Labelled Sequent Calculi and Completeness Theorems for Implicational Relevant Logics

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December 1999

Abstract

It is known that the implicational fragment of the relevant logic **E** is complete with respect to the class of Urquhart's models, where a model consists of a semilattice and a set of possible worlds. This paper shows that some implicational relevant logics, which are obtained from **E** by adding axioms, are complete with respect to the class of Urquhart's models with certain conditions. To show this, we introduce labelled sequent calculi.

1 Introduction

A natural semantic treatment of relevant logics was given by Urquhart [10]. (See [1], [2] and [4] for information on relevant logics.) Let **E** be the implicational fragment of the relevant logic **E**. (In this paper, we treat only "$\to$" (implication) as a logical symbol; therefore, **E**/**R**/**S4**$\cdots$ will denote the implicational fragments of themselves.) An Urquhart's model for **E** (we call it an *E-model*) consists of two structures $\langle I, \cdot, \mathbf{e} \rangle$ and $\langle W, R \rangle$ where the former is an idempotent commutative monoid (i.e., semilattice with identity **e**) and the latter is a quasi-ordered set (i.e., R is a reflexive and transitive relation on W). The structure $\langle I, \cdot, \mathbf{e} \rangle$ is considered to be a "structure of information"—I is a set of pieces of information, $\cdot$ is a binary operator which combines two pieces of information, and **e** is an empty piece of information; and the structure $\langle W, R \rangle$ is, like a Kripke's model for modal logics, considered to be a "structure of possible worlds"—W is a set of worlds, and R is an accessibility relation. The notion "$\alpha, x \models A$" (a formula A holds according to a piece α of information at a world x) is inductively defined by the following.

$$\alpha, x \models B \to C \iff \forall \beta \in I, \forall y \in W[(xRy) \ \&\ (\beta, y \models B) \ \Rightarrow\ (\alpha \cdot \beta, y \models C)].$$

We say that a formula A is *valid in* the model if $\mathbf{e}, x \models A$ for any $x \in W$. The completeness theorem, which was shown in [10] (a proof appeared in [2]), claims that a formula A is provable in **E** if and only if A is valid in any E-model.

The relevant logic **R** and the logic **S4** of "strict implication" are obtained from **E** by adding, respectively, the axiom schemes

C: $(A \to (B \to C)) \to (B \to (A \to C))$, and

K^{01}: $\overrightarrow{A} \to (B \to \overrightarrow{A})$

where $\overrightarrow{A}$ is an abbreviation for $A_1 \to A_2$; and the intuitionistic logic **Int** is obtained from **E** by adding both the schemes C and K^{01} (or, equivalently, **Int** is obtained from **R** by adding the scheme K: $A \to (B \to A)$). (See, e.g., [1] and [9] for axiomatizations of these logics. The name K^{01} comes from [9], where the superscripts represent certain restrictions by "over-arrows".) The completeness theorems for these logics were shown as follows. We define two conditions on E-models:

(Single World Condition) The set of possible worlds is a singleton.

(Hereditary Condition) If $\alpha, x \models A$, then $\alpha \cdot \beta, x \models A$.

Then, the logic (**R** / **S4** / **Int**) is complete with respect to the class of E-models satisfying the condition (Single World / Hereditary / both Single World and Hereditary, respectively). That is, for example, a formula A is provable in **Int** if and only if A is valid in any E-model that satisfies both the Single World and Hereditary Conditions.

Moreover, such completeness results were shown for other two relevant logics—called **E5** and **RM0** (the names come from [5] and [1]).

The logic **E5** is obtained from **E** by adding the axiom scheme

C^{001}: $(A \to (B \to \overrightarrow{C})) \to (B \to (A \to \overrightarrow{C}))$.

(In [5] and [10], the axiom scheme $B \to ((B \to \overrightarrow{C}) \to \overrightarrow{C})$ was adopted where this scheme and C^{001} are mutually inferable over **E**.) The scheme C^{001} is an instance of the scheme C, and therefore **E5** is an intermediate logic between **E** and **R**. It was shown that the logic **E5** is complete with respect to the class of E-models satisfying the following condition.

(Single Cluster Condition) The accessibility relation is universal, that is, xRy for any $x, y \in W$.

Note that this condition is weaker than the Single World Condition.

The logic **RM0** is obtained from **R** by adding the axiom scheme

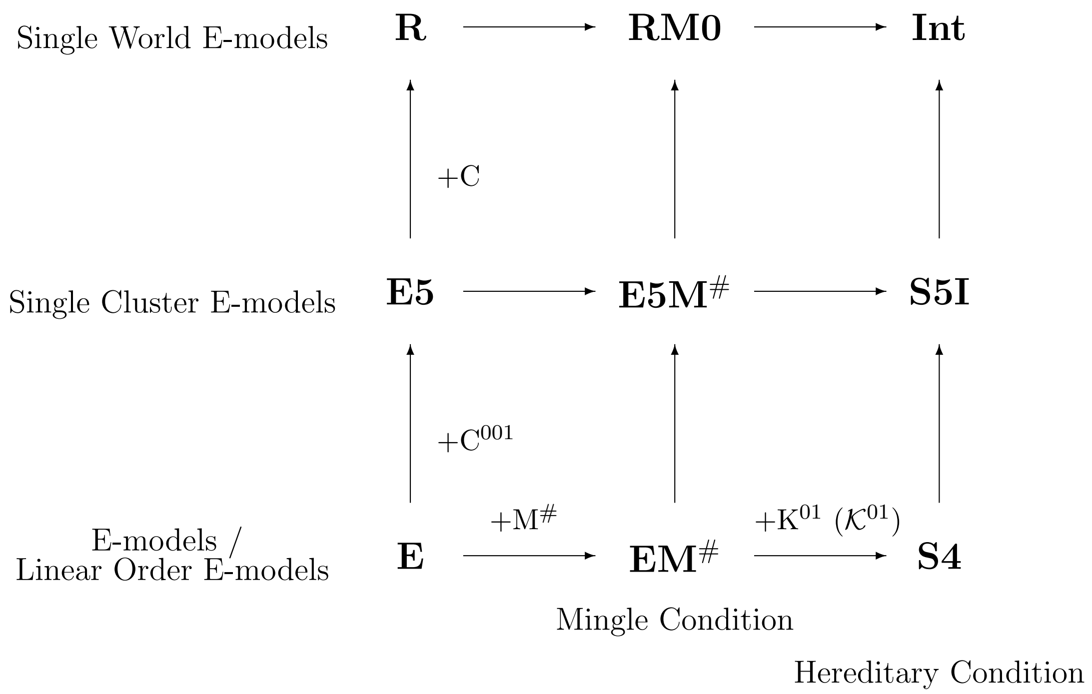
M: $A \to (A \to A)$.

This scheme is an instance of the scheme K, and therefore **RM0** is an intermediate logic between **R** and **Int**. It was shown that the logic **RM0** is complete with respect to the class of E-models satisfying both the Single World Condition and the following condition.

(Mingle Condition) If $\alpha, x \models A$ and $\beta, x \models A$, then $\alpha \cdot \beta, x \models A$.

Note that the Mingle Condition is weaker than the Hereditary Condition.

Figure 1



To sum up, it has been shown that the six logics **E**, **E5**, **R**, **RM0**, **Int**, and **S4**, which are located on the edges of the diagram in Figure 1, are complete with respect to the class of E-models with the additional conditions. (See [2], [5] and [10] for the proofs.) Then, bringing the diagram to completion is a natural requirement. In other words, there are two questions: (Q1) Is there an axiom scheme $\mathcal{A}$ that satisfies the following?

- The logic $\mathbf{E}+\mathcal{A}$ is complete with respect to the class of E-models satisfying the Mingle Condition.

- The logic $\mathbf{E5}+\mathcal{A}$ is complete with respect to the class of E-models satisfying both the Single Cluster Condition and the Mingle Condition.
- The logic $\mathbf{R}+\mathcal{A}$ is equivalent to **RM0**.

(Q2) Is the logic $\mathbf{S5I} = \mathbf{S4}+\mathrm{C}^{001}$ complete with respect to the class of E-models satisfying both the Single Cluster Condition and Hereditary Condition? (We will account for the name **S5I** later.)

This paper gives positive answers to these questions. (It is somewhat surprising that such a natural question has been open.) For Q1, the required scheme $\mathcal{A}$ is

$$\mathrm{M}^{\#}\colon\ (\overrightarrow{A}\to B)\to(\overrightarrow{A}\to(B\to B)),$$

and the logics $\mathbf{E}+\mathrm{M}^{\#}$ and $\mathbf{E5}+\mathrm{M}^{\#}$ are denoted by $\mathbf{EM}^{\#}$ and $\mathbf{E5M}^{\#}$, respectively. (Note that $\mathrm{M}^{\#}$ is an instance of the scheme

$$\mathcal{K}^{01}\colon\ (\overrightarrow{A}\to B)\to(\overrightarrow{A}\to(C\to B))$$

while this scheme and K^{01} are mutually inferable over **E**; see Section 2.) Moreover we show a strong version of the completeness of $\mathbf{E}/\mathbf{EM}^{\#}/\mathbf{S4}$, which claims that these logics are complete with respect to the *linear order* E-models. That is, for example, a formula A is provable in **S4** if and only if A is valid in any E-model in which the accessibility relation is a linear order and the Hereditary Condition holds. These complete the diagram in Figure 1, which shows a nice correspondence between simple axiom schemes and natural conditions on models.

An outline of the proof of the completeness is as follows. For each logic $X = \mathbf{E}, \mathbf{EM}^{\#}, \ldots$, we introduce two systems $\mathcal{G}X$ and $\mathcal{L}X$. The system $\mathcal{G}X$ is an ordinary "sequent calculus", and it is easy to show that a "sequent" $A_1, \ldots, A_n \mapsto B$ is provable in $\mathcal{G}X$ if and only if the formula $A_1\to(\cdots\to(A_n\to B)\cdots)$ is provable in X. The system $\mathcal{L}X$ is a "labelled sequent calculus", that is, a system to treat "labelled sequents"—sequents consisting of labelled formulas where a label reflects the structure of E-models. By a standard way, we show that $\mathcal{L}X$ is complete with respect to the class of E-models satisfying the additional conditions for X. In particular, if a formula A is valid in any E-model of X, then the labelled sequent "$\Rightarrow (\emptyset, 0)\colon A$", which consists of A and the "empty label" $(\emptyset, 0)$, is provable in $\mathcal{L}X$. Then, the following claim completes the proof of the completeness.

> If the labelled sequent $\Rightarrow (\emptyset, 0)\colon A$ is provable in $\mathcal{L}X$, then the unlabelled sequent $\mapsto A$ is provable in $\mathcal{G}X$.

This is proved by showing a stronger claim:

> If a labelled sequent $\mathcal{S}$ is provable in $\mathcal{L}X$, then an unlabelled sequent $\mathcal{T}$ is provable in $\mathcal{G}X$ where $\mathcal{T}$ is "extracted" from $\mathcal{S}$ in a certain way.

Since unlabelled sequents have less expressive power than labelled sequents, this "extraction" is somewhat complicated; and proving this claim is the hardest (and the most interesting) part of the proof of the completeness theorem. There have been a lot of studies on labelled sequent calculi (e.g. see [7] and [8]). Among them, the author thinks, this paper is a good example of an application of labelled sequent calculi to solve important problems on the logics, which are independent from the labelled systems.

The completion of the diagram in Figure 1 induces further questions, for which the author does not have any nontrivial answers now: (1) There are a lot of possible conditions on E-models. Is there a corresponding axiom scheme for each condition? For example, if the models lack the condition $\alpha{\cdot}\alpha = \alpha$ (idempotence), the corresponding logics might be axiomatized by deleting the scheme W: $(A{\to}(A{\to}B)){\to}(A{\to}B)$. Is this true for each logic in the diagram? (2) Introducing other connectives ($\wedge$, $\vee$, $\neg$, ...) is an important problem. It was shown ([3], [6]) that the logic $\mathbf{R}_{\to\wedge\vee}$ with an extra inference rule is complete with respect to the class of models of **R** in which $\wedge$ and $\vee$ are interpreted in a natural way. What happens for the other logics in the diagram?

If we get acceptable answers for these questions, Urquhart's semantics will be a major paradigm of "possible worlds semantics", while Routley-Meyer's "ternary relation semantics" (see, e.g., [2] or [4]) is major now. The diagram in Figure 1 is a starting point.

We make remarks about the scheme $\mathrm{M}^{\#}$ and the logic **S5I**. In literature (see, e.g., [1] or [4]), the axiom scheme M′: $\overrightarrow{A}{\to}(\overrightarrow{A}{\to}\overrightarrow{A})$ was suggested as a "Mingle for **E**". But the author thinks the scheme $\mathrm{M}^{\#}$ is the very axiom of "Mingle for **E**" because of its completeness with respect to the class of E-models satisfying the Mingle Condition. (M′ is inferable from $\mathrm{M}^{\#}$ (by $B := \overrightarrow{A}$), but the author does not know whether the converse holds.) The name **S5I** comes from [8]. In [8], a sequent style system for **S5I** was introduced, whereas the Hilbert style axiomatization $\mathbf{S4}{+}\mathrm{C}^{001}$ did not appear. The name **S5I** reflects the fact that the models of this logic are considered to be the intuitionistic variants of the models for "S5 strict implication". We will discuss this fact in Section 3.

The plan of this paper is as follows. Let $\mathcal{X} = \{\mathbf{E}, \mathbf{EM}^{\#}, \mathbf{S4}, \mathbf{E5}, \mathbf{E5M}^{\#}, \mathbf{S5I}\}$ (the six logics in the bottom and the middle rows in Figure 1). In Section 2, we introduce implicational relevant logics by Hilbert-style axiomatizations; and then we introduce six Gentzen-style sequent calculi named $\mathcal{G}X$ for $X \in \mathcal{X}$. In Section 3, we define the E-models and prove that the Hilbert-style systems are sound with respect to the class of E-models with the additional conditions. In Section 4, we introduce labelled sequent calculi named $\mathcal{L}X$ for $X \in \mathcal{X}$, and prove their completeness with respect to the class of E-models with the additional conditions. In Section 5, we show the "extraction" from the labelled sequent calculi $\mathcal{L}X$ into the unlabelled sequent calculi $\mathcal{G}X$. Finally, in Section 6, we state the main results of this paper—completeness of the six logics in $\mathcal{X}$.

2 Axiom schemes and sequent calculi

In this section we give Hilbert-style axiomatizations of implicational relevant logics, and then introduce Gentzen-style sequent calculi called $\mathcal{G}X$ for $X = \mathbf{E}$, $\mathbf{EM}^{\#}$, $\mathbf{S4}$, $\mathbf{E5}$, $\mathbf{E5M}^{\#}$, and $\mathbf{S5I}$.

This paper treats only the implicational fragments of relevant logics; therefore *formulas* are constructed from propositional variables and $\to$ (implication). We assume that the set of propositional variables is countable. A formula is said to be an *implication* if it is of the form $A_1 \to A_2$, and said to be *atomic* if it is not an implication. We use the metavariables $A, B, \ldots$ for formulas; $p, q, \ldots$ for atomic formulas (i.e., propositional variables); and $\overrightarrow{A}, \overrightarrow{B} \ldots$ for implications. We adopt the convention of association to the right for omitting parentheses; for example, the scheme C appearing Section 1 is written as $(A \to B \to C) \to B \to A \to C$.

The Hilbert-style system $\mathbf{E}$ consists of the following axiom schemes and inference rule:

B: $(B \to C) \to (A \to B) \to A \to C$

C^{010}: $(A \to \overrightarrow{B} \to C) \to \overrightarrow{B} \to A \to C$

I: $A \to A$

W: $(A \to A \to B) \to A \to B$

$\dfrac{A \quad A \to B}{B}$ (modus ponens)

(The superscript "010" represents certain restrictions corresponding to the "structural rules" appearing later.) If a formula A is provable in $\mathbf{E}$, we write $\mathbf{E} \vdash A$. This notation will be used for other systems; that is, $\mathcal{X} \vdash \mathcal{Y}$ will mean *the object* $\mathcal{Y}$ *is provable in the system* $\mathcal{X}$.

We define axiom schemes $\mathrm{M}^{\#}$, $\mathcal{K}^{01}$, K^{01}, C^{001}, and C as follows:

$\mathrm{M}^{\#}$: $(\overrightarrow{A} \to B) \to \overrightarrow{A} \to B \to B$

$\mathcal{K}^{01}$: $(\overrightarrow{A} \to B) \to \overrightarrow{A} \to C \to B$

K^{01}: $\overrightarrow{A} \to B \to \overrightarrow{A}$

C^{001}: $(A \to B \to \overrightarrow{C}) \to B \to A \to \overrightarrow{C}$

C: $(A \to B \to C) \to B \to A \to C$

Note that $\mathrm{M}^{\#}$ is an instance of $\mathcal{K}^{01}$, and C^{010} and C^{001} are instances of C. Note also that the schemes $\mathcal{K}^{01}$ and K^{01} are mutually inferable over $\mathbf{E}$:

[Proof of $\mathbf{E}$+$\mathrm{K}^{01} \vdash \mathcal{K}^{01}$]:

$$\dfrac{\overset{\mathrm{K}^{01}}{X \to Y} \qquad \dfrac{\overset{\mathrm{C}^{010}}{Y \to Z} \quad \overset{\mathrm{B}}{(Y \to Z) \to (X \to Y) \to X \to Z}}{(X \to Y) \to X \to Z}}{X \to Z}$$

where $X = \overrightarrow{A} \to B$, $Y = C \to \overrightarrow{A} \to B$, and $Z = \overrightarrow{A} \to C \to B$.

[Proof of $\mathbf{E}+\mathcal{K}^{01} \vdash \mathrm{K}^{01}$]:

$$\dfrac{\overset{\mathrm{I}}{\overrightarrow{A} \to \overrightarrow{A}} \quad \overset{\mathcal{K}^{01}}{(\overrightarrow{A} \to \overrightarrow{A}) \to \overrightarrow{A} \to B \to \overrightarrow{A}}}{\overrightarrow{A} \to B \to \overrightarrow{A}}$$

Then we define the Hilbert-style systems $\mathbf{EM}^{\#}$, **S4**, **E5**, $\mathbf{E5M}^{\#}$, **S5I**, **R**, **RM0**, and **Int** as follows (see Figure 1).

$\mathbf{EM}^{\#} = \mathbf{E}+\mathrm{M}^{\#}$.

$\mathbf{S4} = \mathbf{EM}^{\#}+\mathrm{K}^{01} = \mathbf{EM}^{\#}+\mathcal{K}^{01} = \mathbf{E}+\mathrm{K}^{01} = \mathbf{E}+\mathcal{K}^{01}$.

$\mathbf{E5} = \mathbf{E}+\mathrm{C}^{001}$.

$\mathbf{E5M}^{\#} = \mathbf{E5}+\mathrm{M}^{\#} = \mathbf{EM}^{\#}+\mathrm{C}^{001}$.

