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Ryo Kashima

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Department of
Mathematical and
Computing Sciences
Tokyo Institute of Technology

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Ryo Kashima*

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Abstract

We give a new proof of the completeness theorem for the smallest normal modal propositional logic K with the additional modal operator representing transitive closure of accessibility relation.

1 Introduction

In Kripke models, the modal operator $\Box$ is interpreted as

$$w \models \Box\varphi \quad \iff \quad x \models \varphi \text{ for any } x \text{ such that } wRx$$

where w and x are possible worlds and R is the accessibility relation. Then we introduce a new modal operator $\Box^+$ by

$$w \models \Box^+\varphi \quad \iff \quad x \models \varphi \text{ for any } x \text{ such that } wR^+x$$

where R^+ is the transitive closure of R. Intuitively $\Box^+\varphi$ means the infinite conjunction as follows:

$$\Box^+\varphi \quad \leftrightarrow \quad \Box\varphi \wedge \Box\Box\varphi \wedge \Box\Box\Box\varphi \wedge \cdots .$$

This paper treats the smallest normal modal propositional logic with the operators $\Box$ and $\Box^+$ as above. This logic will be called $K_{\Box\Box^+}$.

The relationship between $\Box$ and $\Box^+$ in $K_{\Box\Box^+}$ is equal to that between the operators E ("everyone knows") and C ("common knowledge") in the common knowledge logic, since

$$C\varphi \quad \leftrightarrow \quad E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots .$$

Moreover the relationship is similar to that between the operators X ("next time") and G ("globally") in the temporal logic, since

$$G\varphi \quad \leftrightarrow \quad \varphi \wedge X\varphi \wedge XX\varphi \wedge \cdots .$$

There are axiom systems for the common knowledge logic and the temporal logic, and the completeness (i.e., a formula is provable in the system if it is true in every model) have been proved by using cananical models and filtrations (see, e.g., [1] and [2]). Of course the argument can be applied to $K_{\Box\Box^+}$ — an axiom system of $K_{\Box\Box^+}$ is defined similarly to the common knowledge logic or the temporal logic, and the completeness can be shown by using canonical models and filtrations.

The purpose of this paper is to give an alternative proof for the completeness of $K_{\Box\Box^+}$. We use a variant of semantic tableaux.

In general, completeness of a logic is shown by constructing a counter-model for a given unprovable formula. In the canonical model methood, we first construct a *big* model (canonical model, which is an infinite model), then we process it by certain methods (e.g., filtrations), and finally we get a counter-model. In our method, on the other hand, we first make a *small* model (consisting of one world), then we add worlds step by step, and finally we get a counter-model. A point is that no infinite models occur in our method.

This method can be applied to the common knowledge logic and the temporal logic. We hope this will shed a new light on the study of these logics.

*Department of Mathematical and Computing Sciences, Tokyo Institute of Technology. Ookayama, Meguro, Tokyo 152-8552, Japan. E-mail: `kashima@is.titech.ac.jp`

2 Formulas, models, and axomatizations

Formulas of $\mathrm{K}_{\Box\Box^+}$ are constructed from the following symbols:

- Propositional variables. (The set of propositional variables is called **Prop**.)
- Logical connectives $\wedge$ and $\neg$.
- Modal operators $\Box$ and $\Box^+$.

We will use letters $p, q, \ldots$ to denote propositional variables, and letters $\alpha, \beta, \ldots \varphi, \psi, \ldots$ to denote formulas. Other connectives ($\to$, $\vee$) and constants ($\bot, \top$) are defined by the usual abbreviations: $\varphi \vee \psi = \neg((\neg\varphi) \wedge \neg\psi)$, $\varphi \to \psi = \neg(\varphi \wedge \neg\psi)$, $\bot = p \wedge \neg p$, and $\top = \neg(p \wedge \neg p)$. Parentheses are omitted by the convention that $\neg$, $\Box$ and $\Box^+$ bind more stronger than other connectives, $\wedge$ and $\vee$ bind more stronger than $\to$, and that $\alpha_1 \to \alpha_2 \to \cdots \to \alpha_n = \alpha_1 \to (\alpha_2 \to (\cdots \to (\alpha_{n-1} \to \alpha_n) \cdots))$. For example, the axiom scheme (A2) below is $(\Box(\alpha \to \beta)) \to ((\Box\alpha) \to \Box\beta)$, and $\neg\alpha \wedge \beta \to \Box^+\gamma \vee \delta = ((\neg\alpha) \wedge \beta) \to ((\Box^+\gamma) \vee \delta)$.

