambda\rho}$ of Λ_ρ consists of the following rules.
[axiom] (Var) $x : \alpha \vdash x : \alpha$
[Inference rule]

$$\frac{\Gamma_1 \vdash M : \alpha \to \beta,\ \Delta_1 \qquad \Gamma_2 \vdash\ N : \alpha,\ \Delta_2}{\Gamma_1, \Gamma_2 \vdash MN : \beta,\ \Delta_1, \Delta_2}\ (\mathrm{App})$$

$$\frac{\Gamma \vdash M : \beta,\ \Delta}{\Gamma \setminus \{x : \alpha\} \vdash \lambda x.M : \alpha \to \beta,\ \Delta}\ (\mathrm{Abs})$$

$$\frac{\Gamma \vdash M : \alpha,\ \Delta}{\Gamma \vdash aM : \beta,\ \Delta \cup \{a : \alpha\}}\ (\text{App-}\rho) \qquad \frac{\Gamma \vdash M : \alpha,\ \Delta}{\Gamma \vdash \rho a.M : \alpha,\ \Delta \setminus \{a : \alpha\}}\ (\text{Abs-}\rho)$$

Theorem 3.3 ([5])**.** For each $\varphi \in \mathrm{Fml}_\to$, φ is a tautology if and only if there is a closed $\lambda\rho$-term M such that $\vdash M : \varphi$ is derivable in $\mathbf{TA}_{\lambda\rho}$.

We can also construct an intuitionistic fragment of the $\lambda\rho$-calculus in the same way as the previous section.

Definition 3.4. The set $\Lambda_\rho^{\text{Int}}$ is defined as follows.

$$\begin{aligned}
&(\Lambda_\rho^{\text{Int}}0)\ \mathrm{V}_\lambda \subseteq \Lambda_\rho^{\text{Int}}\\
&(\Lambda_\rho^{\text{Int}}1)\ M, N \in \Lambda_\rho^{\text{Int}} \implies MN \in \Lambda_\rho^{\text{Int}}\\
&(\Lambda_\rho^{\text{Int}}2)\ M \in \Lambda_\rho^{\text{Int}},\ x \in \mathrm{V}_\lambda^{\text{Int}}(M) \implies \lambda x.M \in \Lambda_\rho^{\text{Int}}\\
&(\Lambda_\rho^{\text{Int}}3)\ M \in \Lambda_\rho^{\text{Int}}, a \in \mathrm{V}_\rho \implies aM \in \Lambda_\rho^{\text{Int}}\\
&(\Lambda_\rho^{\text{Int}}4)\ M \in \Lambda_\rho^{\text{Int}}, a \in \mathrm{V}_\rho \implies \rho a.M \in \Lambda_\rho^{\text{Int}}
\end{aligned}$$

We call a term in $\Lambda_\rho^{\text{Int}}$ a $\lambda\rho^{\text{Int}}$-term.

Theorem 3.5. Typed $\lambda\rho^{\text{Int}}$-terms correspond to implicational fragment of intuitionistic logic, that is:

(1) If $\varphi \in \mathrm{Fml}_\to$ is intuitionistically valid, then there is a closed $\lambda\rho^{\text{Int}}$-term M_φ such that $\vdash M_\varphi : \varphi$ is provable in $\mathbf{TA}_{\lambda\rho}$.
(2) Let M be a closed $\lambda\rho^{\text{Int}}$-term. If $\vdash M : \varphi$ is provable in $\mathbf{TA}_{\lambda\rho}$, then φ is intuitionistically valid.

Proof. (1) Obvious.

(2) In the same way as the proof of theorem 2.3. We prove only the case when $\Gamma \vdash N : \psi,\ \Delta$ is derived by (Abs-ρ).

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash P : \psi,\ \Delta'\end{array}}{\Gamma \vdash \rho a.P : \psi,\ \Delta' \setminus \{a : \psi\}}\ (\text{Abs-}\rho)$$

$$(N \equiv \rho a.P,\ \Delta = \Delta' \setminus \{a : \psi\})$$

By induction hypothesis, $[\Gamma] \vdash^{T^M(P)} [P] : \psi,\ [\Delta]^P$ is provable in **TNJ**. From lemma 2.5 and the definition of [ ], we obtain:

- $T^M(P) = T^M(\rho a.P)$.
- $[a]^P \preceq [P] = [\rho a.P]$.
- $b \in \mathrm{FV}_\rho(\rho a.P) \implies [b]^{\rho a.P} = [b]^P$.

Then we can construct the following derivation.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ [\Gamma] \vdash^{T^M(\rho a.P)} [\rho a.P] : \psi,\ [\Delta]^P\end{array}}{[\Gamma] \vdash^{T^M(\rho a.P)} [\rho a.P] : \psi,\ [a]^P : \psi,\ [\Delta \setminus \{a : \psi\}]^{\rho a.P}}\ (\text{Weakening})}{[\Gamma] \vdash^{T^M(\rho a.P)} [\rho a.P] : \psi,\ [\Delta \setminus \{a : \psi\}]^{\rho a.P}}\ (\vdash h)$$

□

4. Conclusion and future work

(A) In this paper, we introduced an intuitionistic fragment of $\lambda\mu$-calculus. We think this system has sufficient strength. In fact, we can construct $\lambda\mu$-terms **catch** a **in** M and **throw** M **to** a in Λ_μ^{Int}, if $M \in \Lambda_\mu^{\text{Int}}$.