$\mathbf{S5I} = \mathbf{E5M}^{\#}+\mathrm{K}^{01} = \mathbf{E5M}^{\#}+\mathcal{K}^{01} = \mathbf{E5}+\mathrm{K}^{01} = \mathbf{E5}+\mathcal{K}^{01} = \mathbf{S4}+\mathrm{C}^{001}$.

$\mathbf{R} = \mathbf{E5}+\mathrm{C} = \mathbf{E}+\mathrm{C}$.

$\mathbf{RM0} = \mathbf{R}+\mathrm{M}^{\#} = \mathbf{E5M}^{\#}+\mathrm{C} = \mathbf{EM}^{\#}+\mathrm{C}$.

$\mathbf{Int} = \mathbf{RM0}+\mathrm{K}^{01} = \mathbf{RM0}+\mathcal{K}^{01} = \mathbf{R}+\mathrm{K}^{01} = \mathbf{R}+\mathcal{K}^{01} = \mathbf{S5I}+\mathrm{C} = \mathbf{S4}+\mathrm{C}$.

Remark. The logics **RM0** and **Int** are usually axiomatized as follows.

$\mathbf{RM0} = \mathbf{R}+\mathrm{M}$, where M is the scheme $A \to A \to A$.

$\mathbf{Int} = \mathbf{RM0}+\mathrm{K} = \mathbf{R}+\mathrm{K}$, where K is the scheme $A \to B \to A$.

The equivalence between usual and our axiomatizations can be shown by the method of Theorem 2.3 in [9].

The goal of this paper is to show the completeness of the six logics **E**, $\mathbf{EM}^{\#}$, **S4**, **E5**, $\mathbf{E5M}^{\#}$, and **S5I**. For this purpose, next we introduce sequent calculi.

A *Sequent* is an expression of the form $\Gamma \mapsto A$ where Γ is a finite (possibly empty) sequence of formulas and A is a formula. In this section, we use metavariables $\Gamma, \Delta, \ldots$ for finite sequences of formulas. (In Sections 4 and 5, these letters may denote other objects.) The sequent calculi $\mathcal{G}\mathbf{E}$, which was called L_5^{10} in [9], consists of the following initial sequents and inference rules.

Initial sequents: $A \mapsto A$

Inference rules:

$$\dfrac{\Gamma \mapsto A \quad \Delta, A, \Pi \mapsto B}{\Delta, \Gamma, \Pi \mapsto B}\ (\mathrm{cut})$$

$$\dfrac{\Gamma, A, \overrightarrow{B}, \Delta \mapsto C}{\Gamma, \overrightarrow{B}, A, \Delta \mapsto C}\ (\mathrm{ex}^{010}) \qquad \dfrac{\Gamma, A, A, \Delta \mapsto B}{\Gamma, A, \Delta \mapsto B}\ (\mathrm{contraction})$$

$$\frac{\Gamma \mapsto A \quad \Delta, B, \Pi \mapsto C}{\Delta, A{\rightarrow}B, \Gamma, \Pi \mapsto C}\ (\rightarrow\text{left}) \qquad \frac{\Gamma, A \mapsto B}{\Gamma \mapsto A{\rightarrow}B}\ (\rightarrow\text{right})$$

The name ex^{010} represents that it is a restircted version of the usual rule "exchange". There are a lot of other restricted structural rules (exchange, contraction, and weakening), and [9] investigated certain properties of them (e.g., the cut-elimination of $\mathcal{G}\mathbf{E}$ was proved).

The system $\mathcal{G}\mathbf{E}$ is a sequent calculus for the logic $\mathbf{E}$. For other logics, we introduce the following inference rules, which correspond to the axiom schemes $\text{M}^{\#}$, K^{01}, and C^{001}, respectively:

$$\frac{\Gamma, \overrightarrow{A} \mapsto B}{\Gamma, \overrightarrow{A}, B \mapsto B}\ (\text{m}^{\#}) \qquad \frac{\Gamma, \overrightarrow{\Delta \mapsto A}}{\Gamma, B, \overrightarrow{\Delta \mapsto A}}\ (\text{we}^{01})$$

$$\frac{\Gamma, A, B, \overrightarrow{\Delta \mapsto C}}{\Gamma, B, A, \overrightarrow{\Delta \mapsto C}}\ (\text{ex}^{001})$$

where $\Pi, \overrightarrow{\Sigma \mapsto F}$ denotes a sequent $\Pi, \Sigma \mapsto F$ in which *the formula F is an implication or the sequence Σ is nonempty.* (In other words, $\Pi, \overrightarrow{\Sigma \mapsto F}$ denotes a sequent $\Pi, E_1, \ldots, E_n \mapsto F$ $(n \geq 0)$ where the formula $E_1{\rightarrow}\cdots{\rightarrow}E_n{\rightarrow}F$ is an implication.) Then we define the sequent calculi $\mathcal{G}\mathbf{EM}^{\#}$, $\mathcal{G}\mathbf{S4}$, $\mathcal{G}\mathbf{E5}$, $\mathcal{G}\mathbf{E5M}^{\#}$, and $\mathcal{G}\mathbf{S5I}$ as follows.

$\mathcal{G}\mathbf{EM}^{\#} = \mathcal{G}\mathbf{E} + (\text{m}^{\#})$.

$\mathcal{G}\mathbf{S4} = \mathcal{G}\mathbf{E} + (\text{we}^{01})$.

$\mathcal{G}\mathbf{E5} = \mathcal{G}\mathbf{E} + (\text{ex}^{001})$.

$\mathcal{G}\mathbf{E5M}^{\#} = \mathcal{G}\mathbf{E} + (\text{ex}^{001}) + (\text{m}^{\#})$.

$\mathcal{G}\mathbf{S5I} = \mathcal{G}\mathbf{E} + (\text{ex}^{001}) + (\text{we}^{01})$.

The relationship between these sequent calculi and the Hilbert style systems is stated as follows. For a sequent $\Gamma \mapsto A$, we define the formula $[\Gamma \mapsto A]$ by

$[\Gamma \mapsto A] = C_1{\rightarrow}\cdots{\rightarrow}C_n{\rightarrow}A$ if $\Gamma = (C_1, \ldots, C_n)$.
($[\Gamma \mapsto A] = A$ if Γ is empty.)

Theorem 2.1 *Let X be one of the six logics* $\mathbf{E}$, $\mathbf{EM}^{\#}$, $\mathbf{S4}$, $\mathbf{E5}$, $\mathbf{E5M}^{\#}$, *and* $\mathbf{S5I}$. (1) *If $X \vdash A$, then $\mathcal{G}X \vdash\ \mapsto A$.* (2) *If $\mathcal{G}X \vdash \Gamma \mapsto A$, then $X \vdash [\Gamma \mapsto A]$.*

Proof By induction on the proofs (see Theorem 2.8 in [9]). ∎

In the rest of this section we show some properties of the sequent calculi, which will be used later.

Lemma 2.2 *The rule*

$$\frac{\Gamma \mapsto A{\to}B}{\Gamma, A \mapsto B}\ ({\to}\text{right}^{-1})$$

is derivable in $\mathcal{G}X$, for any sequent calculus $\mathcal{G}X$ in the six. That is, for any Γ, A, and B, there is a derivation from $\Gamma \mapsto A{\to}B$ to $\Gamma, A \mapsto B$ in $\mathcal{G}X$.

Proof Easy, using the cut rule and $A{\to}B, A \mapsto B$. ∎

Let Γ, Δ, and Θ be finite sequences of formulas. If there are sequences $\Gamma_1, \ldots, \Gamma_n$, $\Delta_1, \ldots, \Delta_n$ such that

$$\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$$
$$\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$$
$$\Theta = (\Gamma_1, \Delta_1, \Gamma_2, \Delta_2, \ldots, \Gamma_n, \Delta_n)$$

(each Γ_i and Δ_i may be empty), then Θ is said to be a *merge* of $\langle \Gamma; \Delta \rangle$.

Lemma 2.3 *The rule*

$$\frac{\Gamma \mapsto \overrightarrow{A} \quad \Delta, \overrightarrow{A}, \Pi \mapsto B}{\Theta, \Pi \mapsto B}\ (\text{merge cut})\quad \text{where } \Theta \text{ is a merge of } \langle \Gamma; \Delta \rangle$$

is derivable in $\mathcal{G}X$, for any sequent calculus $\mathcal{G}X$ in the six.

(Such inference rules (i.e., rules described by "merge operator") were investigated in [1].)

Proof We show

$$(\dagger)\ \mathcal{G}X \vdash\ [\Gamma \mapsto \overrightarrow{A}],\ [\Delta, \overrightarrow{A}, \Pi \mapsto B],\ \Theta,\ \Pi\ \mapsto\ B$$

from which the merge cut rule is derivable with (→right) and the cut rule. If Θ is empty, (†) is shown by

$$\frac{\begin{array}{c}\vdots\\ [\overrightarrow{A}, \Pi \mapsto B], \overrightarrow{A}, \Pi \mapsto B\end{array}}{\overrightarrow{A}, [\overrightarrow{A}, \Pi \mapsto B], \Pi \mapsto B.}\ (\text{ex}^{010})$$

If Θ is not empty, (†) is obtained from the above proof by "adding" the elements of Θ one after another, using the following inferences:

$$\frac{[\Gamma' \mapsto A], \overrightarrow{Q}, \Theta', \Pi \mapsto B}{[C, \Gamma' \mapsto A], \overrightarrow{Q}, C, \Theta', \Pi \mapsto B,}\ (\text{to add the element } C \text{ of } \Gamma)$$

$$\frac{P, [\Delta', A, \Pi \mapsto B], \Theta', \Pi \mapsto B}{P, [D, \Delta', A, \Pi \mapsto B], D, \Theta', \Pi \mapsto B}\ (\text{to add the element } D \text{ of } \Delta)$$

where the latter is done by (→left, together with $D \mapsto D$), and the former is done by (→left, together with $C \mapsto C$) and (ex^{010}). ∎

Lemma 2.4 *Let X be one of the logics* **E5**, $\mathbf{E5M^{\#}}$, *and* **S5I** (*i.e., the logics with* C^{001}). *The following inference rules are derivable in $\mathcal{G}X$, where $\{\Sigma\}$ denotes an arbitrary permutation of the sequence Σ.*

$$\frac{\Gamma \mapsto \overrightarrow{A} \quad \Delta, B, \Pi \mapsto C}{\{\Delta, \overrightarrow{A}{\to}B, \Gamma\}, \Pi \mapsto C}\ (\to\text{left}+) \qquad \frac{\Gamma \mapsto A \quad \Delta, \overrightarrow{B}, \Pi \mapsto C}{\{\Delta, A{\to}\overrightarrow{B}, \Gamma\}, \Pi \mapsto C}\ (\to\text{left}+)$$

$$\frac{\Gamma \mapsto \overrightarrow{A} \quad \Delta, \overrightarrow{A}, \Pi \mapsto C}{\{\Delta, \Gamma\}, \Pi \mapsto C}\ (\text{cut}+)$$

Proof Derivability of ($\to$left+) was shown by Theorem 4.2 in [9]. Derivability of (cut+) is shown by ($\to$left+) and (cut, together with $\mapsto \overrightarrow{A}{\to}\overrightarrow{A}$). ∎

Lemma 2.5 *Let X be one of the logics* $\mathbf{EM^{\#}}$, **S4**, $\mathbf{E5M^{\#}}$, *and* **S5I** (*i.e., the logics with* $\mathrm{M}^{\#}$). *The inference rule*

$$\frac{\Gamma, A, \overrightarrow{\Delta \mapsto B}}{\Gamma, A, A, \overrightarrow{\Delta \mapsto B}}$$

is derivable in $\mathcal{G}X$.

Proof If X = **S4** or **S5I**, this inference is an instance of (we^{01}). If $X = \mathbf{EM^{\#}}$ or $\mathbf{E5M^{\#}}$, we have the following, where $\overrightarrow{C} = [\Delta \mapsto B]$.

$$\begin{array}{c}
\dfrac{\begin{array}{c}\Gamma, A, \overrightarrow{\Delta \mapsto B} \\ \vdots\ (\to\text{right}) \\ \Gamma \mapsto A{\to}\overrightarrow{C}\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{\overrightarrow{C} \mapsto \overrightarrow{C}}{\overrightarrow{C}, \overrightarrow{C} \mapsto \overrightarrow{C}}\ (\mathrm{m}^{\#}) \\ \vdots\ (\to\text{left}) \text{ with } A \mapsto A \\ A{\to}\overrightarrow{C}, A, A{\to}\overrightarrow{C}, A \mapsto \overrightarrow{C}\end{array}}{A{\to}\overrightarrow{C}, A{\to}\overrightarrow{C}, A, A \mapsto \overrightarrow{C}}\ (\mathrm{ex}^{010})}{A{\to}\overrightarrow{C}, A, A \mapsto \overrightarrow{C}}\ (\text{contraction})}{\Gamma, A, A, \mapsto \overrightarrow{C}}\ (\text{cut}) \\
\vdots\ (\to\text{right}^{-1})\ (\text{Lemma 2.2}) \\
\Gamma, A, A, \Delta \mapsto B
\end{array}$$

∎

3 Models and soundness

In this section we give the definitions of the models, and show the soundness of the Hilbert-style systems.

A triple $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ is said to be an *E-model* if it satisfies the following conditions.

- $\langle I, \cdot, \mathbf{e}\rangle$ is an idempotent commutative monoid (in other words, a semilattice with identity $\mathbf{e}$); that is, I is a non-empty set, $\cdot$ is a binary operator on I, and $\mathbf{e} \in I$ such that $(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma)$, $\alpha\cdot\beta = \beta\cdot\alpha$, $\alpha\cdot\alpha = \alpha$, and $\alpha\cdot\mathbf{e} = \alpha$ hold for any $\alpha, \beta, \gamma \in I$.
- $\langle W, R\rangle$ is a quasi-ordered set; that is, W is a non-empty set and R is a binary relation on W such that xRx and $(xRy \;\&\; yRz \Rightarrow xRz)$ hold for any $x, y, z \in W$.
- V is a subset of $I \times W \times \text{Atom}$, where Atom is the set of atomic formulas (propositional variables).

As stated in Section 1, the set I, the element $\mathbf{e}$, the set W, and the relation R are said to be "set of pieces of information", "empty piece of information", "set of possible worlds", and "accessibility relation", respectively. For a piece α of information, a possible world x, and a formula A, we define a notion

$$\alpha, x \models_M A$$

(the formula A holds according to the piece α at the world x) inductively as follows.

$\alpha, x \models_M p \iff (\alpha, x, p) \in V.$

$\alpha, x \models_M A{\to}B \iff$
$\forall\beta \in I, \forall y \in W[(xRy) \;\&\; (\beta, y \models_M A) \;\Rightarrow\; (\alpha\cdot\beta, y \models_M B)].$

We say that a formula A is *valid in* the model M if $\mathbf{e}, x \models_M A$ for any $x \in W$.

We introduce some conditions on an E-model $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$. The following three are conditions on the possible worlds.

(Single World Condition) W is a singleton.

(Single Cluster Condition) xRy holds for any $x, y \in W$.

(Linear Order Condition) For any distinct $x, y \in W$, exactly one of xRy or yRx holds.

If M satisfies the Single World Condition, then M is said to be a *Single World E-model*; and a *Single Cluster E-model* and a *Linear Order E-model* are defined similarly.

The following two are conditions on $\models$ and the operator $\cdot$.

(Hereditary Condition) If $\alpha, x \models A$, then $\alpha\cdot\beta, x \models A$, for any $\alpha, \beta \in I$, any $x \in W$ and any formula A.

(Mingle Condition) If $\alpha, x \models A$ and $\beta, x \models A$, then $\alpha\cdot\beta, x \models A$, for any $\alpha, \beta \in I$, any $x \in W$ and any formula A.

We show that these conditions are respectively equivalent to their atomic versions:

(Atomic Hereditary Condition) If $(\alpha, x, p) \in V$, then $(\alpha \cdot \beta, x, p) \in V$, for any $\alpha, \beta \in I$, any $x \in W$ and any atomic formula p.

(Atomic Mingle Condition) If $(\alpha, x, p) \in V$ and $(\beta, x, p) \in V$, then $(\alpha \cdot \beta, x, p) \in V$, for any $\alpha, \beta \in I$, any $x \in W$ and any atomic formula p.

Theorem 3.1
(1) *A model satisfies the Hereditary Condition if and only if it satisfies the Atomic Hereditary Condition.*
(2) *A model satisfies the Mingle Condition if and only if it satisfies the Atomic Mingle Condition.*

Proof Here we show the if-part of (2). (The other parts are similar or trivial.) Let $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ be an E-model satisfying the Atomic Mingle Condition. By induction on the formula A, we show that the following holds for any A:

If $\alpha, x \models_M A$ and $\beta, x \models_M A$, then $\alpha \cdot \beta, x \models_M A$, for any $\alpha, \beta \in I$ and any $x \in W$.

If A is atomic, this is just the Atomic Mingle Condition. For an implication $A = B \to C$, we assume $\alpha, x \models_M B \to C$, $\beta, x \models_M B \to C$, xRy, and $\gamma, y \models_M B$; and we will show $(\alpha \cdot \beta) \cdot \gamma, y \models_M C$. From the assumptions, we have $\alpha \cdot \gamma, y \models_M C$ and $\beta \cdot \gamma, y \models_M C$; then, by the induction hypothesis, we have $(\alpha \cdot \gamma) \cdot (\beta \cdot \gamma), y \models_M C$, which implies $(\alpha \cdot \beta) \cdot \gamma, y \models_M C$ because $(\alpha \cdot \gamma) \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$. ∎

Then we will show the soundness of the Hilbert-style systems introduced in the previous section.