A *Kripke model* for $\mathrm{K}_{\Box\Box^+}$ is a triple $\langle W, R, V \rangle$ as follows.

- W is a nonempty set. (The set of *possible worlds*.)
- R is a binary relation on W. (The *accessibility relation*.)
- V is a function from $W \times$ **Prop** to $\{\mathsf{T}, \mathsf{F}\}$.

The transitive closure of R is denoted by R^+; that is, xR^+y holds if and only if

$$x = a_0 R a_1 R \cdots R a_n = y$$

for some $a_0, a_1, \ldots, a_n$ $(n \geq 1)$. Let $M = \langle W, R, V \rangle$ be a Kripke model. The notion "a formula φ is true at a world w in M", written by "$M, w \models \varphi$", is defined by inducion on φ as follows.

- $M, w \models p \iff V(w, p) = \mathsf{T}$.
- $M, w \models \alpha \wedge \beta \iff M, w \models \alpha$ and $M, w \models \beta$.
- $M, w \models \neg\alpha \iff M, w \not\models \alpha$.
- $M, w \models \Box\alpha \iff M, x \models \alpha$ for any x such that wRx.
- $M, w \models \Box^+\alpha \iff M, x \models \alpha$ for any x such that wR^+x.

We say that a formula φ is *valid* if and only if $M, x \models \varphi$ for any Kripke model M and any world x in M.

A proof system of $\mathrm{K}_{\Box\Box^+}$ is as follows (cf. the axiomatization of linear temporal logic in [2, §9]). The axiom schemes are

(A1) instances of classical tautologies,

(A2) $\Box(\alpha \to \beta) \to \Box\alpha \to \Box\beta$ ('**K** axiom' for $\Box$),

(A3) $\Box^+(\alpha \to \beta) \to \Box^+\alpha \to \Box^+\beta$ ('**K** axiom' for $\Box^+$),

(A4) $\Box^+\alpha \to \Box^+\Box^+\alpha$ ('**4** axiom' for $\Box^+$),

(A5) $\Box^+\alpha \to \Box\alpha$, and

(A6) $\Box\alpha \wedge \Box^+(\alpha \to \Box\alpha) \to \Box^+\alpha$ ('induction axiom')

and the inference rules are

(R1) $\dfrac{\alpha \to \beta \quad \alpha}{\beta}$ (modus ponens), and

(R2) $\dfrac{\alpha}{\Box^+\alpha}$ (generalization for $\Box^+$).

Another aximatization is there (cf. the axiomatization of common knowledge logic in [1, §3.3]): The axiom schemes are (A1), (A2), and

(A7) $\Box^{+}\alpha \to \Box\alpha \wedge \Box\Box^{+}\alpha$,

and the inference rules are (R1) and

(R3) $\dfrac{\alpha}{\Box\alpha}$ (generalization for $\Box$), and

(R4) $\dfrac{\alpha \to \Box(\beta \wedge \alpha)}{\alpha \to \Box^{+}\beta}$ (induction rule).

Theorem 1 *These two systems are equivalent.*

Proof We show the following.

- (♠) The latter scheme (A7) and the rules (R3, R4) are provable and derivable in the formaer system.
- (♣) The former schemes (A3,A4,A5,A6) and the rule (R2) are provable and derivable in the latter system.

(Proof sketch of ♠): (A7) and (R3) are easily shown by using (A4), (A5), (R2) and others. (R4) is shown as follows. First we have

$$\alpha \to \Box(\beta\wedge\alpha) \implies \beta\wedge\alpha \to \Box(\beta\wedge\alpha) \overset{(R2)}{\implies} \Box^{+}(\beta\wedge\alpha\to\Box(\beta\wedge\alpha)) \implies \alpha \to \Box^{+}(\beta\wedge\alpha\to\Box(\beta\wedge\alpha)). \quad (2.1)$$

On the other hand, an instance of (A6) is

$$\Box(\beta \wedge \alpha) \wedge \Box^{+}(\beta \wedge \alpha \to \Box(\beta \wedge \alpha)) \to \Box^{+}(\beta \wedge \alpha). \quad (2.2)$$