However, the question (♯2) has not been solved yet. An answer is going to be given in [1]. Furthermore, in this paper, we do not touch on the reductions of $\lambda\mu$-calculus (or $\lambda\rho$-calculus). We are also going to touch on this topic in [1].

(B) There, of course, exist other subsystems of the $\lambda\mu$-calculus which correspond to intuitionistic logic. For example, we can easily construct more simple one as follows.

Definition 4.1. The subset $\Lambda_\mu^{\mathrm{Int}-}$ of Λ_μ is defined as follows.

$$(\Lambda_\mu^{\mathrm{Int}-}0)\ \mathrm{V}_\lambda \subseteq \Lambda_\mu^{\mathrm{Int}-}$$
$$(\Lambda_\mu^{\mathrm{Int}-}1)\ M, N \in \Lambda_\mu^{\mathrm{Int}-} \implies MN \in \Lambda_\mu^{\mathrm{Int}-}$$
$$(\Lambda_\mu^{\mathrm{Int}-}2)\ M \in \Lambda_\mu^{\mathrm{Int}-},\ \mathrm{BV}_\mu(M) = \emptyset \implies \lambda x.M \in \Lambda_\mu^{\mathrm{Int}-}$$
$$(\Lambda_\mu^{\mathrm{Int}-}3)\ M \in \Lambda_\mu^{\mathrm{Int}-} \implies aM \in \Lambda_\mu^{\mathrm{Int}-}$$
$$(\Lambda_\mu^{\mathrm{Int}-}4)\ M \in \Lambda_\mu^{\mathrm{Int}-} \implies \mu a.M \in \Lambda_\mu^{\mathrm{Int}-}$$

Theorem 4.2. Typed $\lambda\mu^{\mathrm{Int}-}$-terms correspond to implicational fragment of intuitionistic logic, that is:

(1) If $\varphi \in \mathrm{Fml}$ is intuitionistically valid, then there is a closed $\lambda\mu^{\mathrm{Int}-}$-term M_φ such that $\vdash M_\varphi : \varphi$ is provable in $\mathbf{TA}_{\lambda\mu}$.
(2) Let M be a closed $\lambda\mu^{\mathrm{Int}-}$-term. If $\vdash M : \varphi$ is provable in $\mathbf{TA}_{\lambda\mu}$, then φ is intuitionistically valid.

Proof. (2) We can easily see that

$$x_1 : \alpha_1, \ldots, x_n : \alpha_n \vdash M : \varphi, a_1 : \beta_1, \ldots, a_m : \beta_m \text{ is provable in } \mathbf{TA}_{\lambda\mu}$$
$$\implies \alpha_1, \ldots, \alpha_n \vdash \varphi, \beta_1, \ldots, \beta_m \text{ is provable in } \mathbf{LJ}'.$$

□

Although this system is simpler that the $\lambda\mu^{\mathrm{Int}}$-calculus, the $\lambda\mu^{\mathrm{Int}}$-calculus is more flexible than this system. Therefore the $\lambda\mu^{\mathrm{Int}}$-calculus can capture more various program structures than the $\lambda\mu^{\mathrm{Int}-}$-calculus. It therefore must be natural to ask whether there is a natural proof system which is more flexible than $\lambda\mu^{\mathrm{Int}}$-calculus (or $\lambda\mu$-calculus).

$(\sharp 1)'$ Is there a proof system for intuitionistic logic, which system has more flexible inference rules than the $\lambda\mu^{\mathrm{Int}}$-calculus and can capture more program structures?
$(\sharp 2)'$ If it exists, what kind of program structures can it capture?

(C) At last, we should note that Nakano's system [7] has a strong connection to our system. In [1], we are going to touch on the relation between Nakano's system and our system in detail.

References

[1] K. Fujita and R .Kashima and Y. Komori and N. Matsuda. $\lambda\rho$-calculus and catch-thorow mechanism. to appear.
[2] T. G. Griffin. A formulae-as-type notion of control. In *Proceeding of POPL*, pages 47–58, 1990.
[3] I. Hasuo and R. Kashima. Kripke completeness of first-order constructive logics with strong negation. *Logic Journal of IGPL*, 11-6:615–646, 2003.
[4] R. Kashima. An introduction to intuitionistic logic. in Japanese, 2006.
[5] Y. Komori. $\lambda\rho$-calculus ii. *Tsukuba Journal of Mathematics*, 37:307–320, 2013.

[6] N. Matsuda. A proof theoretic study on intuitionistic tree sequent calculus. Technical Report C-281, Tokyo Institute of Technology, 2014.

[7] H. Nakano. The non-deterministic catch and throw mechanism and its subject reduction property. *Lecture Notes in Computer Science*, 792:61–72, 1994.

[8] M. Parigot. $\lambda\mu$-calculus: An algorithmic interpretation of classical natural deduction. In *Lecture Notes in Computer Science*, volume 624, pages 190–201. Springer, 1992.

[9] B. C. Pierce. *Types and Programming Languages*. MIT Press, 2002.

[10] M. H. Sørensen and P. Urzyczyn. *LECTURES ON THE CURRY-HOWARD ISOMORPHISM*. ELSEVIER, 2006.

[11] G. Takeuti. *PROOF THEORY*. NORTH-HOLLAND, 2 edition, 1987.

Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo, Japan

E-mail address: matsuda.naosuke@gmail.com