Lemma 3.2 *Let $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ be an E-model. If $\alpha, x \models_M \overrightarrow{A}$ and xRy, then $\alpha, y \models_M \overrightarrow{A}$.*

Proof Easy, using the transitivity of the accessibility relation. ∎

Lemma 3.3 *The validity is preserved by the rule modus ponens. That is, if both formulas $A \to B$ and A are valid in an E-model M, then also B is valid in M.*

Proof Easy. ∎

Lemma 3.4 *The schemes* B, C^{010}*,* I*, and* W (*the axiom schemes of* **E**) *are valid in any E-model.*

Proof Here we show only the validity of the scheme C^{010}. (The other cases are similar or easy.) We show

$$\mathbf{e}, x_0 \models_M (A \to \overrightarrow{B} \to C) \to \overrightarrow{B} \to A \to C$$

where $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ is an E-model and $x_0 \in W$. Suppose $x_0 R x$ and

(1) $\alpha, x \models_M A \to \overrightarrow{B} \to C.$

Then our goal is to show

(2) $\alpha, x \models_M \overrightarrow{B} \to A \to C.$

For (2), we assume

(3-1) xRy, (3-2) $\beta, y \models_M \overrightarrow{B}$,

(4-1) yRz, and (4-2) $\gamma, z \models_M A$;

and we show $(\alpha{\cdot}\beta){\cdot}\gamma, z \models_M C$. By (1), (4-2) and the fact xRz (induced by (3-1), (4-1) and the transitibity of R), we have

(5) $\alpha{\cdot}\gamma, z \models_M \overrightarrow{B} \to C.$

On the other hand, we have

(6) $\beta, z \models_M \overrightarrow{B}$

by (3-2), (4-1) and Lemma 3.2. Then we have

$(\alpha{\cdot}\beta){\cdot}\gamma, z \models_M C$

by (5), (6) and the facts $(\alpha{\cdot}\gamma){\cdot}\beta = (\alpha{\cdot}\beta){\cdot}\gamma$ and zRz. ∎

Lemma 3.5 *The scheme* C^{001} *is vaild in any Single Cluster E-model.*

Proof We show

$$\mathbf{e}, x_0 \models_M (A \to B \to \overrightarrow{C}) \to B \to A \to \overrightarrow{C}$$

where $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ is a Single Cluster E-model and $x_0 \in W$. Suppose

(1) $\alpha, x \models_M A \to B \to \overrightarrow{C}.$

Then our goal is to show

(2) $\alpha, x \models_M B \to A \to \overrightarrow{C}.$

(Recall that the relation R is universal.) For (2), we assume

(3) $\beta, y \models_M B$,

(4) $\gamma, z \models_M A$;

and we show $(\alpha{\cdot}\beta){\cdot}\gamma, z \models_M \overrightarrow{C}$. By (1), (4) and (3), we have

(6) $(\alpha{\cdot}\gamma){\cdot}\beta, y \models_M \overrightarrow{C}.$

Then we have

$(\alpha\cdot\beta)\cdot\gamma, z \models_M \overrightarrow{C}$

by Lemma 3.2 and the fact $(\alpha\cdot\gamma)\cdot\beta = (\alpha\cdot\beta)\cdot\gamma$. ∎

Lemma 3.6 *The scheme* C *is vaild in any Single World E-model.*

Proof Easy. ∎

Lemma 3.7 *The scheme* $\mathrm{M}^{\#}$ *is vaild in any E-model that satisfies the Mingle Condition.*

Proof We show

$$\mathbf{e}, x_0 \models_M (\overrightarrow{A}\to B)\to\overrightarrow{A}\to B\to B$$

where $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ is an E-model satisfying the Mingle Condition, and $x_0 \in W$. Suppose $x_0 R x$ and

(1) $\alpha, x \models_M \overrightarrow{A}\to B$.

Then our goal is to show

(2) $\alpha, x \models_M \overrightarrow{A}\to B\to B$.

For (2), we assume

(3-1) xRy, (3-2) $\beta, y \models_M \overrightarrow{A}$,
(4-1) yRz, and (4-2) $\gamma, z \models_M B$;

and we show $(\alpha\cdot\beta)\cdot\gamma, z \models_M B$. By (3-2), (4-1), and Lemma 3.2, we have

$\beta, z \models_M \overrightarrow{A}$;

and, by (1), (3-1), (4-1) and the transitivily of R, we have

(5) $\alpha\cdot\beta, z \models_M B$.

Then (4-2), (5) and the Mingle Condition imply

$(\alpha\cdot\beta)\cdot\gamma, z \models_M B$.

∎

Lemma 3.8 *The scheme* K^{01} *is vaild in any E-model that satisfies the Hereditary Condition.*

Proof Easy. ∎

Theorem 3.9 (Soundness Theorem) *The nine logics* **E**, **EM**$^{\#}$, **S4**, **E5**, **E5M**$^{\#}$, **S5I**, **R**, **RM0**, *and* **Int** *are sound with respcet to the class of E-models described in Figure 1. That is, for example, if* **EM**$^{\#}$ $\vdash A$, *then* A *is valid in any E-model that satisfies the Mingle Condition (and of course valid in any Linear Order E-model that satisfies the Mingle Condition); and if* **E5M**$^{\#}$ $\vdash A$, *then* A *is valid in any Single Cluster E-model that satisfies the Mingle Condition.*

Proof By Lemmas 3.3, 3.4, 3.5, 3.6, 3.7, and 3.8. ∎

In the rest of this section, we investigate another model for **S5I**.

A tripple $M = \langle\langle K, \leq\rangle, W, V\rangle$ is said to be an *S5I-model* if $\langle K, \leq\rangle$ is a quasi-ordered set, W is a non-empty set, and V is a subset of $K \times W \times$ Atom such that

$$((\alpha, x, p) \in V) \;\&\; (\alpha \leq \beta) \;\Rightarrow\; ((\beta, x, p) \in V)$$

holds for any $\alpha, \beta \in K$, any $x \in W$, and any propositional variable p. We define a notion

$$\alpha, x \models_M A$$

($\alpha \in K$, $x \in W$, and A is a formula) inductively as follows.

$\alpha, x \models_M p \iff (\alpha, x, p) \in V.$

$\alpha, x \models_M A{\rightarrow}B \iff$
$\forall\beta \in K, \forall y \in W[(\alpha \leq \beta) \;\&\; (\beta, y \models_M A) \;\Rightarrow\; (\beta, y \models_M B)].$

We say that a formula A is *valid in* the model M if $\alpha, x \models_M A$ for any $\alpha \in K$ and any $x \in W$.

If K is a singleton, then M is just a usual model for "S5 strict implication"; and if W is a singleton, then M is just a usual model for intuitionistic logic. Therefore S5I-models are considered to be intuitionistic variants of models for S5 strict implication. (This is the reason for the name **S5I**).

Let $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ be an E-model of **S5I**—a Single Cluster E-model satisfying the Hereditary Condition. Then an S5I-model M^*, which are naturally induced from M, is defined as follows.

$$M^* = \langle\langle I, \leq\rangle, W, V\rangle \text{ where } \alpha \leq \beta \iff \exists\gamma[\beta = \alpha{\cdot}\gamma].$$

(It is easy to verify that M^* satisfy the conditions of S5I-model.) These two models are equivalent in the following sense:

Theorem 3.10 $\alpha, x \models_M A$ *if and only if* $\alpha, x \models_{M^*} A$.

Proof Easy, by induction on A. ∎

This implies the following: (†) If A is valid in any S5I-model, than A is valid in any Single Cluster E-model that satisfies the Hereditary Condition. On the other hand, it is easy to show the soundness: (‡) If $\mathbf{S5I} \vdash A$, then A is valid in any S5I-model. By (†), (‡), and Theorem 6.6(completeness of **S5I** with respect to E-models), we have the following:

Theorem 3.11 (Completeness Theorem of S5I with respect to S5I-model) $\mathbf{S5I} \vdash A$ *if and only if* A *is valid in any S5I-model.*

Remark "Intuitionistic variants of the models of strict implications" were introduced in [8], where an S5I-model was defined as $\langle\langle K, \leq\rangle, \{W_k \mid k \in K\}, V\rangle$ ($W_k \subseteq W_{k'}$ if $k \leq k'$). The completeness theorem of **S5I** also holds for the S5I-models of this definition. ([8] did not mention the Hilbert-style axiomatization.)

4 Labelled sequent calculi and completeness

In this section we introduce six labelled sequent calculi (i.e., systems which derive labelled sequents) called $\mathcal{L}X$ ($X \in \{\mathbf{E}, \mathbf{EM}^{\#}, \mathbf{S4}, \mathbf{E5}, \mathbf{E5M}^{\#}, \mathbf{S5I}\}$). A labelled sequent consists of labelled formulas where the labels reflect the structure of E-models. We prove that the system $\mathcal{L}X$ is complete with respect to the class of E-models corresponding to the logic X.

A *label* is an ordered pair (α, x) where α is a finite set of natural numbers and x is a rational number. If (α, x) is a label and A is a formula, then the expression

$$(\alpha, x) : A$$

is called a *labelled formula*. In this section, letters $\Gamma, \Delta, \ldots$ will denote (possibly infinite) a multiset of labelled formulas; and, for example, $(\Gamma, \Delta, (\alpha, x) : A, (\beta, y) : B)$ denotes the multiset $\Gamma \cup_m \Delta \cup_m \{(\alpha, x) : A\} \cup_m \{(\beta, y) : B)\}$ where $\cup_m$ is the "multiset union".

If Γ and Δ are finite multiset of labelled formulas, then the expression

$$\Gamma \Rightarrow \Delta$$

is called a *labelled sequent*. Intuitively, a label (α, x) represents a pair of "piece α of information" and "possible world x" in an E-model M; and the meaning of a labelled sequent

$$(\alpha_1, x_1) : A_1, \ldots, (\alpha_m, x_m) : A_m \Rightarrow (\beta_1, y_1) : B_1, \ldots, (\beta_n, y_n) : B_n$$

is, like the sequent calculus **LK** for classical logics, the following:

$(\alpha_1, x_1 \not\models_M A_1)$ or $\cdots$ or $(\alpha_m, x_m \not\models_M A_m)$ or $(\beta_1, y_1 \models B_1)$ or $\cdots$ or $(\beta_n, y_n \models_M B)$.

If Γ is a multiset of labelled formulas, then we define a set $\mathrm{NSet}(\Gamma)$ of finite sets of natural numbers and a set $\mathrm{Rat}(\Gamma)$ of rational numbers as follows.

$$\mathrm{NSet}(\Gamma) = \{\alpha \mid \text{ there is a labelled formula } (\alpha, x)\!:\!A \text{ in } \Gamma\}.$$
$$\mathrm{Rat}(\Gamma) = \{x \mid \text{ there is a labelled formula } (\alpha, x)\!:\!A \text{ in } \Gamma\}.$$

That is, $\mathrm{NSet}(\Gamma)$ (or $\mathrm{Rat}(\Gamma)$) is the set of first (or second, resp.) elements of the labels in Γ. Let $\mathcal{X}$ be a set of rational numbers and $x, y \in \mathcal{X}$. If $x < y$ and if there is no rational number z in $\mathcal{X}$ such that $x < z < y$, then we say that x is the predecessor of y in $\mathcal{X}$ and that y is the successor of x in $\mathcal{X}$.

We define a labelled sequent calculus $\mathcal{L}\mathbf{E}$ as follows. Axioms of $\mathcal{L}\mathbf{E}$ are

$$\Gamma,\ (\alpha, x)\!:\!A \Rightarrow (\alpha, x)\!:\!A,\ \Delta.$$

Inference rules of $\mathcal{L}\mathbf{E}$ are (contraction left/right), ($\rightarrow$ label down), ($\rightarrow$ left 0/1), ($\rightarrow$ right) as follows:

$$\frac{\Gamma,\ (\alpha, x)\!:\!A,\ (\alpha, x)\!:\!A \Rightarrow \Delta}{\Gamma,\ (\alpha, x)\!:\!A \Rightarrow \Delta}\ \text{(contraction left)}$$

$$\frac{\Gamma \Rightarrow (\alpha, x)\!:\!A,\ (\alpha, x)\!:\!A,\ \Delta}{\Gamma \Rightarrow (\alpha, x)\!:\!A,\ \Delta}\ \text{(contraction right)}$$

$$\frac{\Gamma,\ (\alpha, y)\!:\!\overrightarrow{A} \Rightarrow \Delta}{\Gamma,\ (\alpha, x)\!:\!\overrightarrow{A} \Rightarrow \Delta}\ (\rightarrow \text{label down}) \text{ with a condition:}$$

(Label Condition): $x, y \in \mathrm{Rat}(\Gamma, \Delta)$ and x is the predecessor of y in $\mathrm{Rat}(\Gamma, \Delta)$.

$$\frac{\Gamma,\ (\alpha{\cup}\beta, x)\!:\!B \Rightarrow \Delta}{\Gamma,\ (\alpha, x)\!:\!p{\rightarrow}B,\ (\beta, x)\!:\!p \Rightarrow \Delta}\ (\rightarrow \text{left } 0)$$

$$\frac{\Gamma \Rightarrow (\beta, x)\!:\!\overrightarrow{A},\ \Delta \qquad \Gamma,\ (\alpha{\cup}\beta, x)\!:\!B \Rightarrow \Delta}{\Gamma,\ (\alpha, x)\!:\!\overrightarrow{A}{\rightarrow}B \Rightarrow \Delta}\ (\rightarrow \text{left } 1)$$

$$\frac{\Gamma,\ (\{a\}, y)\!:\!A \Rightarrow (\alpha{\cup}\{a\}, y)\!:\!B,\ \Delta}{\Gamma \Rightarrow (\alpha, x)\!:\!A{\rightarrow}B,\ \Delta}\ (\rightarrow \text{right}) \text{ with two conditions:}$$

(Label Condition 1): $a \notin \bigcup \mathrm{NSet}(\Gamma, \Delta) \cup \{\alpha\}$ (i.e., a does not appear in the lower sequent).

(Label Condition 2): $x \in \mathrm{Rat}(\Gamma, \Delta)$, $y \notin \mathrm{Rat}(\Gamma, \Delta)$, and x is the predecessor of y in $\mathrm{Rat}(\Gamma, \Delta) \cup \{y\}$.

Note that the rules

$$\frac{\Gamma,\ (\alpha,x)\!:\!A,\ (\beta,y)\!:\!B,\ \Delta \Rightarrow \Pi}{\Gamma,\ (\beta,y)\!:\!B,\ (\alpha,x)\!:\!A,\ \Delta \Rightarrow \Pi}\ \text{(exchange left)}$$

$$\frac{\Gamma \Rightarrow \Delta,\ (\alpha,x)\!:\!A,\ (\beta,y)\!:\!B,\ \Pi}{\Gamma \Rightarrow \Delta,\ (\beta,y)\!:\!B,\ (\alpha,x)\!:\!A,\ \Delta}\ \text{(exchange right)}$$

are implicitly available because a labelled sequent consists of multisets.

We will show that $\mathcal{L}\mathbf{E}$ is complete in the sense that $\mathcal{L}\mathbf{E} \vdash\ \Rightarrow (\emptyset,0)\!:\!A$ if A is valid in any Linear Order E-model. To show this, first we give some definitions, and then we prove Lemma 4.1.

If $\mathcal{X}$ is a set of finite set of natural numbers, then a set $\mathcal{X}^*$ is defined by

$$\mathcal{X}^* = \{\alpha \mid \exists n \geq 0, \exists \beta_1, \ldots, \beta_n \in \mathcal{X}[\alpha = \emptyset \cup \beta_1 \cup \cdots \cup \beta_n]\}.$$

In other words, $\mathcal{X}^*$ is the smallest set $\mathcal{Y}$ such that $\mathcal{X} \subseteq \mathcal{Y}$, $\emptyset \in \mathcal{Y}$, and $\mathcal{Y}$ is closed under the binary operator $\cup$. An E-model $M = \langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ is said to be a *label model* if it satisfies the following conditions.

- I is a set of finite sets of natural numbers such that $I^* = I$.
- $\alpha\cdot\beta = \alpha \cup \beta$.
- $\mathbf{e} = \emptyset$.
- W is a set of rational numbers.
- $xRy \iff x \leq y$.

Obviously this is a Liner Order E-model. We say that a labelled formula $(\alpha,x)\!:\!A$ is *true* (or *false*) *in* M if $\alpha \in I$, $x \in W$, and $\alpha, x \models_M A$ (or $\alpha, x \not\models_M A$, resp.).

Let Γ and Δ be multisets of labelled formulas. We say that the pair $\langle \Gamma, \Delta\rangle$ is $\mathcal{L}\mathbf{E}$-*saturated* if it satisfies the following four conditions.

(1) If $[(\alpha,x)\!:\!\overrightarrow{A} \in \Gamma$, $x < y$, and $y \in \mathrm{Rat}(\Gamma,\Delta)]$, then $(\alpha,y)\!:\!\overrightarrow{A} \in \Gamma$. (Converse of iteration of ($\rightarrow$ label down))

(2) If $[(\alpha,x)\!:\!p{\rightarrow}B \in \Gamma$ and $(\beta,x)\!:\!p \in \Gamma]$, then $(\alpha{\cup}\beta,x)\!:\!B \in \Gamma$. (Converse of ($\rightarrow$ left 0))

(3) If $[(\alpha,x)\,:\,\overrightarrow{A}{\rightarrow}B \in \Gamma$ and $\beta \in (\mathrm{NSet}(\Gamma,\Delta))^*]$, then $[(\beta,x)\,:\,\overrightarrow{A} \in \Delta$ or $(\alpha{\cup}\beta,x)\!:\!B \in \Gamma]$. (Converse of ($\rightarrow$ left 1))

(4) If $(\alpha,x)\!:\!A{\rightarrow}B \in \Delta$, then $[(\{a\},y)\!:\!A \in \Gamma$ and $(\alpha{\cup}\{a\},y)\!:\!B \in \Delta$ for some a and for some $y > x]$. (Converse of ($\rightarrow$ right))

Lemma 4.1 *Let Γ and Δ be finite multisets of labelled formulas. If $\mathcal{L}\mathbf{E} \nvdash \Gamma \Rightarrow \Delta$, then there exists a label model M such that any labelled formula in Γ is true in M and any labelled formula in Δ is false in M.*

Proof We call a triple $\langle(\varphi, u) : F, \psi, v\rangle$ a *seed for saturation* where $(\varphi, u) : F$ is a labelled formula, ψ is a finite set of natural numbers, and v is a rational number. Since the set of finite sets of natural numbers, the set of rational numbers, and the set of formulas are countable, we can enumerate all seeds for saturations as

$$\langle(\varphi_1, u_1) : F_1, \psi_1, v_1\rangle,\ \langle(\varphi_2, u_2) : F_2, \psi_2, v_2\rangle,\ \ldots$$

so that every seed occurs infinitely often in the enumeration. Using this enumeration, we define a sequence $\Gamma_i \Rightarrow \Delta_i$ $(i = 0, 1, 2, \ldots)$ of unprovable labelled sequents as follows.

[Step 0] $(\Gamma_0 \Rightarrow \Delta_0) = (\Gamma \Rightarrow \Delta)$

[Step k] Suppose that $\Gamma_{k-1} \Rightarrow \Delta_{k-1}$ is already defined and is not provable in $\mathcal{L}\mathbf{E}$. In the following, we define four unprovable labelled sequents $\Pi_1 \Rightarrow \Sigma_1, \ldots, \Pi_4 \Rightarrow \Sigma_4$ (for each $i \in \{1, \ldots, 4\}$, the labelled sequent $\Pi_i \Rightarrow \Sigma_i$ is constructed to satisfy the condition (i) of $\mathcal{L}\mathbf{E}$-saturatedness); and finally we take $\Pi_4 \Rightarrow \Sigma_4$ as $\Gamma_k \Rightarrow \Delta_k$ which is the goal of this step k.

(1) If [F_k is an implication, $(\varphi_k, u_k) : \overrightarrow{F_k} \in \Gamma_{k-1}$, $u_k < v_k$, and $v_k \in \mathrm{Rat}(\Gamma_{k-1}, \Delta_{k-1})$], then

$$(\Pi_1 \Rightarrow \Sigma_1) = (\Gamma_{k-1},\ (\varphi_k, v_k) : \overrightarrow{F_k} \Rightarrow \Delta_{k-1});$$

and otherwise $(\Pi_1 \Rightarrow \Sigma_1) = (\Gamma_{k-1} \Rightarrow \Delta_{k-1})$. The fact $\mathcal{L}\mathbf{E} \nvdash \Pi_1 \Rightarrow \Sigma_1$ is guaranteed by the fact $\mathcal{L}\mathbf{E} \nvdash \Gamma_{k-1} \Rightarrow \Delta_{k-1}$ and the rules ($\rightarrow$ label down) and (contraction left).