Then

$$\alpha \to \Box(\beta \wedge \alpha) \overset{(2.1),(2.2)}{\implies} \alpha \to \Box^{+}(\beta \wedge \alpha) \implies \alpha \to \Box^{+}\beta.$$

(Proof sketch of ♣): (A3) is inferred by (R4) from $\Box^{+}(\alpha \to \beta) \wedge \Box^{+}\alpha \to \Box(\beta \wedge \Box^{+}(\alpha \to \beta) \wedge \Box^{+}\alpha)$, which is implied by (A2), (A7) and others. (A4) is inferred by (R4) from $\Box^{+}\alpha \to \Box(\Box^{+}\alpha \wedge \Box^{+}\alpha)$, which is implied by (A7) and others. (A5) is easy. (A6) is shown as follows. By (A2), we have

$$\Box\alpha \wedge \Box(\alpha \to \Box\alpha) \to \Box\Box\alpha. \quad (2.3)$$

An instance of (A7) is

$$\Box^{+}(\alpha \to \Box\alpha) \to \Box(\alpha \to \Box\alpha) \wedge \Box\Box^{+}(\alpha \to \Box\alpha). \quad (2.4)$$

Then (2.3) and (2.4) imply

$$\Box\alpha \wedge \Box^{+}(\alpha \to \Box\alpha) \to \Box\Box\alpha, \quad (2.5)$$

and (2.4), (2.5) and others imply

$$\Box\alpha \wedge \Box^{+}(\alpha \to \Box\alpha) \to \Box(\alpha \wedge \Box\alpha \wedge \Box^{+}(\alpha \to \Box\alpha)), \quad (2.6)$$

which induces (A6) using (R4). (R2) is shown as follows.

$$\alpha \implies \alpha \wedge \top \overset{(R3)}{\implies} \Box(\alpha \wedge \top) \implies \top \to \Box(\alpha \wedge \top) \overset{(R4)}{\implies} \top \to \Box^{+}\alpha \implies \Box^{+}\alpha.$$

QED

By "$\vdash \varphi$", we mean "φ is provable in (any one of) the above systems". The purpose of this paper is to show

$$(\vdash \varphi) \iff (\varphi \text{ is valid}).$$

The soundness ($\Rightarrow$) is easily shown: all the axioms are valid, and all the inference rules preserve the validity of formulas. The rest of this paper will be devoted to proving the completeness ($\Leftarrow$).

3 Special formulas

In this section, we show provability of certain formulas which will be used in the next section.

Two formulas α and β are said to be *provably equivalent* when $\vdash (\alpha \to \beta) \wedge (\beta \to \alpha)$. If $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a finite set of formulas, then "$\vdash \langle\ \Gamma\ \rangle \Rightarrow \varphi$" means "$\vdash (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) \to \varphi$". Note that we do not mind permutations or duplications in $\langle \gamma_1, \ldots, \gamma_n \rangle$ because, for example, $((\gamma_1 \wedge \gamma_2) \wedge \gamma_3) \to \varphi$ and $((\gamma_2 \wedge \gamma_1) \wedge (\gamma_3 \wedge \gamma_1)) \to \varphi$ are provably equivalent.

Lemma 2 *(1) If* $\vdash \langle\ \varphi_1, \varphi_2, \ldots, \varphi_n\ \rangle \Rightarrow \psi$, *then* $\vdash \langle\ \varphi_1 \vee \rho, \varphi_2 \vee \rho, \ldots, \varphi_n \vee \rho\ \rangle \Rightarrow \psi \vee \rho$.

(2) If $\vdash \langle\ \varphi_1, \varphi_2, \ldots, \varphi_n\ \rangle \Rightarrow \psi$, *then* $\vdash \langle\ \rho \to \varphi_1, \rho \to \varphi_2, \ldots, \rho \to \varphi_n\ \rangle \Rightarrow \rho \to \psi$.

(3) If $\vdash \langle\ \varphi_1, \varphi_2, \ldots, \varphi_n\ \rangle \Rightarrow \psi$, *then* $\vdash \langle\ \Box\varphi_1, \Box\varphi_2, \ldots, \Box\varphi_n\ \rangle \Rightarrow \Box\psi$.

(4) If $\vdash \langle\ \varphi_1, \varphi_2, \ldots, \varphi_n\ \rangle \Rightarrow \psi$, *then* $\vdash \langle\ \Box^+\varphi_1, \Box^+\varphi_2, \ldots, \Box^+\varphi_n\ \rangle \Rightarrow \Box^+\psi$.