(2) If [F_k is of the form $p \rightarrow B$, $(\varphi_k, u_k) : p \rightarrow B \in \Pi_1$, and $(\psi_k, u_k) : p \in \Pi_1$], then

$$(\Pi_2 \Rightarrow \Sigma_2) = (\Pi_1,\ (\varphi_k \cup \psi_k, u_k) : B \Rightarrow \Sigma_1);$$

and otherwise $(\Pi_2 \Rightarrow \Sigma_2) = (\Pi_1 \Rightarrow \Sigma_1)$. The fact $\mathcal{L}\mathbf{E} \nvdash \Pi_2 \Rightarrow \Sigma_2$ is guaranteed by the fact $\mathcal{L}\mathbf{E} \nvdash \Pi_1 \Rightarrow \Sigma_1$ and the rules ($\rightarrow$ left 0) and (contraction left).

(3) If [F_k is of the form $\overrightarrow{A} \rightarrow B$, $(\varphi_k, u_k) : \overrightarrow{A} \rightarrow B \in \Pi_2$, and $\psi_k \in \mathrm{NSet}(\Pi_2, \Sigma_2)^*$], then

$$(\Pi_3 \Rightarrow \Sigma_3) = \begin{cases} (\Pi_2 \Rightarrow (\psi_k, u_k) : \overrightarrow{A},\ \Sigma_2) & \text{(if this labelled sequent is not provable in } \mathcal{L}\mathbf{E}\text{)}, \\ (\Pi_2,\ (\varphi_k \cup \psi_k, u_k) : B \Rightarrow \Sigma_2) & \text{(otherwise)}; \end{cases}$$

and otherwise $(\Pi_3 \Rightarrow \Sigma_3) = (\Pi_2 \Rightarrow \Sigma_2)$. The fact $\mathcal{L}\mathbf{E} \not\vdash \Pi_3 \Rightarrow \Sigma_3$ is guaranteed by the fact $\mathcal{L}\mathbf{E} \not\vdash \Pi_2 \Rightarrow \Sigma_2$ and the rules ($\rightarrow$ left 1) and (contraction left).

(4) If $[F_k$ is an implication, say $A \rightarrow B$, and $(\varphi_k, u_k) : A \rightarrow B \in \Sigma_3]^{(\dagger)}$, then we take a natural number a and a rational number y such that

- $a \notin \bigcup \mathrm{NSet}(\Pi_3, \Sigma_3)$,
- $y \notin \mathrm{Rat}(\Pi_3, \Sigma_3)$, and
- u_k is the predecessor of y in $\mathrm{Rat}(\Pi_4, \Sigma_4)$;

and we define

$$(\Pi_4 \Rightarrow \Sigma_4) = (\Pi_3,\ (\{a\}, y) : A \Rightarrow (\varphi_k \cup \{a\}, y) : B,\ \Sigma_3).$$

Since the elements of (Π_3, Σ_3) is finite, we can take such a and y. If the condition ($\dagger$) fails, then $(\Pi_4 \Rightarrow \Sigma_4) = (\Pi_3 \Rightarrow \Sigma_3)$. The fact $\mathcal{L}\mathbf{E} \not\vdash \Pi_4 \Rightarrow \Sigma_4$ is guaranteed by the fact $\mathcal{L}\mathbf{E} \not\vdash \Pi_3 \Rightarrow \Sigma_3$ and the rules ($\rightarrow$ right) and (contraction right).

This completes the construction of the infinite sequence $\Gamma_i \Rightarrow \Delta_i$ $(i = 0, 1, 2, \ldots)$ such that $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$, and $\Delta = \Delta_0 \subseteq \Delta_1 \subseteq \cdots$. Then we define $\Gamma_\infty = \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta_\infty = \bigcup_{i=0}^{\infty} \Delta_i$, and we show the following: (1) $\langle \Gamma_\infty, \Delta_\infty \rangle$ is $\mathcal{L}\mathbf{E}$-saturated. (2) $\Gamma_\infty \cap \Delta_\infty = \emptyset$.

[Proof of (1)] We show the condition (3) of the definition of $\mathcal{L}\mathbf{E}$-saturatedness ((1), (2) and (4) are similar). Assume that

(i) $(\alpha, x) : \overrightarrow{A} \rightarrow B \in \Gamma_\infty$, and

(ii) $\beta \in (\mathrm{NSet}(\Gamma_\infty, \Delta_\infty))^*$.

By (i), there is a natural number p such that $(\alpha, x) : \overrightarrow{A} \rightarrow B \in \Gamma_p$; and by (ii) and the definition of $(\mathrm{NSet}(\Gamma_\infty, \Delta_\infty))^*$, there is a natural number q such that $\beta \in (\mathrm{NSet}(\Gamma_q, \Delta_q))^*$. Since the seed $\langle (\alpha, x) : \overrightarrow{A} \rightarrow B, \beta, v \rangle$ (v is arbitrary) occurs infinitely often in the enumaration, there is a natural number $k \geq p, q$ that "hits" the above construction of $\Pi_3 \Rightarrow \Sigma_3$; that is, there is a natural number k such that $(\beta, x) : \overrightarrow{A} \in \Delta_k$ or $(\alpha \cup \beta, x) : B \in \Gamma_k$. This means $(\beta, x) : \overrightarrow{A} \in \Delta_\infty$ or $(\alpha \cup \beta, x) : B \in \Gamma_\infty$.

[Proof of (2)] If there is a labled formula $(\alpha, x) : A$ in $\Gamma_\infty \cap \Delta_\infty$, then there is a natural number k such that $(\alpha, x) : A \in \Gamma_k \cap \Delta_k$, and this means $\Gamma_k \Rightarrow \Delta_k$ is an axiom of $\mathcal{L}\mathbf{E}$. This contradicts the fact $\mathcal{L}\mathbf{E} \not\vdash \Gamma_k \Rightarrow \Delta_k$.

Now we define a label model M to be $\langle \langle I, \cup, \emptyset \rangle, \langle W, \leq \rangle, V \rangle$ where

$I = (\mathrm{NSet}(\varGamma_\infty, \varDelta_\infty))^*$,
$W = \mathrm{Rat}(\varGamma_\infty, \varDelta_\infty)$,
$V = \{(\alpha, x, p) \mid (\alpha, x)\!:\!p \in \varGamma_\infty\}$.

Then we prove that the following conditions hold for any labelled formula $(\alpha, x)\!:\!A$.

- If $(\alpha, x)\!:\!A \in \varGamma_\infty$, then it is true in M.
- If $(\alpha, x)\!:\!A \in \varDelta_\infty$, then it is false in M.

Since $\varGamma \subseteq \varGamma_\infty$ and $\varDelta \subseteq \varDelta_\infty$, this implies that M is the required model. The above claim is proved by induction on the complexity of the formula A. Here we show two cases. (The other cases are similar.)

(Case 1): $(\alpha, x) : p \in \varDelta_\infty$. In this case, $(\alpha, x) : p \notin \varGamma_\infty$ because of the fact $\varGamma_\infty \cap \varDelta_\infty = \emptyset$. Therefore $(\alpha, x)\!:\!p$ is false in M by the definition of V.

(Case 2): $(\alpha, x)\!:\!p{\to}B \in \varGamma_\infty$ $^{(\ddagger)}$. To show that $(\alpha, x)\!:\!p{\to}B$ is true in M, we first assume

(i) $x \le y$ and $y \in W = \mathrm{Rat}(\varGamma_\infty, \varDelta_\infty)$, and

(ii) $(\beta, y)\!:\!p$ is true in M (therefore $(\beta, y)\!:\!p \in \varGamma_\infty$ by the definition of V);

and we show that $(\alpha{\cup}\beta, y) : B$ is true in M. By ($\ddagger$), (i), and the condition (1) of $\mathcal{L}$**E**-saturatedness of $\langle \varGamma_\infty, \varDelta_\infty \rangle$, we have $(\alpha, y)\!:\!p{\to}B \in \varGamma_\infty$. Then, (ii) and the condition (2) of $\mathcal{L}$**E**-saturatedness imply $(\alpha{\cup}\beta, y) : B \in \varGamma_\infty$. This means, by the induction hypothesis for B, that $(\alpha{\cup}\beta, y)\!:\!B$ is true in M. ▮

Theorem 4.2 *If a formula A is valid in any Linear Order E-model, then* $\mathcal{L}$**E** $\vdash\ \Rightarrow (\emptyset, 0)\!:\!A$.

Proof By the previous lemma. (If $\mathcal{L}$**E** $\nvdash\ \Rightarrow (\emptyset, 0)\!:\!A$, then A is not valid in a label model, which is a Liner Order E-model.) ▮

Next we introduce labelled sequent calculi for **EM**$^\#$ and **S4**, and prove their completeness. The system $\mathcal{L}$**EM**$^\#$ is obtained from $\mathcal{L}$**E** by adding the rule

$$\frac{\varGamma,\ (\alpha{\cup}\beta, x)\!:\!p \Rightarrow \varDelta}{\varGamma,\ (\alpha, x)\!:\!p,\ (\beta, x)\!:\!p \Rightarrow \varDelta,}\ \text{(atomic minglement)}$$

and the system $\mathcal{L}$**S4** is obtained from $\mathcal{L}$**E** by adding the rule

$$\frac{\varGamma,\ (\alpha{\cup}\beta, x)\!:\!p \Rightarrow \varDelta}{\varGamma,\ (\alpha, x)\!:\!p \Rightarrow \varDelta.}\ \text{(atomic heredity)}$$

A pair $\langle \varGamma, \varDelta \rangle$ of multiset of labelled formulas is said to be $\mathcal{L}$**EM**$^\#$*-saturated* if it is $\mathcal{L}$**E**-saturated and satisfies the additional condition:

(5) (for $\mathcal{L}\mathbf{EM}^{\#}$) If $[(\alpha, x) : p \in \Gamma$ and $(\beta, x) : p \in \Gamma]$, then $(\alpha \cup \beta, x) : p \in \Gamma$. (Converse of (atomic minglement))

Moreover $\langle \Gamma, \Delta \rangle$ is said to be $\mathcal{L}\mathbf{S4}$-*saturated* if it is $\mathcal{L}\mathbf{E}$-saturated and satisfies the additional condition:

(5) (for $\mathcal{L}\mathbf{S4}$) If $[(\alpha, x) : p \in \Gamma$ and $\beta \in (\mathrm{NSet}(\Gamma, \Delta))^*]$, then $(\alpha \cup \beta, x) : p \in \Gamma$. (Converse of (atomic heredity))

Lemma 4.3 *Let Γ and Δ be finite multisets of labelled formulas. If $\mathcal{L}\mathbf{EM}^{\#}$ (or $\mathcal{L}\mathbf{S4}$) $\not\vdash \Gamma \Rightarrow \Delta$, then there exists a label model M such that any labelled formula in Γ is true in M, any labelled formula in Δ is false in M, and M satisfies the Atomic Mingle Condition (the Atomic Hereditary Condition, resp.).*

Proof The proof is similar to the proof of Lemma 4.1. First we construct an $\mathcal{L}\mathbf{EM}^{\#}$-saturated (or $\mathcal{L}\mathbf{S4}$-saturated) pair $\langle \Gamma_\infty, \Delta_\infty \rangle$. The construction is the same as that in Lemma 4.1 except that we define five unprovable labelled sequents $\Pi_1 \Rightarrow \Sigma_1, \ldots, \Pi_5 \Rightarrow \Sigma_5$ in the step k, and $(\Gamma_k \Rightarrow \Delta_k) = (\Pi_5 \Rightarrow \Sigma_5)$. The definition of $\Pi_5 \Rightarrow \Sigma_5$ is described as follows.

(5) (for $\mathcal{L}\mathbf{EM}^{\#}$) If $[F_k$ is an atomic formula, say p, $(\varphi_k, u_k) : p \in \Pi_4$, and $(\psi_k, u_k) : p \in \Pi_4]$, then

$$(\Pi_5 \Rightarrow \Sigma_5) = (\Pi_4,\ (\varphi_k \cup \psi_k, u_k) : p \Rightarrow \Sigma_4);$$

and otherwise $(\Pi_5 \Rightarrow \Sigma_5) = (\Pi_4 \Rightarrow \Sigma_4)$. The fact $\mathcal{L}\mathbf{EM}^{\#} \not\vdash \Pi_5 \Rightarrow \Sigma_5$ is guaranteed by the fact $\mathcal{L}\mathbf{EM}^{\#} \not\vdash \Pi_4 \Rightarrow \Sigma_4$ and the rules (atomic minglement) and (contraction left).

(5) (for $\mathcal{L}\mathbf{S4}$) If $[F_k$ is an atomic formula, say p, $(\varphi_k, u_k) : p \in \Pi_4$, and $\psi_k \in (\mathrm{NSet}(\Pi_4, \Sigma_4))^*]$, then

$$(\Pi_5 \Rightarrow \Sigma_5) = (\Pi_4,\ (\varphi_k \cup \psi_k, u_k) : p \Rightarrow \Sigma_4);$$

and otherwise $(\Pi_5 \Rightarrow \Sigma_5) = (\Pi_4 \Rightarrow \Sigma_4)$. The fact $\mathcal{L}\mathbf{S4} \not\vdash \Pi_5 \Rightarrow \Sigma_5$ is guaranteed by the fact $\mathcal{L}\mathbf{S4} \not\vdash \Pi_4 \Rightarrow \Sigma_4$ and the rules (atomic heredity) and (contraction left).

Then we construct a label model M by the same way as Lemma 4.1. The condition (5) of $\mathcal{L}\mathbf{EM}^{\#}$-saturatedness implies the Atomic Mingle Condition of M, and the condition (5) of $\mathcal{L}\mathbf{S4}$-saturatedness implies the Atomic Hereditary Condition of M. ∎

Theorem 4.4
(1) *If a formula A is valid in any Linear Order E-model that satisfies the Mingle Condition, then $\mathcal{L}\mathbf{EM}^{\#} \vdash\ \Rightarrow (\emptyset, 0) : A$.*
(2) *If a formula A is valid in any Linear Order E-model that satisfies the Hereditary Condition, then $\mathcal{L}\mathbf{S4} \vdash\ \Rightarrow (\emptyset, 0) : A$.*

Proof By the previous lemma and Theorem 3.1. ∎

We have introduced three labelled sequent calculi for **E**, $\mathbf{EM}^{\#}$, and **S4**, and prove their completeness. In the rest of this section, we show similar results for **E5**, $\mathbf{E5M}^{\#}$, and **S5I**.

A labelled sequent calculus ($\mathcal{L}\mathbf{E5}$/ $\mathcal{L}\mathbf{E5M}^{\#}$/ $\mathcal{L}\mathbf{S5I}$) is obtained from ($\mathcal{L}\mathbf{E}$/ $\mathcal{L}\mathbf{EM}^{\#}$/ $\mathcal{L}\mathbf{S4}$, resp.) by replacing the rule ($\rightarrow$ label down) with the stronger rule

$$\frac{\Gamma,\ (\alpha, y)\colon \overrightarrow{A} \Rightarrow \Delta}{\Gamma,\ (\alpha, x)\colon \overrightarrow{A} \Rightarrow \Delta}\ (\rightarrow \text{label}) \quad \text{where } x, y \in \mathrm{Rat}(\Gamma, \Delta) \text{ and } x \neq y.$$

(Therefore $\mathcal{L}\mathbf{E5M}^{\#}$= $\mathcal{L}\mathbf{E5}$ + (atomic minglement), and $\mathcal{L}\mathbf{S5I}$= $\mathcal{L}\mathbf{E5}$ + (atomic heredity).) A pair $\langle \Gamma, \Delta\rangle$ of multiset of labelled formulas is said to be ($\mathcal{L}\mathbf{E5}$-*saturated* / $\mathcal{L}\mathbf{E5M}^{\#}$-*saturated* / $\mathcal{L}\mathbf{S5I}$-*saturated*) if it is ($\mathcal{L}\mathbf{E}$-saturated / $\mathcal{L}\mathbf{EM}^{\#}$-saturated / $\mathcal{L}\mathbf{S4}$-saturated, resp.) and satisfies the additional condition, which is stronger than the condition (1):

(1^+) If $[(\alpha, x)\colon \overrightarrow{A} \in \Gamma$ and $y \in \mathrm{Rat}(\Gamma, \Delta)]$, then $(\alpha, y)\colon \overrightarrow{A} \in \Gamma$. (Converse of ($\rightarrow$ label))

A model $\langle\langle I, \cdot, \mathbf{e}\rangle, \langle W, R\rangle, V\rangle$ is said to be a *single cluster label model* if it satisfies the following conditions.

- I is a set of finite sets of natural numbers such that $I^* = I$.
- $\alpha{\cdot}\beta = \alpha \cup \beta$.
- $\mathbf{e} = \emptyset$.
- W is a set of rational numbers.
- xRy for any x and y.

In other words, a single cluster label model is obtained from a label model by extending the accessibility relation to the universal relation.

Lemma 4.5 *Let Γ and Δ be finite multisets of labelled formulas, and $\mathcal{L}$ be one of the systems $\mathcal{L}\mathbf{E5}$, $\mathcal{L}\mathbf{E5M}^{\#}$, and $\mathcal{L}\mathbf{S5I}$. If $\mathcal{L} \nvdash \Gamma \Rightarrow \Delta$, then there exists a single cluster label model M such that any labelled formula in Γ is true in M and any labelled formula in Δ is false in M. Moreover, M satisfies the Atomic Mingle Condition if $\mathcal{L} = \mathcal{L}\mathbf{E5M}^{\#}$, and satisfies the Atomic Hereditary Condition if $\mathcal{L} = \mathcal{L}\mathbf{S5I}$.*

Proof The proof is similar to the proofs of Lemmas 4.1 and 4.3. The difference is the construction of $\Pi_1 \Rightarrow \Sigma_1$. We replace the item (1) in the proofs of the lemmas to the following:

(1$^+$) If [F_k is an implication, $(\varphi_k, u_k):\overrightarrow{F_k} \in \Gamma_{k-1}$, and $v_k \in \mathrm{Rat}(\Gamma_{k-1}, \Delta_{k-1})$], then

$$(\Pi_1 \Rightarrow \Sigma_1) = (\Gamma_{k-1},\ (\varphi_k, v_k):\overrightarrow{F_k} \Rightarrow \Delta_{k-1});$$

and otherwise $(\Pi_1 \Rightarrow \Sigma_1) = (\Gamma_{k-1} \Rightarrow \Delta_{k-1})$. The fact $\mathcal{L} \not\vdash \Pi_1 \Rightarrow \Sigma_1$ is guaranteed by The fact $\mathcal{L} \not\vdash \Gamma_{k-1} \Rightarrow \Delta_{k-1}$ and the rules ($\rightarrow$ label) and (contraction left).

Then we get an $\mathcal{L}$-saturated pair $\langle \Gamma_\infty, \Delta_\infty \rangle$, and we construct the required single cluster label model M by the same way as the proofs of the lemmas. ∎

Theorem 4.6
(1) *If a formula A is valid in any Single Cluster E-model then* $\mathcal{L}\mathbf{E5} \vdash\ \Rightarrow (\emptyset, 0):A$.
(2) *If a formula A is valid in any Single Cluster E-model that satisfies the Mingle Condition, then* $\mathcal{L}\mathbf{E5M}^\# \vdash\ \Rightarrow (\emptyset, 0):A$.
(3) *If a formula A is valid in any Single Cluster E-model that satisfies the Hereditary Condition, then* $\mathcal{L}\mathbf{S5I} \vdash\ \Rightarrow (\emptyset, 0):A$.

Proof By the previous lemma and Theorem 3.1. ∎

5 From labelled sequents to unlabelled sequents

In this section we prove the following: *If a labelled sequent $\mathcal{S}$ is provable in $\mathcal{L}X$, then an unlabelled sequent $\mathcal{T}$ is provable in $\mathcal{G}X$ where $\mathcal{T}$ is "extracted" from $\mathcal{S}$ in a certain way.* This includes the following claim as a special case: *If $\mathcal{L}X \vdash\ \Rightarrow (\emptyset, 0):A$, then $\mathcal{G}X \vdash\ \mapsto A$.* Before giving detailed argument, we explain an outline of the "extraction".

In the case of relevant logic **R**, the situation is relatively simple. The sequent calculus $\mathcal{G}\mathbf{R}$ is obtained from $\mathcal{G}\mathbf{E}$ by adding the ordinary exchange rule

$$\frac{\Gamma, A, B, \Delta \mapsto C}{\Gamma, B, A, \Delta \mapsto C.}$$

The labelled sequent calculus $\mathcal{L}\mathbf{R}$ is defined as follows, where a label is a finite set of natural numbers. (Since the model of **R** is "single world", labels reflect only the structure of "pieces of information".) Axioms of $\mathcal{L}\mathbf{R}$ are

$$\Gamma,\ \alpha:A \Rightarrow \alpha:A,\ \Delta.$$

Inference rules of $\mathcal{L}\mathbf{R}$ are "contraction left/right" and the following.

$$\frac{\Gamma \Rightarrow \beta:A,\ \Delta \qquad \Gamma,\ \alpha\cup\beta:B \Rightarrow \Delta}{\Gamma,\ \alpha:A\rightarrow B \Rightarrow \Delta}\ (\rightarrow \text{left})$$

$$\frac{\Gamma,\ \{a\}:A \Rightarrow \alpha\cup\{a\}:B,\ \Delta}{\Gamma \Rightarrow \alpha:A\rightarrow B,\ \Delta}\ (\rightarrow \text{right}) \text{ with a condition:}$$

(Label Condition) a does not appear in the lower sequent. Then the "extraction" from $\mathcal{L}\mathbf{R}$ is stated as follows.

If $\mathcal{L}\mathbf{R} \vdash \Gamma \Rightarrow \Delta$, then there exists a labelled formula $\beta : B \in \Delta$ and there exist labelled formulas $\alpha_1 : A_1, \ldots, \alpha_n : A_n \in \Gamma$ $(n \geq 0)$ such that

- $\alpha_1 \cup \cdots \cup \alpha_n = \beta$; *and*
- $\mathcal{G}\mathbf{R} \vdash A_1, \ldots, A_n \mapsto B$.

This is proved by induction on the proofs in $\mathcal{L}\mathbf{R}$.

The "extraction" from the systems introduced in the previous section basically follows the above one for $\mathcal{L}\mathbf{R}$. However, there is a counterexample which shows the above simple strategy does not work for $\mathcal{L}\mathbf{EM}^{\#}$: The labelled sequent $(\{1\}, 0) : p, (\{2\}, 0) : p \Rightarrow (\{1, 2\}, 0) : p$ is provable in $\mathcal{L}\mathbf{EM}^{\#}$, but the sequent $p, p \mapsto p$ is not provable in $\mathcal{G}\mathbf{EM}^{\#}$. To deal with such a case, the "extraction" from $\mathcal{L}\mathbf{EM}^{\#}$ is established as follows.

If $\mathcal{L}\mathbf{EM}^{\#} \vdash \Gamma \Rightarrow \Delta$, then there exists a labelled formula $(\beta, y) : B \in \Delta$ and there exist labelled formulas $(\alpha_1, x_1) : A_1, \ldots, (\alpha_n, x_n) : A_n \in \Gamma$ $(n \geq 0)$ such that

- $\alpha_1 \cup \cdots \cup \alpha_n = \beta$;
- $x_1, \ldots, x_n \leq y$; *and*
- $\mathcal{G}\mathbf{EM}^{\#} \vdash C_1, \ldots, C_k \mapsto B$ *for any "normal" sequence* $(C_1, \ldots, C_k)$ *obtained from* $(A_1, \ldots, A_n)$ *by certain "rules".*

This is a sketch, and the precise descriptions will be given by Main Lemmas 5.8 (for $\mathcal{L}\mathbf{E}$, $\mathcal{L}\mathbf{EM}^{\#}$, and $\mathcal{L}\mathbf{S4}$) and 5.10 (for $\mathcal{L}\mathbf{E5}$, $\mathcal{L}\mathbf{E5M}^{\#}$, and $\mathcal{L}\mathbf{S5I}$). Here we only note some facts, which indicate how to handle the above counterexample.

- (p, p) is not "normal".
- $(\overrightarrow{A} \to p, \overrightarrow{A}, p)$ is "normal", and it is obtained from (p, p) by the "rules".
- $\mathcal{G}\mathbf{EM}^{\#} \vdash \overrightarrow{A} \to p, \overrightarrow{A}, p \mapsto p$.

Now we start detailed proofs. In this section, the letters $\Gamma, \Delta, \ldots$ will be used as metavariables for finite sequences of formulas, for finite multisets of formulas, or for finite multisets of labelled formulas, depending on the context.

First we introduce some rules which transform finite multisets of formulas:

$\dfrac{\Gamma, A, A}{\Gamma, A}$ (contraction)("co", for short)

$\dfrac{\Gamma, B}{\Gamma, A \to B, A}$ (implication)("imp", for short)

$\dfrac{\Gamma, B}{\Gamma, A \to B, A, B}$ (copy-implication)("c-imp", for short)

$\dfrac{\Gamma}{\Gamma, A}$ (weakening)("we", for short)

That is, for exmple, the multiset $(W, X, Y, X \to Z)$ of formulas can be obtained from the multiset (W, Y, Z) by "implication". Then we define three sets of the rules as follows.

$$\rhd_{\mathbf{E}} \; = \{\text{co, imp}\}.$$
$$\rhd_{\mathbf{EM}^{\#}} \; = \{\text{co, imp, c-imp}\}.$$
$$\rhd_{\mathbf{S4}} \; = \{\text{co, imp, we}\}.$$

(The subscripts $\mathbf{E}/\mathbf{EM}^{\#}/\mathbf{S4}$ represent that these are devices to show the "extraction" for the denoted logics.) Let $\rhd_X$ be one of these sets, and Γ and Δ be finite multisets of formulas. We write

$$\Gamma \; \rhd_X \; \Delta$$

if Δ is obtained from Γ by finitely successive (posiibly zero) applications of the rules in $\rhd_X$. The following lemmas are obvious by the definition, and they will be implicitly used later.

Lemma 5.1
(1) $\emptyset \; \rhd_X \; \emptyset$, *for* $X \in \{\mathbf{E}, \mathbf{EM}^{\#}, \mathbf{S4}\}$.
(2) $\Gamma \; \rhd_X \; \emptyset$ *only if* $\Gamma = \emptyset$, *for* $X \in \{\mathbf{E}, \mathbf{EM}^{\#}, \mathbf{S4}\}$.
(3) $\emptyset \; \rhd_X \; \Delta$ *only if* $\Delta = \emptyset$, *for* $X \in \{\mathbf{E}, \mathbf{EM}^{\#}\}$.

Lemma 5.2 *If* $\Gamma_1 \; \rhd_X \; \Delta_1$ *and* $\Gamma_2 \; \rhd_X \; \Delta_2$, *then* $(\Gamma_1, \Gamma_2) \; \rhd_X \; (\Delta_1, \Delta_2)$, *for* $X \in \{\mathbf{E}, \mathbf{EM}^{\#}, \mathbf{S4}\}$.

Let Γ be a finite sequence of formulas or be a finite multiset of formulas. We say that Γ is *atom-normal* if (1) Γ contains at most one atomic formula; and (2) the atomic formula exists at the right most position if Γ is a sequence which contains an atomic formula. In other words, Γ is atom-normal if either Γ does not contain an atomic formula or $\Gamma = (\Gamma', p)$ where Γ' does not contain an atomic formula.

If Γ is an atom-normal multiset of formulas, then Γ° denotes an atom-normal sequence of formulas such that the members of Γ° are exactly the same as Γ (this means that the atomic formula, if exists, is at the right most position in Γ°). For example, both the sequences $(\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{A}, \overrightarrow{B}, p)$ and $(\overrightarrow{B}, \overrightarrow{B}, \overrightarrow{A}, \overrightarrow{A}, p)$ are $\{p, \overrightarrow{A}, \overrightarrow{A}, \overrightarrow{B}, \overrightarrow{B}\}^\circ$.

Let Γ and Δ be atom-normal multisets whose atomic formulas, if exit, are identical. Then we define an atom-normal multiset $\Gamma \cup^\circ \Delta$ as follows.

$$\Gamma \cup^\circ \Delta = \begin{cases} (\Gamma', \Delta', p) & \text{(if } \Gamma = (\Gamma', p) \text{ and } \Delta = (\Delta', p)\text{)}, \\ (\Gamma, \Delta) & \text{(otherwise)}. \end{cases}$$

Let Γ and Δ be finite multisets of formulas. We say that Δ is a *contraction* of Γ if Δ is obtained from Γ by finitely successive (possibly zero) applications of "contraction".

Lemma 5.3 *Let X be one of* **E**, $\mathbf{EM}^{\#}$, *and* **S4**. *If $n \geq 1$, $(A_1, \ldots, A_n) \rhd_X \Gamma$, and Γ is atom-normal, then there exist atom-normal multisets $\Gamma_1, \ldots, \Gamma_n$ that satisfy the following.*

(1) *Γ is a contraction of $(\Gamma_1, \ldots, \Gamma_n)$. (Therefore, the atomic formulas in $\Gamma_1, \ldots, \Gamma_n$ are, if exist, identical because of the atom-normalness of Γ.)*

(2) $A_i \rhd_X \Gamma_i$ $(i = 1, \ldots, n)$.

(3) $\mathcal{G}X \vdash \Gamma_i^\circ \mapsto A_i$ $(i = 1, \ldots, n)$.

(*We will call Γ_i the* descendant *of A_i.*)

Note that in the item (3) above, the arbitrariness of the order of implications in Γ_i° does not cause a problem since $\mathcal{G}X$ has the rule (ex^{010}).

Proof By induction on the number of the steps of transformation (i.e., the number of applications of the rules) from $(A_1, \ldots, A_n)$ to Γ.

(Case 0): The number is 0; that is, $\Gamma = (A_1, \ldots, A_n)$. In this case, the descendant of A_i is the singleton A_i.

(Case 1): The number is $k > 0$; that is, there exists a multiset Γ' that satisfies the following.

- Γ' is obtained from $(A_1, \ldots, A_n)$ by one application of a rule, say R.
- $\Gamma' \rhd_X \Gamma$ with $(k-1)$-steps. (Therefore, the induction hypothesis is available for Γ'.)

(Subcase 1-1): R = contraction. In this case, $n \geq 2$, $\Gamma' = (A_1, \ldots, A_{n-1})$ and $A_n = A_{n-1}$ (by arbitrariness of the order in a multiset). Then, by the induction hypothesis, there are descendants $\Gamma_1, \ldots, \Gamma_{n-1}$ of $A_1, \ldots, A_{n-1}$; and the required descendants of $A_1, \ldots, A_{n-1}, A_n$ are $\Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_{n-1}$, respectively. The conditions (1), (2) and (3) are easily verified by the induction hypothesis.

(Subcase 1-2): R = implication. In this case, $\Gamma' = (A_1, \ldots, A_{n-1}, C {\to} A_n, C)$, and there are descendants $\Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_{C\to A_n}, \Gamma_C$ of, respectively, $A_1, \ldots, A_{n-1}$, $C {\to} A_n$, C by the induction hypothesis. Then the required descendants of $A_1, \ldots, A_{n-1}, A_n$ are $\Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_{C\to A_n} \cup^\circ \Gamma_C$. The condition (1) is shown by the definition of $\Gamma_{C\to A_n} \cup^\circ \Gamma_C$ and the induction hypothesis. The condition (2) is shown by the fact

$$A_n \rhd_X (C {\to} A_n, C) \rhd_X (\Gamma_{C\to A_n}, \Gamma_C) \rhd_X \Gamma_{C\to A_n} \cup^\circ \Gamma_C$$

which is guaranteed by Lemma 5.2 and the induction hypothesis. The condition (3) is shown by the induction hypothesis and the following proof in $\mathcal{G}X$.

$$\dfrac{\Gamma_C^\circ \mapsto C \quad \dfrac{\Gamma_{C\to A_n}^\circ \mapsto C {\to} A_n}{\Gamma_{C\to A_n}^\circ, C \mapsto A_n}\,(\to \text{right}^{-1})\ (\text{Lemma 2.2})}{\dfrac{\Gamma_{C\to A_n}^\circ, \Gamma_C^\circ \mapsto A_n}{(\Gamma_{C\to A_n} \cup^\circ \Gamma_C)^\circ \mapsto A_n}\,\vdots\ (\text{ex}^{010}), (\text{contraction})}\,(\text{cut})$$

(Subcase 1-3): R = copy-implication, and $X = \mathbf{EM}^{\#}$. In this case, $\Gamma' = (A_1, \ldots, A_{n-1}, C{\to}A_n, C, A_n)$, and there are descendants $\Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_{C\to A_n}, \Gamma_C, \Gamma_{A_n}$ of, respectively, $A_1, \ldots, A_{n-1}, C{\to}A_n, C, A_n$ by the induction hypothesis. Then the required descendants of $A_1, \ldots, A_{n-1}, A_n$ are $\Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_{C\to A_n} \cup^\circ \Gamma_C \cup^\circ \Gamma_{A_n}$. The condition (1) is shown by the definition of $\cup^\circ$ and the induction hypothesis. The condition (2) is shown by the fact

$$A_n \vartriangleright_{\mathbf{EM}^{\#}} (C{\to}A_n, C, A_n) \vartriangleright_{\mathbf{EM}^{\#}} (\Gamma_{C\to A_n}, \Gamma_C, \Gamma_{A_n}) \vartriangleright_{\mathbf{EM}^{\#}} \Gamma_{C\to A_n} \cup^\circ \Gamma_C \cup^\circ \Gamma_{A_n}$$

which is guaranteed by Lemma 5.2 and the induction hypothesis. The condition (3) is shown by the induction hypothesis and the following proofs in $\mathcal{G}\mathbf{EM}^{\#}$.

[In case that Γ_{A_n} contains no atomic formula.]

$$\dfrac{\Gamma^\circ_{C\to A_n} \mapsto C{\to}A_n \qquad \dfrac{\Gamma^\circ_C \mapsto C \qquad \dfrac{\Gamma^\circ_{A_n} \mapsto A_n}{\Gamma^\circ_{A_n}, A_n \mapsto A_n}\,(\mathrm{m}^{\#})^{\dagger}}{\Gamma^\circ_{A_n}, C{\to}A_n, \Gamma^\circ_C \mapsto A_n}\,(\to \text{left})}{\Gamma^\circ_{A_n}, \Gamma^\circ_{C\to A_n}, \Gamma^\circ_C \mapsto A_n}\,(\text{cut})$$

$$\vdots\ (\text{ex}^{010}),\ (\text{contraction})$$

$$(\Gamma_{C\to A_n} \cup^\circ \Gamma_C \cup^\circ \Gamma_{A_n})^\circ \mapsto A_n$$

(†): There is an implication in the right most position in $\Gamma^\circ_{A_n}$ because of the induction hypothesis (2) and Lemma 5.1(2).

[In case that Γ_C contains no atomic formula.]

$$\dfrac{\Gamma^\circ_{A_n} \mapsto A_n \qquad \dfrac{\Gamma^\circ_{C\to A_n} \mapsto C{\to}A_n \qquad \dfrac{\dfrac{\Gamma^\circ_C \mapsto C \qquad A_n \mapsto A_n}{C{\to}A_n, \Gamma^\circ_C \mapsto A_n}\,(\to \text{left})}{C{\to}A_n, \Gamma^\circ_C, A_n \mapsto A_n}\,(\mathrm{m}^{\#})}{\Gamma^\circ_{C\to A_n}, \Gamma^\circ_C, A_n \mapsto A_n}\,(\text{cut})}{\Gamma^\circ_{C\to A_n}, \Gamma^\circ_C, \Gamma^\circ_{A_n} \mapsto A_n}\,(\text{cut})$$

$$\vdots\ (\text{ex}^{010}),\ (\text{contraction})$$

$$(\Gamma_{C\to A_n} \cup^\circ \Gamma_C \cup^\circ \Gamma_{A_n})^\circ \mapsto A_n$$

[In case that $\Gamma_{A_n} = (\Phi, p)$ and $\Gamma_C = (\Psi, p)$.]

$$\dfrac{\Gamma^\circ_{C\to A_n} \mapsto C{\to}A_n \qquad \dfrac{\dfrac{\dfrac{\Gamma^\circ_{A_n} \mapsto A_n}{\Phi \mapsto p{\to}A_n}\,(\to \text{right}) \qquad \dfrac{\dfrac{\dfrac{\Gamma^\circ_C \mapsto C \qquad A_n \mapsto A_n}{C{\to}A_n, \Gamma^\circ_C \mapsto A_n}\,(\to \text{left})}{C{\to}A_n, \Psi \mapsto p{\to}A_n}\,(\to \text{right})}{C{\to}A_n, \Psi, p{\to}A_n \mapsto p{\to}A_n}\,(\mathrm{m}^{\#})}{C{\to}A_n, \Psi, \Phi \mapsto p{\to}A_n}\,(\text{cut})}{C{\to}A_n, \Psi, \Phi, p \mapsto A_n}\,(\to \text{right}^{-1})\ (\text{Lemma 2.2})}{\Gamma^\circ_{C\to A_n}, \Psi, \Phi, p \mapsto A_n}\,(\text{cut})$$

$$\vdots\ (\text{ex}^{010}),\ (\text{contraction})$$

$$(\Gamma_{C\to A_n} \cup^\circ \Gamma_C \cup^\circ \Gamma_{A_n})^\circ \mapsto A_n$$

(Subcase 1-4): R = weakening, and $X = \mathbf{S4}$. In this case, $\Gamma' = (A_1, \ldots, A_n, B)$, and there are descendants $\Gamma_1, \ldots, \Gamma_n, \Gamma_B$ of, respectively, $A_1, \ldots, A_n, B$ by the induction hypothesis. Then the required descendants of $A_1, \ldots, A_{n-1}, A_n$ are $\Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_n \cup^\circ \Gamma_B$. The condition (1) is shown by the definition of $\cup^\circ$ and the induction hypothesis. The condition (2) is shown by the fact

$$A_n \ \rhd_{\mathbf{S4}} \ (A_n, B) \ \rhd_{\mathbf{S4}} \ (\Gamma_n, \Gamma_B) \ \rhd_{\mathbf{S4}} \ \Gamma_n \cup^\circ \Gamma_B$$

which is guaranteed by Lemma 5.2 and the induction hypothesis. The condition (3) is shown by the induction hypothesis and the following proof in $\mathcal{G}\mathbf{S4}$.