Proof (1) and (2) are properties of classical logic. (3) and (4) are properties of normal modal logics (i.e., systems containing the axioms scheme A1, A2, and A3, and the inference rules R1, R2, and R3). **QED**

Lemma 3 *Define formulas (1),(2),...,(5) as follows.*

(1) $\Box\tau$. (2) $\Box^+(\neg\alpha \to \Box\tau)$. (3) $\Box^+(\alpha \to \Box\beta)$.
(4) $\Box^+(\tau \to \omega)$. (5) $\Box^+(\tau \to \alpha \to \Box^+(\beta \to \omega))$.

Then we have

$$\vdash \langle\ (1), (2), (3), (4), (5)\ \rangle \Rightarrow \Box^+\omega.$$

Proof We define $\xi = \alpha \to \Box^+(\beta \to \omega)$ (therefore $(5) = \Box^+(\tau \to \xi)$). Since $\vdash \Box^+(\beta \to \omega) \to \Box\Box^+(\beta \to \omega)$ ($\because$ A7), we get

$$\vdash \Box^+(\Box^+(\beta \to \omega) \to \Box\xi) \tag{3.1}$$

by (R2) and others. Let $(5') = \Box(\tau \to \alpha \to \Box^+(\beta \to \omega))$ and $(5'') = \Box^+(\Box\tau \to \Box(\alpha \to \Box^+(\beta \to \omega)))$. We have

$$\vdash \langle\ (1), (5')\ \rangle \Rightarrow \Box\xi, \text{ and} \tag{3.2}$$

$$\vdash \langle\ (2), (5'')\ \rangle \Rightarrow \Box^+(\neg\alpha \to \Box\xi); \tag{3.3}$$

hence, by (3.3) and (3.1), we get

$$\vdash \langle\ (2), (5'')\ \rangle \Rightarrow \Box^+(\neg\alpha \vee \Box^+(\beta \to \omega) \to \Box\xi),$$

which is equivalent to

$$\vdash \langle\ (2), (5'')\ \rangle \Rightarrow \Box^+(\xi \to \Box\xi). \tag{3.4}$$

Then (3.2), (3.4) and the induction axiom (A6) imply

$$\vdash \langle\ (1), (2), (5'), (5'')\ \rangle \Rightarrow \Box^+\xi. \tag{3.5}$$

Next we define $\xi' = \alpha \to \Box(\beta \to \omega)$ and $(4') = \Box^+\Box(\tau \to \omega)$. We have

$$\vdash \langle\ (3), \Box^+\xi'\ \rangle \Rightarrow \Box^+(\alpha \to \Box\omega), \text{ and} \tag{3.6}$$

$$\vdash \langle\ (2), (4')\ \rangle \Rightarrow \Box^+(\neg\alpha \to \Box\omega); \tag{3.7}$$

then, by (3.6) and (3.7), we get

$$\vdash \langle\ (2), (3), (4'), \Box^+\xi'\ \rangle \Rightarrow \Box^+\Box\omega. \tag{3.8}$$

On the other hand,

$$\vdash \langle\ (1), (4'')\ \rangle \Rightarrow \Box\omega \tag{3.9}$$

where $(4'') = \Box(\tau \to \omega)$. Therefore, by (3.8), (3.9) and the fact $\vdash \Box\omega \wedge \Box^+\Box\omega \to \Box^+\omega$ ($\because$ induction axiom A6), we get

$$\vdash \langle\ (1), (2), (3), (4'), (4''), \Box^+\xi'\ \rangle \Rightarrow \Box^+\omega. \tag{3.10}$$

Finally (3.5), (3.10), and the facts

$$\vdash (4) \to (4'), \quad \vdash (4) \to (4''), \quad \vdash (5) \to (5'), \quad \vdash (5) \to (5'')$$

imply the conclusion:

$$\vdash \langle\ (1), (2), (3), (4), (5)\ \rangle \Rightarrow \Box^+\omega.$$

QED

Lemma 4 *Suppose $\sigma, \sigma', \tau, \tau'$ and ω are formulas such that*

(a) $\vdash \sigma \to \Box\tau$,

(b) $\vdash \sigma' \to \Box\tau'$, *and*

(c) $\vdash \neg\sigma' \to \Box\tau$.