$$\begin{array}{c} \Gamma_n^\circ \mapsto A_n \\ \vdots \ (\mathrm{we}^{01})^\dagger \\ \Gamma_B^\circ, \Gamma_n^\circ \mapsto A_n \\ \vdots \ (\mathrm{ex}^{010}), (\mathrm{contraction}) \\ (\Gamma_n \cup^\circ \Gamma_B)^\circ \mapsto A_n \end{array}$$

(†): $\Gamma_n \neq \emptyset$ by the induction hypothesis (2) and Lemma 5.1(2). ∎

Lemma 5.4 *Let X be one of* **E**, $\mathbf{EM}^\#$, *and* **S4**. *If $A \rhd_X \Gamma$ and Γ is atom-normal, then $\mathcal{G}X \vdash \Gamma^\circ \Rightarrow A$.*

Proof By Lemma 5.3 and the contraction rule in $\mathcal{G}X$. ∎

Lemma 5.5 *Let X be one of* **E**, $\mathbf{EM}^\#$, *and* **S4**, *and Δ and Γ be finite multisets of formulas. If $\Delta \rhd_X \Gamma$ and Γ is atom-normal, then the following inference rule is derivable in $\mathcal{G}X$.*

$$\frac{\cdots, \Delta, \overrightarrow{\cdots \mapsto \cdots}}{\cdots, \Gamma^\circ, \overrightarrow{\cdots \mapsto \cdots}} \ (\text{conditional replace } \Delta \text{ by } \Gamma^\circ)$$

(*this sequence Δ is an arbitrary permutation of the multiset Δ*). *That is, for any Φ, Ψ and F such that $[\Psi \mapsto F]$ is an implication, there is a derivation from the sequent $\Phi, \Delta, \Psi \mapsto F$ to the sequent $\Phi, \Gamma^\circ, \Psi \mapsto F$ in $\mathcal{G}X$.*

Proof (Case 1): $\Delta = (A_1, \ldots, A_n)$ and $n \geq 1$. We apply Lemma 5.3, and a derivation from $(\Phi, \Delta, \Psi \mapsto F)$ to $(\Phi, \Gamma^\circ, \Psi \mapsto F)$ is obtained as follows.

$$\dfrac{\begin{array}{c}\vdots \ \text{Lemma 5.3(3)} \\ \Gamma_n^\circ \mapsto A_n\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\vdots \ \text{Lemma 5.3(3)} \\ \Gamma_1^\circ \mapsto A_1\end{array} \qquad \Phi, A_1, \ldots, A_n, \Psi \mapsto F}{\Phi, \Gamma_1^\circ, A_2, \ldots, A_n, \Psi \mapsto F} \ (\text{cut}) \\ \vdots \\ \Phi, \Gamma_1^\circ, \ldots, \Gamma_{n-1}^\circ, A_n, \Psi \mapsto F \end{array}}{\begin{array}{c}\Phi, \Gamma_1^\circ, \ldots, \Gamma_n^\circ, \Psi \mapsto F \\ \vdots \ (\mathrm{ex}^{010}), (\mathrm{contraction}) \\ \Phi, \Gamma^\circ, \Psi \mapsto F. \end{array}} \ (\text{cut})$$

(Case 2): Δ is empty. If $X = \mathbf{E}$ or $\mathbf{EM}^\#$, then Γ must be empty by Lemma 5.1 (3). If $X = \mathbf{S4}$, then the rule (conditional replace Δ by Γ°) is obtained by (we^{01}). ∎

Lemma 5.6 *Let X be one of* **E**, $\mathbf{EM}^{\#}$, *and* **S4**. *If Φ, Ψ and Σ are finite multisets of formulas such that $(\Phi, \Psi) \rhd_X \Sigma$ and Σ is atom-normal, then there exist atom-normal multisets Θ and Λ such that $(\Phi \rhd_X \Theta)$, $(\Psi \rhd_X \Lambda)$ and that the rule*

$$\frac{\cdots, \Theta^\circ, \Lambda^\circ, \cdots \mapsto \cdots}{\cdots, \Sigma^\circ, \cdots \mapsto \cdots} \text{ (replace } (\Theta^\circ, \Lambda^\circ) \text{ by } \Sigma^\circ)$$

is derivable in the sequent calculus $\mathcal{G}\mathbf{E}$.

Proof If $\Phi = \emptyset$, then $\Theta = \emptyset$ and $\Lambda = \Sigma$. If $\Psi = \emptyset$, then $\Theta = \Sigma$ and $\Lambda = \emptyset$. If $\Phi = (A_1, \ldots, A_m)$, $\Psi = (B_1, \ldots, B_n)$ and $m, n \geq 1$, then we apply Lemma 5.3, and we get atom-normal multisets $\Phi_1, \ldots, \Phi_m, \Psi_1, \ldots, \Psi_n$ that satisfy the following.

(1) Σ is a contraction of $(\Phi_1, \ldots, \Phi_m, \Psi_1, \ldots, \Psi_n)$. If $(\Phi_1, \ldots, \Phi_m, \Psi_1, \ldots, \Psi_n)$ contains atomic formulas, then all these atomic formulas are identical.

(2) $A_i \rhd_X \Phi_i$ $(i = 1, \ldots, m)$ and $B_i \rhd_X \Psi_i$ $(i = 1, \ldots, n)$.

Then we take $\Theta = \Phi_1 \cup^\circ \cdots \cup^\circ \Phi_m$ and $\Lambda = \Psi_1 \cup^\circ \cdots \cup^\circ \Psi_n$. The rule (replace $(\Theta^\circ, \Lambda^\circ)$ by Σ°) is obtained by the rules (ex^{010}) and (contraction). The required properties $\Phi \rhd_X \Theta$ and $\Psi \rhd_X \Lambda$ are shown by the facts

$\Phi = (A_1, \ldots, A_m) \rhd_X (\Phi_1, \ldots, \Phi_m) \rhd_X (\Phi_1 \cup^\circ \cdots \cup^\circ \Phi_m) = \Theta$, and

$\Psi = (B_1, \ldots, B_n) \rhd_X (\Psi_1, \ldots, \Psi_n) \rhd_X (\Psi_1 \cup^\circ \cdots \cup^\circ \Psi_n) = \Lambda$,

which are guaranteed by Lemma 5.2. ∎

Lemma 5.7 *If $(\Gamma, p, p) \rhd_{\mathbf{EM}^{\#}} \Delta$ and Δ is atom-normal, then $(\Gamma, p) \rhd_{\mathbf{EM}^{\#}} \Delta$.*

Proof By induction on the number of the steps of transformation from (Γ, p, p) to Δ.

(Case 0): The number is 0. This case does not happen because (Γ, p, p) is not atom-normal.

(Case 1): The number is $k > 0$; that is, there exists a multiset Φ that satisfies the following.

- Φ is obtained from (Γ, p, p) by one application of a rule, say R.
- $\Phi \rhd_{\mathbf{EM}^{\#}} \Delta$ with $(k-1)$-steps. (Therefore, if Φ is of the form (Φ', p', p'), the induction hypothesis is available.)

(Subcase 1-1): R operates on Γ; that is, $\Phi = (\Gamma', p, p)$ and $\Gamma \rhd_{\mathbf{EM}^{\#}} \Gamma'$. In this case, we have

$(\Gamma, p) \rhd_{\mathbf{EM}^{\#}} (\Gamma', p) \rhd_{\mathbf{EM}^{\#}} \Delta$ (by the induction hypothesis).

(Subcase 1-2): R = contraction, and it operates on p. In this case, Φ is of the form (Γ, p); and $(\Gamma, p) \ \triangleright_{\mathbf{EM}^\#} \ \Delta$ with $(k-1)$-steps.

(Subcase 1-3): R = implication, and it operates on p. In this case, Φ is of the form $(\Gamma, A{\rightarrow}p, A, p)$, which is also obtained from (Γ, p) by one application of the copy-implication rule. Hence we have $(\Gamma, p) \ \triangleright_{\mathbf{EM}^\#} \ \Phi \ \triangleright_{\mathbf{EM}^\#} \ \Delta$.

(Subcase 1-4): R = copy-implication, and it operates on p. In this case, Φ is of the form $(\Gamma, A{\rightarrow}p, A, p, p)$. Then, by the induction hypothesis, we have $(\Gamma, A{\rightarrow}p, A, p) \ \triangleright_{\mathbf{EM}^\#} \ \Delta$; and therefore we have $(\Gamma, p) \ \triangleright_{\mathbf{EM}^\#} \ (\Gamma, A{\rightarrow}p, A, p) \ \triangleright_{\mathbf{EM}^\#} \ \Delta$. ∎

If Γ is a finite multiset of labelled formulas, then $\Gamma{\downarrow}$ is defined to be the multiset of formulas that is obtained from Γ by deleting the labels. For example,

$$\{(\alpha, x){:}A,\ (\alpha, x){:}A,\ (\alpha, x){:}B,\ (\beta, y){:}B\}{\downarrow} = \ \{A, A, B, B\}.$$

If Γ is a finite multiset of labelled formulas and x is a rational number, then by $\Gamma_{\langle x\rangle}$ (or $\Gamma_{\langle -x\rangle}$), we mean the multisubset of Γ in which each rational number of the label is x (or, is not x, resp.). For example, if Δ is

$$\{(\alpha, x){:}A, (\beta, y){:}B, (\beta, y){:}B, (\gamma, y){:}C, (\gamma, z){:}C, (\delta, z){:}C\},$$

$(y \neq x,\ y \neq z)$ then

$$\Delta_{\langle y\rangle} = \{(\beta, y){:}B, (\beta, y){:}B, (\gamma, y){:}C\},$$

$$\Delta_{\langle -y\rangle} = \{(\alpha, x){:}A, (\gamma, z){:}C, (\delta, z){:}C\}.$$

Let $\Gamma \Rightarrow \Delta$ be a labelled sequent, and $x_1, \ldots, x_n$ be the rational numbers such that $\mathrm{Rat}(\Gamma, \Delta) = \{x_1, \ldots, x_n\}$ and $x_1 < x_2 < \cdots < x_n$. Then the expression

$$\Gamma_{\langle x_1\rangle}; \Gamma_{\langle x_2\rangle}; \cdots; \Gamma_{\langle x_n\rangle} \Rightarrow \Delta_{\langle x_1\rangle}; \Delta_{\langle x_2\rangle}; \cdots; \Delta_{\langle x_n\rangle}$$

is said to be the *linear partition* of $\Gamma \Rightarrow \Delta$.

Now we are ready to show the precise description of the "extraction".

Lemma 5.8 (Main Lemma for $\mathcal{L}$E, $\mathcal{L}$EM$^\#$, and $\mathcal{L}$S4) *Let X be one of* **E**, **EM**$^\#$, *and* **S4**, *and let $\Gamma_1; \cdots; \Gamma_N \Rightarrow \Delta_1; \cdots; \Delta_N$ be the linear partition of a labelled sequent $\Gamma \Rightarrow \Delta$. If $\mathcal{L}X \vdash \Gamma \Rightarrow \Delta$, then, for some $k \in \{1, \ldots, N\}$, there exists a labelled formula $(\varphi, x_k){:}F$ in Δ_k and there exist multisubsets $\Pi_1, \Pi_2, \ldots, \Pi_k$ of, respectively, $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ that satisfy the following conditions.*

(1) $\bigcup \mathrm{NSet}(\Pi_1, \ldots, \Pi_k) = \varphi$, *if* X = **E** *or* **EM**$^\#$.
$\bigcup \mathrm{NSet}(\Pi_1, \ldots, \Pi_k) \subseteq \varphi$, *if* X = **S4**.

(2) $\mathcal{G}X \vdash \Sigma_1^\circ, \ldots, \Sigma_k^\circ \mapsto F$ *for any atom-normal multisets $\Sigma_1, \ldots, \Sigma_k$ such that $\Pi_i{\downarrow} \ \triangleright_X \ \Sigma_i$ $(i = 1, \ldots, k)$.*

(*The labelled sequent $\Pi_1; \Pi_2; \cdots; \Pi_k \Rightarrow (\varphi, x_k){:}F$ will be called an* extract *of $\Gamma \Rightarrow \Delta$.*)

Proof By induction on the proof of $\Gamma \Rightarrow \Delta$ in $\mathcal{L}X$. We divide cases according to the last inference of the proof of $\Gamma \Rightarrow \Delta$.

(Case 1): $\Gamma \Rightarrow \Delta$ is an axiom $\Gamma', (\alpha, x_n){:}A \Rightarrow (\alpha, x_n){:}A, \Delta'$. In this case, the required extract is

$$\emptyset; \cdots; \emptyset; (\alpha, x_n){:}A \Rightarrow (\alpha, x_n){:}A.$$

The condition (1) obviously holds. If $X = \mathbf{E}$ or $\mathbf{EM}^{\#}$, then the condition (2) is shown by Lemma 5.1(3) and Lemma 5.4. If $X = \mathbf{S4}$, then the condition (2) is shown by Lemma 5.4 and the rule (we^{01}). (Note that Σ_n is not empty by Lemma 5.1(2).)

(Case 2): The last inference is

$$\frac{\Gamma',\ (\alpha, x_n){:}A,\ (\alpha, x_n){:}A \Rightarrow \Delta}{\Gamma',\ (\alpha, x_n){:}A \Rightarrow \Delta.}\ \text{(contraction left)}$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract contains at most one occurrence of $(\alpha, x_n){:}A$, then we take the extract as the required one of the lower sequent. If the the extract of the upper sequent is of the form

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha, x_n){:}A,\ (\alpha, x_n){:}A;\ \Phi_{n+1}; \cdots; \Phi_m \Rightarrow (\beta, x_m){:}B$$

$(m \geq n)$, then the required extract is

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha, x_n){:}A;\ \Phi_{n+1}; \cdots; \Phi_m \Rightarrow (\beta, x_m){:}B.$$

The condition (1) obviously holds by the induction hypothesis; and (2) is shown by the induction hypothesis and the fact that $(\Phi_n{\downarrow}, A) \rhd_X \Sigma$ implies $(\Phi_n{\downarrow}, A, A) \rhd_X \Sigma$.

(Case 3): The last inference is "contraction right". In this case, the extract obtained by the induction hypothesis is just the required extract.

(Case 4): The last inference is

$$\frac{\Gamma',\ (\alpha, x_{n+1}){:}\overrightarrow{A} \Rightarrow \Delta}{\Gamma',\ (\alpha, x_n){:}\overrightarrow{A} \Rightarrow \Delta.}\ (\rightarrow \text{label down})$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract does not contain $(\alpha, x_{n+1}){:}\overrightarrow{A}$, then we take the extract as the required one of the lower sequent. If the the extract of the upper sequent is of the form

$$\Phi_1; \cdots; \Phi_{n-1}; \Phi_n;\ \Phi_{n+1},\ (\alpha, x_{n+1}){:}\overrightarrow{A};\ \Phi_{n+2}; \cdots; \Phi_m \Rightarrow (\beta, x_m){:}B$$

$(m \geq n+1)$, then the required extract is

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha, x_n){:}\overrightarrow{A};\ \Phi_{n+1}; \Phi_{n+2}; \cdots; \Phi_m \Rightarrow (\beta, x_m){:}B.$$

(Note that $x_n \in \mathrm{Rat}(\Gamma', \Delta)$ by the Label Condition on the inference rule. This guarantees $\mathrm{Rat}(\Phi_n) = \{x_n\}$.) The condition (1) obviously holds by the induction hypothesis. We will show the condition (2). Let $\Sigma_1, \ldots, \Sigma_m$ be atom-normal multisets such that

(i) $\Phi_i{\downarrow} \ \rhd_X \ \Sigma_i$, for $i = 1, \dots, n-1, n+1, \dots, m$; and

(ii) $(\Phi_n{\downarrow}, \overrightarrow{A}) \ \rhd_X \ \Sigma_n$.

By (ii) and Lemma 5.6, there are atom-normal multisets Θ and Λ such that

(iii) $\Phi_n{\downarrow} \ \rhd_X \ \Theta$;

(iv) $\overrightarrow{A} \ \rhd_X \ \Lambda$; and

(v) the rule (replace $(\Theta^\circ, \Lambda^\circ)$ by Σ_n°) is derivable in $\mathcal{G}X$.

By (i) $(i = n+1)$, we have $(\Phi_{n+1}{\downarrow}, \overrightarrow{A}) \ \rhd_X \ (\Sigma_{n+1}, \overrightarrow{A})$ where $(\Sigma_{n+1}, \overrightarrow{A})$ is atom-normal. Therefore (i), (iii) and the induction hypothesis imply

$$\mathcal{G}X \vdash \Sigma_1^\circ, \dots, \Sigma_{n-1}^\circ, \Theta^\circ, (\Sigma_{n+1}, \overrightarrow{A})^\circ, \Sigma_{n+2}^\circ, \dots, \Sigma_m^\circ \mapsto B,$$

and then

$$\mathcal{G}X \vdash \Sigma_1^\circ, \dots, \Sigma_{n-1}^\circ, \Theta^\circ, \overrightarrow{A}, \Sigma_{n+1}^\circ, \Sigma_{n+2}^\circ, \dots, \Sigma_m^\circ \mapsto B$$

by the rule (ex^{010}). On the other hand, we have

$$\mathcal{G}X \vdash \Lambda^\circ \mapsto \overrightarrow{A}$$

by (iv) and Lemma 5.4. Hence, by the cut rule and the condition (v), we have

$$\mathcal{G}X \vdash \Sigma_1^\circ, \dots, \Sigma_{n-1}^\circ, \Sigma_n^\circ, \Sigma_{n+1}^\circ, \dots, \Sigma_m^\circ \mapsto B.$$

(Case 5): The last inference is

$$\frac{\Gamma',\ (\alpha{\cup}\beta, x_n){:}B \Rightarrow \Delta}{\Gamma',\ (\alpha, x_n){:}p{\rightarrow}B,\ (\beta, x_n){:}p \Rightarrow \Delta.}\ (\rightarrow \text{left } 0)$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract does not contain $(\alpha{\cup}\beta, x_n){:}B$, then we take the extract as the required one of the lower sequent. If the the extract of the upper sequent is of the form

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha{\cup}\beta, x_n){:}B;\ \Phi_{n+1}; \cdots; \Phi_m \Rightarrow (\gamma, x_m){:}C$$

$(m \geq n)$, then the required extract is

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha, x_n){:}p{\rightarrow}B,\ (\beta, x_n){:}p;\ \Phi_{n+1}; \cdots; \Phi_m \Rightarrow (\gamma, x_m){:}C.$$

The condition (1) obviously holds by the induction hypothesis; and (2) is shown by the induction hypothesis and the fact that $(\Phi_n{\downarrow}, p{\rightarrow}B, p) \ \rhd_X \ \Sigma$ implies $(\Phi_n{\downarrow}, B) \ \rhd_X \ \Sigma$.

(Case 6): The last inference is

$$\frac{\Gamma' \Rightarrow (\beta, x_n) : \overrightarrow{A},\ \Delta \quad \Gamma',\ (\alpha \cup \beta, x_n) : B \Rightarrow \Delta}{\Gamma',\ (\alpha, x_n) : \overrightarrow{A} \to B \Rightarrow \Delta.} (\to \text{ left } 1)$$

By the induction hypotheses, there are extracts of the left and of the right upper sequents. If the right-part of the extract of the left upper sequent is not $(\beta, x_n) : \overrightarrow{A}$, then we take this as the required extract of the lower sequent. Similarly, if the left-part of the extract of the right upper sequent does not contain $(\alpha \cup \beta, x_n) : B$, then we take this as the required extract. The remaining case is that the extracts are of the forms

$$\Phi_1; \cdots; \Phi_n \Rightarrow (\beta, x_n) : \overrightarrow{A}, \text{ and}$$
$$\Psi_1; \cdots; \Psi_{n-1};\ \Psi_n,\ (\alpha \cup \beta, x_n) : B;\ \Psi_{n+1}; \cdots; \Psi_m \Rightarrow (\gamma, x_m) : C$$

where $m \geq n$. In this case, the required extract is

$$\Phi_1 \oplus \Psi_1; \cdots; \Phi_{n-1} \oplus \Psi_{n-1};\ (\Phi_n \oplus \Psi_n), (\alpha, x_n) : \overrightarrow{A} \to B;\ \Psi_{n+1}; \cdots; \Psi_m \Rightarrow (\gamma, x_m) : C$$

where $\Phi \oplus \Psi$ is the least (with respect to the number of elements) contraction of $\Phi \cup_m \Psi$. (For example, $\{(\alpha, x) : A,\ (\alpha, x) : A,\ (\beta, y) : B,\ (\beta, y) : B\} \oplus \{(\beta, y) : B,\ (\gamma, z) : C\} = \{(\alpha, x) : A,\ (\beta, y) : B,\ (\gamma, z) : C\}$.) The condition (1) is easily verified by the induction hypotheses. We will show the condition (2). Let $\Sigma_1, \ldots, \Sigma_m$ be atom-normal multisets such that

(i) $(\Phi_i \oplus \Psi_i)\!\downarrow\ \rhd_X\ \Sigma_i$, for $i = 1, \ldots, n-1$;

(ii) $((\Phi_n \oplus \Psi_n)\!\downarrow, \overrightarrow{A} \to B)\ \rhd_X\ \Sigma_n$; and

(iii) $\Psi_i\!\downarrow\ \rhd_X\ \Sigma_i$, for $i = n+1, \ldots, m$.

By (i) and the definitions of $\oplus$ and $\rhd_X$, we have

$$((\Phi_i\!\downarrow), (\Psi_i\!\downarrow))\ \rhd_X\ \Sigma_i$$

for $i = 1, \ldots, n-1$; and similarly, by (ii) and the definitions, we have

$$((\Phi_n\!\downarrow), (\Psi_n\!\downarrow, \overrightarrow{A} \to B))\ \rhd_X\ \Sigma_n.$$

Then we apply Lemma 5.6 to these, and we get atom-normal multisets $\Theta_1, \ldots, \Theta_n$ and $\Lambda_1, \ldots, \Lambda_n$ that satisfy the following.

(iv) $\Phi_i\!\downarrow\ \rhd_X\ \Theta_i$, for $i = 1, \ldots, n$.

(v) $\Psi_i\!\downarrow\ \rhd_X\ \Lambda_i$, for $i = 1, \ldots, n-1$.

(vi) $(\Psi_n\!\downarrow, \overrightarrow{A} \to B)\ \rhd_X\ \Lambda_n$.

(vii) The rule (replace $(\Theta_i^\circ, \Lambda_i^\circ)$ by Σ_i°) is derivable in $\mathcal{G}X$, for $i = 1, \ldots, n$.

The definition of $\rhd_X$ and (vi) imply the fact

(viii) $(\Psi_n{\downarrow}, B) \ \rhd_X \ (\overrightarrow{A}, \Lambda_n)$,

where $(\overrightarrow{A}, \Lambda_n)$ is atom-normal. Now the induction hypotheses are available by (iii), (iv), (v), and (viii); and we have

$$\mathcal{G}X \vdash \Theta_1^\circ, \ldots, \Theta_n^\circ \mapsto \overrightarrow{A}, \text{ and}$$
$$\mathcal{G}X \vdash \Lambda_1^\circ, \ldots, \Lambda_{n-1}^\circ, \overrightarrow{A}, \Lambda_n^\circ, \Sigma_{n+1}^\circ, \ldots, \Sigma_m^\circ \mapsto C.$$

Then, by the merge cut rule (Lemma 2.3) and "replacing each $(\Theta_i^\circ, \Lambda_i^\circ)$ by Σ_i°", we have

$$\mathcal{G}X \vdash \Sigma_1^\circ, \ldots, \Sigma_m^\circ \mapsto C.$$

(Case 7): The last inference is

$$\frac{\Gamma,\ (\{a\}, x_{n+1}){:}A \Rightarrow (\alpha \cup \{a\}, x_{n+1}){:}B,\ \Delta'}{\Gamma \Rightarrow (\alpha, x_n){:}A {\to} B,\ \Delta'.}\ (\to \text{right})$$

By the induction hypothesis, there is an extract of the upper sequent.

(Subcase 7-1) The right-part of the extract is $(\beta, x_m){:}C$, which is not equal to $(\alpha \cup \{a\}, x_{n+1}){:}B$. In this case, by the condition (1) of the induction hypothesis and the Label Condition 1 on the inference rule, the left-part of the extract does not contain $(\{a\}, x_{n+1}){:}A$. If $x_m \le x_n$, then the extract is just the required one of the lower sequent. If $x_m > x_{n+1}$, then the extract of the upper sequent is of the form

$$\Phi_1; \cdots; \Phi_n;\ \emptyset;\ \Phi_{n+2}; \cdots; \Phi_m \Rightarrow (\beta, x_m){:}C,$$

and the required extract is

$$\Phi_1; \cdots; \Phi_n;\ \Phi_{n+2}; \cdots; \Phi_m \Rightarrow (\beta, x_m){:}C.$$

(Note that $x_{n+1} \notin \mathrm{Rat}(\Gamma, (\alpha, x_n){:}A {\to} B, \Delta')$.) The condition (1) obviously holds by the induction hypothesis; and (2) is shown by the induction hypothesis and the fact that $\emptyset \ \rhd_X \ \emptyset$.

(Subcase 7-2) The right-part of the extract of the upper sequent is $(\alpha \cup \{a\}, x_{n+1})$: B. If $X = \mathbf{E}$ or $\mathbf{EM}^{\#}$, then the extract of the upper sequent is of the form

$$\Phi_1; \cdots; \Phi_n;\ (\{a\}, x_{n+1}){:}A \Rightarrow (\alpha \cup \{a\}, x_{n+1}){:}B$$

because of the condition (1) of the induction hypothesis and the Label Condition 1 on the inference rule. If $X = \mathbf{S4}$, then the extract of the upper sequent is either the above one or of the form

$$\Phi_1; \cdots; \Phi_n; \emptyset \Rightarrow (\alpha \cup \{a\}, x_{n+1}){:}B.$$

In any case, the required extract is

$$\Phi_1; \cdots; \Phi_n \Rightarrow (\alpha, x_n) : A \rightarrow B.$$

The condition (1) is easily verified by the induction hypothesis and the Label Condition 1; and (2) is shown by the induction hypothesis and the facts $A \ \rhd_X \ A$ and $\emptyset \ \rhd_{\mathbf{S4}} \ A$.

(Case 8): $X = \mathbf{EM}^{\#}$, and the last inference is

$$\frac{\Gamma',\ (\alpha \cup \beta, x_n) : p \Rightarrow \Delta}{\Gamma',\ (\alpha, x_n) : p,\ (\beta, x_n) : p \Rightarrow \Delta.} \text{(atomic minglement)}$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract does not contain $(\alpha \cup \beta, x_n) : p$, then we take the extract as the required one of the lower sequent. If the the extract of the upper sequent is of the form

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha \cup \beta, x_n) : p;\ \Phi_{n+1}; \cdots; \Phi_m \Rightarrow (\gamma, x_m) : C$$

$(m \geq n)$, then the required extract is

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha, x_n) : p,\ (\beta, x_n) : p;\ \Phi_{n+1}; \cdots; \Phi_m \Rightarrow (\gamma, x_m) : C.$$

The condition (1) obviously holds by the induction hypothesis; and (2) is shown by the induction hypothesis and Lemma 5.7.

(Case 9): $X = \mathbf{S4}$, and the last inference is

$$\frac{\Gamma',\ (\alpha \cup \beta, x_n) : p \Rightarrow \Delta}{\Gamma',\ (\alpha, x_n) : p \Rightarrow \Delta.} \text{(atomic heredity)}$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract does not contain $(\alpha \cup \beta, x_n) : p$, then we take the extract as the required one of the lower sequent. If the the extract of the upper sequent is of the form

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha \cup \beta, x_n) : p;\ \Phi_{n+1}; \cdots; \Phi_m \Rightarrow (\gamma, x_m) : C$$

$(m \geq n)$, then the required extract is

$$\Phi_1; \cdots; \Phi_{n-1};\ \Phi_n,\ (\alpha, x_n) : p;\ \Phi_{n+1}; \cdots; \Phi_m \Rightarrow (\gamma, x_m) : C.$$

The conditions (1) and (2) obviously hold by the induction hypothesis. ∎

Theorem 5.9 *Let* X *be one of* **E**, $\mathbf{EM}^{\#}$, *and* **S4**. *If* $\mathcal{L}X \vdash \ \Rightarrow (\emptyset, 0) : A$, *then* $\mathcal{G}X \vdash\ \mapsto A$.

Proof By the previous lemma and the fact $\emptyset \ \rhd_X \ \emptyset$. ∎

Next we show the extraction from $\mathcal{L}\mathbf{E5}$, $\mathcal{L}\mathbf{E5M}^{\#}$ and $\mathcal{L}\mathbf{S5I}$.

Lemma 5.10 (Main Lemma for $\mathcal{L}$E5, $\mathcal{L}$E5M$^{\#}$, and $\mathcal{L}$S5I) *Let X be one of* **E5**, **E5M**$^{\#}$*, and* **S5I**. *If a labelled sequent $\Gamma \Rightarrow \Delta$ is provable in $\mathcal{L}X$, then there exists a labelled formula $(\varphi, z)\!:\!F$ in Δ and there exists a multisubset Π of Γ that satisfy the following conditions.*

(1) $\bigcup \mathrm{NSet}(\Pi) = \varphi$, *if X =* **E5** *or* **E5M**$^{\#}$.
$\bigcup \mathrm{NSet}(\Pi) \subseteq \varphi$, *if X =* **S5I**.

(2) *If F is an implication, then $\mathcal{G}X \vdash \Pi\!\downarrow\, \mapsto F$ for any sequence $\Pi\!\downarrow$ (i.e., an arbitrary permutation of $\Pi\!\downarrow$). If F is an atomic formula, then the following two conditions hold.*

(2-1) *$\mathcal{G}X \vdash \Pi_{\langle -z\rangle}\!\downarrow, \Sigma^{\circ} \mapsto F$ for any sequence $\Pi_{\langle -z\rangle}\!\downarrow$ and any atom-normal multisets Σ such that $\Pi_{\langle z\rangle}\!\downarrow\ \rhd_X \Sigma$, where $\rhd_{\mathbf{E5}}$ is equal to $\rhd_{\mathbf{E}}$, $\rhd_{\mathbf{E5M^{\#}}}$ is equal to $\rhd_{\mathbf{EM^{\#}}}$, and $\rhd_{\mathbf{S5I}}$ is equal to $\rhd_{\mathbf{S4}}$.*

(2-2) *$\Pi_{\langle z\rangle}$ is not empty, or $\Pi_{\langle -z\rangle}$ contains at least one implication.*

(*The labelled sequent $\Pi \Rightarrow (\varphi, z)\!:\!F$ will be called an* extract *of $\Gamma \Rightarrow \Delta$.*)

Proof By induction on the proof of $\Gamma \Rightarrow \Delta$ in $\mathcal{L}X$. We divide cases according to the last inference of the proof of $\Gamma \Rightarrow \Delta$.

(Case 1): $\Gamma \Rightarrow \Delta$ is an axiom $\Gamma', (\alpha, x)\!:\!A \Rightarrow (\alpha, x)\!:\!A, \Delta'$. In this case,

$$(\alpha, x)\!:\!A \Rightarrow (\alpha, x)\!:\!A$$

is the required extract. The conditions (1) and (2) are easily verified. (When A is atomic, we use Lemma 5.4 to show (2-1).)

(Case 2): The last inference is

$$\frac{\Gamma',\ (\alpha, x)\!:\!A,\ (\alpha, x)\!:\!A \Rightarrow \Delta}{\Gamma',\ (\alpha, x)\!:\!A \Rightarrow \Delta.}\ \text{(contraction left)}$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract contains at most one occurrence of $(\alpha, x)\!:\!A$, then we take the extract as the required one of the lower sequent. If the extract of the upper sequent is of the form

$$\Phi,\ (\alpha, x)\!:\!A,\ (\alpha, x)\!:\!A \Rightarrow (\beta, y)\!:\!B,$$

then the required extract is

$$\Phi,\ (\alpha, x)\!:\!A \Rightarrow (\beta, y)\!:\!B.$$

The conditions (1) and (2) are easily verified by the induction hypothesis. (When $x = y$ and B is atomic, we use the fact that $(\Phi_{\langle y\rangle}\!\downarrow, A)\ \rhd_X\ \Sigma$ implies $(\Phi_{\langle y\rangle}\!\downarrow, A, A)\ \rhd_X\ \Sigma$.)

(Case 3): The last inference is "contraction right". In this case, the extract obtained by the induction hypothesis is just the required extract.

(Case 4): The last inference is

$$\frac{\Gamma',\ (\alpha, y)\colon \overrightarrow{A} \Rightarrow \Delta}{\Gamma',\ (\alpha, x)\colon \overrightarrow{A} \Rightarrow \Delta.}\ (\to \text{label})$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract does not contain $(\alpha, y)\colon \overrightarrow{A}$, then we take the extract as the required one of the lower sequent. If the extract of the upper sequent is of the form

$$\Phi,\ (\alpha, y)\colon \overrightarrow{A} \Rightarrow (\beta, z)\colon B,$$

then the required extract is

$$\Phi,\ (\alpha, x)\colon \overrightarrow{A} \Rightarrow (\beta, z)\colon B.$$

The condition (1) obviously holds by the induction hypothesis. If B is an implication, also the condition (2) is obvious. In the following, we assume B is atomic, and we will show the condition (2-1). (The condition (2-2) is obvious.) We consider three cases according to the values of x, y, z.

(Case A): $y \neq z$ and $x \neq z$. In this case, the condition (2-1) obviously holds by the induction hypothesis because both $(\alpha, x)\colon \overrightarrow{A}$ and $(\alpha, y)\colon \overrightarrow{A}$ belong to $\Phi_{\langle -z \rangle}$.

(Case B): $y = z$ and $x \neq z$. In this case, the condition (2-1) is shown by the induction hypothesis, the rule (ex^{010}), and the fact

$$\Phi_{\langle z \rangle}{\downarrow}\ \rhd_X\ \Sigma \text{ implies } (\overrightarrow{A}, \Phi_{\langle z \rangle}{\downarrow})\ \rhd_X\ (\overrightarrow{A}, \Sigma),$$

where $(\overrightarrow{A}, \Sigma)$ is atom-normal.

(Case C): $y \neq z$ and $x = z$. In this case, the condition (2-1) is shown as follows. Let Σ be an atom-normal multiset such that

(i) $(\overrightarrow{A}, \Phi_{\langle z \rangle}{\downarrow})\ \rhd_X\ \Sigma$.

By (i) and Lemma 5.6, there are atom-normal multisets Θ and Λ such that

(ii) $\overrightarrow{A}\ \rhd_X\ \Theta$;

(iii) $\Phi_{\langle z \rangle}{\downarrow}\ \rhd_X\ \Lambda$; and

(iv) the rule (replace $(\Theta^\circ, \Lambda^\circ)$ by Σ°) is derivable in $\mathcal{G}X$.

By (iii) and the induction hypothesis, we have

$$\mathcal{G}X \vdash \Phi_{\langle -z \rangle}{\downarrow}, \overrightarrow{A}, \Lambda^\circ \mapsto B.$$

On the other hand, we have

$$\mathcal{G}X \vdash \Theta^\circ \mapsto \overrightarrow{A}$$

by (ii) and Lemma 5.4. Hence, by the cut rule and the condition (iv), we have

$$\mathcal{G}X \vdash \Phi_{\langle -z \rangle}\downarrow, \Sigma^\circ \mapsto B.$$

(Case 5): The last inference is

$$\frac{\Gamma',\ (\alpha{\cup}\beta, x){:}B \Rightarrow \Delta}{\Gamma',\ (\alpha, x){:}p{\rightarrow}B,\ (\beta, x){:}p \Rightarrow \Delta.}\ (\rightarrow \text{left } 0)$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract does not contain $(\alpha{\cup}\beta, x){:}B$, then we take the extract as the required one of the lower sequent. If the extract of the upper sequent is of the form

$$\Phi,\ (\alpha{\cup}\beta, x){:}B \Rightarrow (\gamma, y){:}C,$$

then the required extract is

$$\Phi,\ (\alpha, x){:}p{\rightarrow}B,\ (\beta, x){:}p \Rightarrow (\gamma, y){:}C.$$

The condition (1) obviously holds by the induction hypothesis. If C is an implication, the condition (2) is shown by the induction hypothesis and the rule (ex^{001}). In the following, we assume C is atomic, and we will show the condition (2-1). (The condition (2-2) is obvious.) We consider two cases according to the values of x, y.

(Case A): $x \neq y$. Let Σ be an atom-normal multiset such that $\Phi_{\langle y \rangle}\downarrow\ \rhd_X \Sigma$, and let $\Pi\downarrow$ be an arbitrary permutation of $(\Phi_{\langle -y \rangle}\downarrow, p{\rightarrow}B, p)$. We show that the sequent $\Pi\downarrow, \Sigma^\circ \mapsto C$ is provable in $\mathcal{G}X$. By the induction hypothesis (2-2), we have

(i) $\Phi_{\langle y \rangle}\downarrow$ is not empty; or

(ii) $(\Phi_{\langle -y \rangle}\downarrow, B)$ contains at least one implication.

We divide cases according to these conditions.