Then we have

(d) $\vdash \langle\ \sigma \to \Box^+(\tau \to \omega),\ \sigma \to \Box^+(\tau \to \sigma' \to \Box^+(\tau' \to \omega))\ \rangle \Rightarrow \sigma \to \Box^+\omega$.

Proof By Lemma 3 ($\alpha = \sigma'$ and $\beta = \tau'$), we get

$$\vdash \langle\ \Box\tau,\ \Box^+(\neg\sigma' \to \Box\tau),\ \Box^+(\sigma' \to \Box\tau'),\ \Box^+(\tau \to \omega),\ \Box^+(\tau \to \sigma' \to \Box^+(\tau' \to \omega))\ \rangle \Rightarrow \Box^+\omega,$$

which implies

$$\vdash \langle\ \Box\tau,\ \Box^+(\tau \to \omega),\ \Box^+(\tau \to \sigma' \to \Box^+(\tau' \to \omega))\ \rangle \Rightarrow \Box^+\omega$$

because of the facts $\vdash \Box^+(\neg\sigma' \to \Box\tau)$ ($\because$ (c)) and $\vdash \Box^+(\sigma' \to \Box\tau')$ ($\because$ (b)). Then (d) is obtained by (a) and Lemma 2(2). **QED**

In the rest of this section, a natural number $N \geq 2$ and formulas ω, σ_i, τ_i $(i = 1, 2, \ldots, N)$ are fixed. A formula is called a *spaecial formula* if and only if it is of the form

$$\sigma_{f(1)} \to \Box^+\Big(\tau_{f(1)} \to \sigma_{f(2)} \to \Box^+\Big(\tau_{f(2)} \to \cdots \to \sigma_{f(m)} \to \Box^+(\tau_{f(m)} \to \omega)\cdots\Big)\Big)$$

for some natural number m and some function f that satisfy the following conditions.

- $1 \leq m \leq N$.
- f is an injection (one-to-one) from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, N\}$.
- $f(1) = 1$.

The set of spacial formulas is called **SP**, which is a finite set. For example, if $N = 3$, then

$$\begin{aligned}\mathbf{SP} = \{\ & \sigma_1 \to \Box^+(\tau_1 \to \omega),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_2 \to \omega)),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \omega)),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_2 \to \sigma_3 \to \Box^+(\tau_3 \to \omega))),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \sigma_2 \to \Box^+(\tau_2 \to \omega)))\ \}.\end{aligned} \tag{3.11}$$

SP consists of sixteen formulas if $N = 4$.

Theorem 5 (Main Theorem on Special Formulas) *Suppose that*

(1) $\vdash \sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_N$*, and*

(2) $\vdash \sigma_i \to \Box\tau_i$*, for* $i = 1, 2, \ldots, N$*,*

where $N \geq 2$*. Then we have*

$$\vdash \langle\ \mathbf{SP}\ \rangle \Rightarrow (\sigma_1 \to \Box^+\omega).$$

This theorem, which is the goal of this section, will be proved later.

A formula is called a *key formula of type I* if and only if it is of the form

$$\sigma_{f(1)} \to \Box^+\Big(\tau_{g(1)} \to \sigma_{f(2)} \to \Box^+\Big(\tau_{g(2)} \to \cdots \to \sigma_{f(m)} \to \Box^+(\underline{\tau_{g(m)} \to}\ \omega)\cdots\Big)\Big) \tag{3.12}$$

(the underline will be used later) for some natural number m and some functions f and g that satisfy the following conditions.

- $1 \leq m \leq N$.
- f is an injection from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, N\}$.
- g is a function (not limited to injection) from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, N\}$.
- $f(1) = g(1) = 1$.

♡ $(\forall i \in \{1, \ldots, m\})(\exists j \leq i)(f(j) = g(i))$.

The set of key formulas of type I is called **KeyI**, which is a finite superset of **SP**. For example, if $N = 3$, then **KeyI** consists of seventeen formulas as follows.

$$\begin{aligned}\mathbf{KeyI} \ = \ \mathbf{SP} \text{ (see (3.11))} \ \cup \ \big\{ \ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_1 \to \omega)),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_1 \to \omega)),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_i \to \omega)))\ (i = 1, 2, 3),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_2 \to \sigma_3 \to \Box^+(\tau_j \to \omega)))\ (j = 1, 2),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_i \to \omega)))\ (i = 1, 2, 3),\\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \sigma_2 \to \Box^+(\tau_k \to \omega)))\ (k = 1, 3)\big\}.\end{aligned}$$

Lemma 6 $\vdash \langle\ \mathbf{SP}\ \rangle \Rightarrow \varphi$ *for any key formula* φ *of type I.*

Proof For any key formula φ of type I, there is a special formula $\varphi*$ which is embedded in φ and $\vdash \langle\ \varphi*\ \rangle \Rightarrow \varphi$. For example, if φ is

$$\sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \sigma_4 \to \Box^+(\tau_3 \to \sigma_5 \to \Box^+(\tau_1 \to \sigma_6 \to \Box^+(\tau_5 \to \omega)))))),$$

then $\varphi*$ is

$$\sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \sigma_5 \to \Box^+(\tau_5 \to \omega)))),$$

which is embedded in φ as

$$\underline{\sigma_1 \to} \Box^+(\tau_1 \to \sigma_2 \to \underline{\Box^+(\tau_1 \to \sigma_3 \to} \Box^+(\tau_3 \to \sigma_4 \to \underline{\Box^+(\tau_3 \to \sigma_5 \to} \Box^+(\tau_1 \to \sigma_6 \to \underline{\Box^+(\tau_5 \to \omega)}))))).$$

In general, $\varphi*$ is defined as follows. Let φ be the fomrula as (3.12). Without loss of generality, we suppose $f(i) = i$ for all i. Then, by the property ♡, we have

(♡′) $i \geq g(i)$.

Now we define a sequence $a_1, a_2, \ldots$ of natural numbers by

$$a_1 = g(m), \quad a_{x+1} = g(a_x - 1) \text{ for } x = 1, 2, \ldots$$

By $\heartsuit'$, this sequence is strictly decreasing, and $\varphi*$ is

$$\sigma_{a_z} \to \Box^+\Big(\tau_{a_z} \to \sigma_{a_{z-1}} \to \Box^+\Big(\tau_{a_{z-1}} \to \cdots \to \sigma_{a_2} \to \Box^+(\tau_{a_2} \to \sigma_{a_1} \to \Box^+(\tau_{a_1} \to \omega)) \cdots \Big)\Big)$$

where $a_z = 1$. The fact $\vdash \langle\ \varphi*\ \rangle \Rightarrow \varphi$ is obtained from $\vdash \Box^+(\tau_{g(m)} \to \omega) \to \Box^+(\tau_{g(m)} \to \omega)$ by appropriate applications of Lemma 2(2), 2(4) and the fact "$\vdash \langle\ \Box^+\alpha\ \rangle \Rightarrow \Box^+\beta$ implies $\vdash \langle\ \Box^+\alpha\ \rangle \Rightarrow \Box^+(\tau \to \sigma \to \Box^+\beta)$". **QED**

A formula φ is called a *key formula of type II* if and only if there is a formula ψ which satisfies the following conditions.

- ψ is a key formula of type I as (3.12) where $m \leq (N-1)$.
- φ is obtained from ψ by deleting the underlined '$\tau_{g(m)} \to$' in (3.12).

The natural number m is called the *depth of* φ. For example, if $N = 3$, then there are just three key formulas of type II:

$$\sigma_1 \to \Box^+\omega. \qquad \text{(depth = 1)}$$
$$\sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+\omega). \qquad \text{(depth = 2)}$$
$$\sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+\omega). \qquad \text{(depth = 2)}$$

Lemma 7 *Suppose that*

(1) $\vdash \sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_N$*, and*

(2) $\vdash \sigma_i \to \Box\tau_i$*, for* $i = 1, 2, \ldots, N$*,*

where $N \geq 2$*. Then*

$$\vdash \langle\ \mathbf{KeyI}\ \rangle \Rightarrow \varphi$$

for any key formula φ *of type II.*

Proof φ is of the form

$$\sigma_{f(1)} \to \Box^+\Big(\tau_{g(1)} \to \cdots \to \sigma_{f(m-1)} \to \Box^+\Big(\tau_{g(m-1)} \to \sigma_{f(m)} \to \Box^+\omega\Big) \cdots\Big).$$

We will abbreviate this to

$$\bullet \to \sigma_{f(m)} \to \Box^+\omega.$$

That is, "$\bullet$" denotes the context " $\sigma_{f(1)} \to \Box^+(\tau_{g(1)} \to \cdots \to \sigma_{f(m-1)} \to \Box^+(\tau_{g(m-1)} \to$ ". Therefore, for example, $\bullet \to \sigma_{f(m)} \to \Box^+(\tau_{g(m)} \to \omega)$ is the fomrula (3.12), and $\bullet \to \sigma_1 \to \Box^+\omega$ is just $\sigma_1 \to \Box^+\omega$ when $m = 1$.