(Subcase A-1): $\Phi_{\langle y \rangle}\downarrow$ is not empty. In this case, $\Sigma^\circ \mapsto C$ is an implication because of Lemma 5.1(2). Then we have

$$\dfrac{p \mapsto p \quad \begin{array}{c}\vdots\ \text{i.h.}\\ \Phi_{\langle -y \rangle}\downarrow, B, \Sigma^\circ \mapsto C\end{array}}{\begin{array}{c}\Phi_{\langle -y \rangle}\downarrow, p{\rightarrow}B, p, \Sigma^\circ \mapsto C\\ \vdots\ (\text{ex}^{001})\\ \Pi\downarrow, \Sigma^\circ \mapsto C.\end{array}}\ (\rightarrow \text{left})$$

(Subcase A-2): $\Phi_{\langle -y \rangle}\downarrow = (\Phi'\downarrow, \overrightarrow{D})$. Then we have

$$\dfrac{p \mapsto p \quad \begin{array}{c}\vdots\ \text{i.h.}\\ \Phi'\downarrow, B, \overrightarrow{D}, \Sigma^\circ \mapsto C\end{array}}{\begin{array}{c}\Phi'\downarrow, p{\rightarrow}B, p, \overrightarrow{D}, \Sigma^\circ \mapsto C\\ \vdots\ (\text{ex}^{001})\\ \Pi'\downarrow, \overrightarrow{D}, \Sigma^\circ \mapsto C\\ \vdots\ (\text{ex}^{010})\\ \Pi\downarrow, \Sigma^\circ \mapsto C\end{array}}\ (\rightarrow \text{left})$$

where $\Pi'\!\downarrow$ is the sequence obtained from $\Pi\!\downarrow$ by deleting an occurrence of $\overrightarrow{D}$.

(Subcase A-3): B is an implication. Then we have

$$\dfrac{p \mapsto p \quad \dfrac{\vdots\ \text{i.h.}}{\Phi_{\langle -y\rangle}\!\downarrow, \overrightarrow{B}, \Sigma^\circ \mapsto C}}{\Pi\!\downarrow, \Sigma^\circ \mapsto C.} \ (\to\text{left}+)\ (\text{Lemma 2.4})$$

(Case B): $x = y$. In this case, the condition (2-2) is shown by the induction hypothesis and the fact that $(\Phi_{\langle y\rangle}\!\downarrow, p{\to}B, p) \ \rhd_X\ \Sigma$ implies $(\Phi_{\langle y\rangle}\!\downarrow, B) \ \rhd_X\ \Sigma$.

(Case 6): The last inference is

$$\dfrac{\Gamma' \Rightarrow (\beta, x){:}\overrightarrow{A},\ \Delta \qquad \Gamma',\ (\alpha{\cup}\beta, x){:}B \Rightarrow \Delta}{\Gamma',\ (\alpha, x){:}\overrightarrow{A}{\to}B \Rightarrow \Delta.} \ (\to \text{left } 1)$$

By the induction hypotheses, there are extracts of the left and of the right upper sequents. If the right-part of the extract of the left upper sequent is not $(\beta, x){:}\overrightarrow{A}$, or the left-part of the extract of the right upper sequent does not contain $(\alpha{\cup}\beta, x){:}B$, then one of the extracts of the upper sequents is the required one of the lower sequent. If the extracts of upper sequents are

$\Phi \Rightarrow (\beta, x){:}\overrightarrow{A}$, and

$\Psi,\ (\alpha{\cup}\beta, x){:}B \Rightarrow (\gamma, y){:}C$,

then the required extract is

$$\Phi \oplus \Psi,\ (\alpha, x){:}\overrightarrow{A}{\to}B \Rightarrow (\gamma, y){:}C$$

where the operator $\oplus$ was defined in the proof of Lemma 5.8, Case 6. The condition (1) is easily verified by the induction hypotheses. If C is an implication, the condition (2) is shown by the induction hypotheses and the rules ($\to$left), (ex^{001}), and (contraction). In the following, we assume C is atomic, and we will show the condition (2-1). (The condition (2-2) is obvious.) We consider two cases according to the values of x, y.

(Case A): $x \neq y$. Let Σ be an atom-normal multiset such that

(i) $(\Phi_{\langle y\rangle} \oplus \Psi_{\langle y\rangle})\!\downarrow \ \rhd_X\ \Sigma$,

and let $\Pi\!\downarrow$ be an arbitrary permutation of $((\Phi_{\langle -y\rangle} \oplus \Psi_{\langle -y\rangle})\!\downarrow, \overrightarrow{A}{\to}B)$. We show that the sequent $\Pi\!\downarrow, \Sigma^\circ \mapsto C$ is provable in $\mathcal{G}X$. By (i) and the definitions of $\oplus$ and $\rhd_X$, we have $((\Phi_{\langle y\rangle}\!\downarrow), (\Psi_{\langle y\rangle}\!\downarrow)) \ \rhd_X\ \Sigma$. Then, by Lemma 5.6, we get atom-normal multisets Θ and Λ that satisfy the following.

(ii) $\Phi_{\langle y\rangle}\!\downarrow \ \rhd_X\ \Theta$.

(iii) $\Psi_{\langle y\rangle}\!\downarrow \ \rhd_X\ \Lambda$.

(iv) The rule (replace $(\Theta^\circ, \Lambda^\circ)$ by Σ°) is derivable in $\mathcal{G}X$.

Then we have

$$\dfrac{\dfrac{\dfrac{\vdots\ \text{i.h.(2)}}{\Phi_{\langle -y\rangle}{\downarrow}, \Phi_{\langle y\rangle}{\downarrow} \mapsto \overrightarrow{A}}}{\Phi_{\langle -y\rangle}{\downarrow}, \Theta^\circ \mapsto \overrightarrow{A}}\ \text{(ii) and Lemma 5.5} \qquad \dfrac{\vdots\ \text{(iii) and i.h.(2-1)}}{\Psi_{\langle -y\rangle}{\downarrow}, B, \Lambda^\circ \mapsto C}}{(\Pi{\downarrow})^+, \Theta^\circ, \Lambda^\circ \mapsto C}\ (\to\text{left}+)\ \text{(Lemma 2.4)}$$

$$\vdots\ \text{(contraction)}$$

$$\dfrac{\Pi{\downarrow}, \Theta^\circ, \Lambda^\circ \mapsto C}{\Pi{\downarrow}, \Sigma^\circ \mapsto C}\ \text{(iv)}$$

where $(\Pi{\downarrow})^+$ is a permutation of $(\Phi_{\langle -y\rangle}{\downarrow}, \Psi_{\langle -y\rangle}{\downarrow}, \overrightarrow{A}{\to}B)$ such that $(\Pi{\downarrow})$ is a contraction of it.

(Case B): $x = y$. Let Σ be an atom-normal multiset such that

(i) $((\Phi_{\langle y\rangle} \oplus \Psi_{\langle y\rangle}){\downarrow}, \overrightarrow{A}{\to}B)\ \rhd_X\ \Sigma$,

and let $\Pi{\downarrow}$ be an arbitrary permutation of $(\Phi_{\langle -y\rangle} \oplus \Psi_{\langle -y\rangle}){\downarrow}$. We show that the sequent $\Pi{\downarrow}, \Sigma^\circ \mapsto C$ is provable in $\mathcal{G}X$. By (i) and the definitions of $\oplus$ and $\rhd_X$, we have $(\Phi_{\langle y\rangle}{\downarrow}, (\Psi_{\langle y\rangle}{\downarrow}, \overrightarrow{A}{\to}B))\ \rhd_X\ \Sigma$. Then we apply Lemma 5.6 to these, and we get atom-normal multisets Θ and Λ that satisfy the following.

(ii) $\Phi_{\langle y\rangle}{\downarrow}\ \rhd_X\ \Theta$.

(iii) $(\Psi_{\langle y\rangle}{\downarrow}, \overrightarrow{A}{\to}B)\ \rhd_X\ \Lambda$.

(iv) The rule (replace $(\Theta^\circ, \Lambda^\circ)$ by Σ°) is derivable in $\mathcal{G}X$.

The definition of $\rhd_X$ and (iii) imply the fact

(v) $(\Psi_{\langle y\rangle}{\downarrow}, B)\ \rhd_X\ (\overrightarrow{A}, \Lambda)$,

where $(\overrightarrow{A}, \Lambda)$ is atom-normal. Then we have

$$\dfrac{\dfrac{\dfrac{\vdots\ \text{i.h.(2)}}{\Phi_{\langle -y\rangle}{\downarrow}, \Phi_{\langle y\rangle}{\downarrow} \mapsto \overrightarrow{A}}}{\Phi_{\langle -y\rangle}{\downarrow}, \Theta^\circ \mapsto \overrightarrow{A}}\ \text{(ii) and Lemma 5.5} \qquad \dfrac{\vdots\ \text{(v) and i.h.(2-1)}}{\Psi_{\langle -y\rangle}{\downarrow}, \overrightarrow{A}, \Lambda^\circ \mapsto C}}{(\Pi{\downarrow})^+, \Theta^\circ, \Lambda^\circ \mapsto C}\ (\text{cut}+)\ \text{(Lemma 2.4)}$$

$$\vdots\ \text{(contraction)}$$

$$\dfrac{\Pi{\downarrow}, \Theta^\circ, \Lambda^\circ \mapsto C}{\Pi{\downarrow}, \Sigma^\circ \mapsto C}\ \text{(iv)}$$

where $(\Pi{\downarrow})^+$ is a permutation of $(\Phi_{\langle -y\rangle}{\downarrow}, \Psi_{\langle -y\rangle}{\downarrow})$ such that $(\Pi{\downarrow})$ is a contraction of it.

(Case 7): The last inference is

$$\frac{\Gamma,\ (\{a\},y)\!:\!A \Rightarrow (\alpha\cup\{a\},y)\!:\!B,\ \Delta'}{\Gamma \Rightarrow (\alpha,x)\!:\!A\to B,\ \Delta'.}\ (\to \text{right})$$

By the induction hypotheses, there is an extract of the upper sequent.

(Subcase 7-1): The right-part of the extract is not $(\alpha\cup\{a\},y)\!:\!B$. In this case, the left-part of the extract does not contain $(\{a\},y)\!:\!A$ because of the condition (1) of the induction hypothesis and the Label Condition 1 on the inference rule. Then the extract is just the required one of the lower sequent.

(Subcase 7-2): The right-part of the extract of the upper sequent is $(\alpha\cup\{a\},y)\!:\!B$. If $X = \mathbf{E5}$ or $\mathbf{E5M}^{\#}$, the extract is of the form

$$\Phi,\ (\{a\},y)\!:\!A \Rightarrow (\alpha\cup\{a\},y)\!:\!B$$

because of the condition (1) of the induction hypothesis and the Label Condition 1 on the inference rule. In this case, the required extract is

$$\Phi \Rightarrow (\alpha,x)\!:\!A\to B.$$

The conditions (1) are obvious by the induction hypothesis; and (2) is shown by the induction hypothesis and the fact $A \rhd_X A$ (in the case that B is atomic). If $X = \mathbf{S5I}$, the extract of the upper sequent is either the above one or of the form

$$\Phi \Rightarrow (\alpha\cup\{a\},y)\!:\!B$$

where $(\{a\},y)\!:\!A \notin \Phi$. In any case, the required extract is also

$$\Phi \Rightarrow (\alpha,x)\!:\!A\to B.$$

In the former case, the proof of the required condition is the same as above. In the latter case, the condition (1) is obvious by the induction hypothesis; and (2) is shown by the induction hypothesis, the rule (we^{01}) (in the case that B is an implication), and the fact $\Phi_{\langle y\rangle} = \emptyset \rhd_{\mathbf{S4}} A$ (in the case that B is atomic).

(Case 8): $X = \mathbf{E5M}^{\#}$, and the last inference is

$$\frac{\Gamma',\ (\alpha\cup\beta,x)\!:\!p \Rightarrow \Delta}{\Gamma',\ (\alpha,x)\!:\!p,\ (\beta,x)\!:\!p \Rightarrow \Delta.}\ (\text{atomic minglement})$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract does not contain $(\alpha\cup\beta,x)\!:\!p$, then we take the extract as the required one of the lower sequent. If the extract of the upper sequent is of the form

$$\Phi,\ (\alpha\cup\beta,x)\!:\!p \Rightarrow (\gamma,y)\!:\!C,$$

then the required extract is

$$\Phi,\ (\alpha,x)\!:\!p,\ (\beta,x)\!:\!p \Rightarrow (\gamma,y)\!:\!C.$$

The condition (1) obviously holds by the induction hypothesis. If C is an implication, the condition (2) is shown by the induction hypothesis, Lemma 2.5, and the rule (ex^{001}). In the following, we assume C is atomic, and we will show the condition (2-1). (The condition (2-2) is obvious by the induction hypothesis.) We consider two cases according to the values of x, y.

(Case A): $x \neq y$. Let Σ be an atom-normal multiset such that $\Phi_{\langle y\rangle}\!\downarrow\ \rhd_{\mathbf{EM}^{\#}} \Sigma$, and let $\Pi\!\downarrow$ be an arbitrary permutation of $(\Phi_{\langle -y\rangle}\!\downarrow, p, p)$. We show that the sequent $\Pi\!\downarrow, \Sigma^{\circ} \mapsto C$ is provable in $\mathcal{G}\mathbf{E5M}^{\#}$. By the induction hypothesis (2-2), we have

(i) $\Phi_{\langle y\rangle}\!\downarrow$ is not empty; or

(ii) $(\Phi_{\langle -y\rangle}\!\downarrow, p)$ contains at least one implication.

We divide cases according to these conditions.

(Subcase A-1): $\Phi_{\langle y\rangle}\!\downarrow$ is not empty. In this case, $\Sigma^{\circ} \mapsto C$ is an implication because of Lemma 5.1(2). Then we have

$$\dfrac{\begin{array}{c}\vdots\ \text{i.h.}\\ \Phi_{\langle -y\rangle}\!\downarrow, p, \Sigma^{\circ} \mapsto C\end{array}}{\begin{array}{c}\Phi_{\langle -y\rangle}\!\downarrow, p, p, \Sigma^{\circ} \mapsto C\\ \vdots\ (\mathrm{ex}^{001})\\ \Pi\!\downarrow, \Sigma^{\circ} \mapsto C.\end{array}}\ \text{Lemma 2.5}$$

(Subcase A-2): $\Phi_{\langle -y\rangle}\!\downarrow = (\Phi'\!\downarrow, \overrightarrow{D})$. Then we have

$$\dfrac{\begin{array}{c}\vdots\ \text{i.h.}\\ \Phi'\!\downarrow, p, \overrightarrow{D}, \Sigma^{\circ} \mapsto C\end{array}}{\begin{array}{c}\Phi'\!\downarrow, p, p, \overrightarrow{D}, \Sigma^{\circ} \mapsto C\\ \vdots\ (\mathrm{ex}^{001})\\ \Pi'\!\downarrow, \overrightarrow{D}, \Sigma^{\circ} \mapsto C\\ \vdots\ (\mathrm{ex}^{010})\\ \Pi\!\downarrow, \Sigma^{\circ} \mapsto C\end{array}}\ \text{Lemma 2.5}$$

where $\Pi'\!\downarrow$ is the sequence obtained from $\Pi\!\downarrow$ by deleting an occurrence of $\overrightarrow{D}$.

(Case B): $x = y$. In this case, the condition (2-1) is shown by the induction hypothesis and Lemma 5.7.

(Case 9): $X = \mathbf{S5I}$, and the last inference is

$$\frac{\Gamma',\ (\alpha\cup\beta, x):p \Rightarrow \Delta}{\Gamma',\ (\alpha, x):p \Rightarrow \Delta.}\ \text{(atomic heredity)}$$

By the induction hypothesis, there is an extract of the upper sequent. If the left-part of the extract does not contain $(\alpha\cup\beta, x):p$, then we take the extract as the required one of the lower sequent. If the the extract of the upper sequent is of the form

$$\Phi,\ (\alpha\cup\beta, x):p \Rightarrow (\gamma, y):C,$$

then the required extract is

$$\Phi, (\alpha, x) : p \Rightarrow (\gamma, y) : C.$$

The conditions (1) and (2) obviously hold by the induction hypothesis. ▮

Theorem 5.11 *Let X be one of* **E5**, **E5M**$^{\#}$, *and* **S5I**. *If $\mathcal{L}X \vdash \Rightarrow (\emptyset, 0) : A$, then $\mathcal{G}X \vdash \mapsto A$.*

Proof By the previous lemma. ▮

6 Conclusion

By Theorems 2.1, 3.9, 4.2, 4.4, 4.6, 5.9, and 5.11, we get the main results of this paper:

Theorem 6.1 (Main Theorem for E) *For any formula A, the following five conditions are equivalent.*

- $\mathbf{E} \vdash A$.
- $\mathcal{G}\mathbf{E} \vdash \mapsto A$.
- $\mathcal{L}\mathbf{E} \vdash \Rightarrow (\emptyset, 0) : A$.
- *A is valid in any Linear Order E-model.*
- *A is valid in any E-model.*

Theorem 6.2 (Main Theorem for EM$^{\#}$) *For any formula A, the following five conditions are equivalent.*

- $\mathbf{EM}^{\#} \vdash A$.
- $\mathcal{G}\mathbf{EM}^{\#} \vdash \mapsto A$.
- $\mathcal{L}\mathbf{EM}^{\#} \vdash \Rightarrow (\emptyset, 0) : A$.
- *A is valid in any Linear Order E-model that satisfies the Mingle Condition.*
- *A is valid in any E-model that satisfies the Mingle Condition.*

Theorem 6.3 (Main Theorem for S4) *For any formula A, the following five conditions are equivalent.*

- $\mathbf{S4} \vdash A$.
- $\mathcal{G}\mathbf{S4} \vdash\ \mapsto A$.
- $\mathcal{L}\mathbf{S4} \vdash\ \Rightarrow (\emptyset, 0) : A$.
- *A is valid in any Linear Order E-model that satisfies the Hereditary Condition.*
- *A is valid in any E-model that satisfies the Hereditary Condition..*

Theorem 6.4 (Main Theorem for E5) *For any formula A, the following four conditions are equivalent.*

- $\mathbf{E5} \vdash A$.
- $\mathcal{G}\mathbf{E5} \vdash\ \mapsto A$.
- $\mathcal{L}\mathbf{E5} \vdash\ \Rightarrow (\emptyset, 0) : A$.
- *A is valid in any Single Cluster E-model.*

Theorem 6.5 (Main Theorem for $\mathbf{E5M}^{\#}$) *For any formula A, the following four conditions are equivalent.*

- $\mathbf{E5M}^{\#} \vdash A$.
- $\mathcal{G}\mathbf{E5M}^{\#} \vdash\ \mapsto A$.
- $\mathcal{L}\mathbf{E5M}^{\#} \vdash\ \Rightarrow (\emptyset, 0) : A$.
- *A is valid in any Single Cluster E-model that satisfies the Mingle Condition.*

Theorem 6.6 (Main Theorem for S5I) *For any formula A, the following four conditions are equivalent.*

- $\mathbf{S5I} \vdash A$.
- $\mathcal{G}\mathbf{S5I} \vdash\ \mapsto A$.
- $\mathcal{L}\mathbf{S5I} \vdash\ \Rightarrow (\emptyset, 0) : A$.
- *A is valid in any Single Cluster E-model that satisfies the Hereditary Condition.*

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