We define a set U of natural numbers by

$$U = \{1, 2, \ldots, N\} - \{f(1), f(2), \ldots, f(m)\}.$$

U is not empty because of the definition of key formula of type II. Then we prove this Lemma 7 by induction on $|U|$; in other words, we prove this lemma for any φ of depth $(N-1)$, φ of depth $(N-2)$, $\ldots$, φ of depth 1, successively.

(Case 1: $|U| = 1$; depth of φ is $N-1$.) For any $i \in \{1, \ldots, m\}$, the formula

$$\bullet \to \sigma_{f(m)} \to \Box^+(\tau_{f(i)} \to \omega) \tag{3.13}$$

is a key formula of type I. Therefore we have

$$\vdash \langle\ \mathbf{KeyI}\ \rangle \Rightarrow\ \bullet \to \sigma_{f(m)} \to \Box^+((\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)}) \to \omega) \tag{3.14}$$

because of the fact

$$\vdash \langle\ \tau_{f(1)} \to \omega,\ \tau_{f(2)} \to \omega,\ \ldots,\ \tau_{f(m)} \to \omega\ \rangle \Rightarrow (\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)}) \to \omega$$

and Lemma 2(4) and 2(2). Let u be the only element of U. Similarly to (3.14), we have

$$\vdash \langle\ \mathbf{KeyI}\ \rangle \Rightarrow\ \bullet \to \sigma_{f(m)} \to \Box^{+}\big((\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)}) \to \sigma_u \to \Box^{+}(\tau_u \to \omega)\big) \tag{3.15}$$

because the formula

$$\bullet \to \sigma_{f(m)} \to \Box^{+}(\tau_{f(i)} \to \sigma_u \to \Box^{+}(\tau_u \to \omega))$$

is a key formula of type I for any $i \in \{1, \ldots, m\}$. On the other hand, by Lemma 4 ($\sigma = \sigma_{f(m)}$, $\sigma' = \sigma_u$, $\tau = (\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)})$, $\tau' = \tau_u$), we get

$$\begin{aligned} \vdash \langle\ &\sigma_{f(m)} \to \Box^{+}((\tau_{f(1)} \vee \cdots \vee \tau_{f(m)}) \to \omega), \\ &\sigma_{f(m)} \to \Box^{+}((\tau_{f(1)} \vee \cdots \vee \tau_{f(m)}) \to \sigma_u \to \Box^{+}(\tau_u \to \omega))\ \rangle \Rightarrow \sigma_{f(m)} \to \Box^{+}\omega. \end{aligned} \tag{3.16}$$

Note that the hypotheses (a), (b), and (c) of Lemma 4 are shown by the hypotheses (1) and (2) of this Lemma 7. Then (3.14), (3.15), (3.16) and Lemma 2 imply

$$\vdash \langle\ \mathbf{KeyI}\ \rangle \Rightarrow\ \bullet \to \sigma_{f(m)} \to \Box^{+}\omega, \tag{3.17}$$

which is the required formula.

(Case 2: $|U| > 1$; depth of φ is less then $N-1$.) By the same argument of (3.14), we obtain

$$\vdash \langle\ \mathbf{KeyI}\ \rangle \Rightarrow\ \bullet \to \sigma_{f(m)} \to \Box^{+}((\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)}) \to \omega). \tag{3.18}$$

On the other hand, the formula

$$\bullet \to \sigma_{f(m)} \to \Box^{+}(\tau_{f(i)} \to \sigma_u \to \Box^{+}\omega))$$

is a key formulas of type II with greater depth for any $i \in \{1, \ldots, m\}$ and any $u \in U$. Therefore by the induction hypothesis,

$$\vdash \langle\ \mathbf{KeyI}\ \rangle \Rightarrow\ \bullet \to \sigma_{f(m)} \to \Box^{+}(\tau_{f(i)} \to \sigma_u \to \Box^{+}\omega)),$$

and then

$$\vdash \langle\ \mathbf{KeyI}\ \rangle \Rightarrow\ \bullet \to \sigma_{f(m)} \to \Box^{+}\big((\tau_{f(1)} \vee \cdots \vee \tau_{f(m)}) \to (\sigma_{u_1} \vee \cdots \vee \sigma_{u_k}) \to \Box^{+}(\top \to \omega)\big) \tag{3.19}$$

where $U = \{u_1, \ldots, u_k\}$. Now (3.18), (3.19), and Lemma 4 ($\sigma = \sigma_{f(m)}$, $\sigma' = (\sigma_{u_1} \vee \cdots \vee \sigma_{u_k})$, $\tau = (\tau_{f(1)} \vee \cdots \vee \tau_{f(m)})$, $\tau' = \top$) imply

$$\vdash \langle\ \mathbf{KeyI}\ \rangle \Rightarrow\ \bullet \to \sigma_{f(m)} \to \Box^{+}\omega$$

similarly to (3.17). **QED**

Proof of Theorem 5 Since $(\sigma_1 \to \Box^{+}\omega)$ is a key formula of type II, we get Theorem 5 by Lemmas 6 and 7. **QED**

4 Making a countermodel

If φ is a frommula, then the expressions '$\varphi : \mathsf{T}$' and '$\varphi : \mathsf{F}$' are called *signed formulas.* A *semantic diagram* is a finite tree whose nodes are associated with finite sets of signed formulas and whose edges are labeled '$\Box$' or '$\Box^{+}$'. Set(a) denotes the set of signed formulas that is associated with the node a. If a node b is a $\Box$-successor (or $\Box^{+}$-successor) of a node a, then we write $\mathsf{a} <^{\Box} \mathsf{b}$ (or $\mathsf{a} <^{\Box^{+}} \mathsf{b}$, respectively). Moreover we write $\mathsf{a} < \mathsf{b}$ if and only if $\mathsf{a} <^{\Box} \mathsf{b}$ or $\mathsf{a} <^{\Box^{+}} \mathsf{b}$. The transitive closure of $<$ is written by $\ll$. Figure 1 is an example of a semantic diagram, in which Set(a) $= \{\alpha : \mathsf{T}, \beta : \mathsf{T}, \gamma : \mathsf{F}\}$, Set($\mathsf{b}$) $= \emptyset$, $\mathsf{a} <^{\Box} \mathsf{b}$, $\mathsf{b} <^{\Box^{+}} \mathsf{e}$, $\mathsf{a} < \mathsf{b}$, $\mathsf{b} < \mathsf{e}$, $\mathsf{a} \not< \mathsf{e}$, $\mathsf{a} \ll \mathsf{b}$, $\mathsf{a} \ll \mathsf{e}$, and $\mathsf{a} \not\ll \mathsf{a}$ hold. In the following, $\Gamma, \Delta, \ldots$ will denote sets of signed formulas,

Figure 1: A semantic diagram.

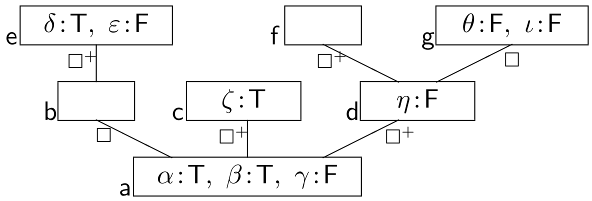
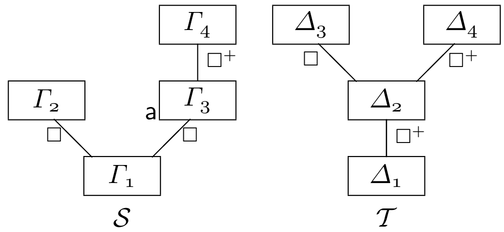


Figure 2: Semantic diagrams $\mathcal{S}$ and $\mathcal{T}$.



$\mathcal{S}, \mathcal{T}, \ldots$ will denote semantic diagrams, and $\mathsf{a}, \mathsf{b}, \ldots$ will denote nodes of diagrams. By "$\varphi \in_{\mathsf{T}} \mathsf{x}$" (or "$\varphi \in_{\mathsf{F}} \mathsf{x}$"), we mean "$(\varphi : \mathsf{T}) \in \mathrm{Set}(\mathsf{x})$" (or "$(\varphi : \mathsf{F}) \in \mathrm{Set}(\mathsf{x})$", respectively).

For each diagram $\mathcal{S}$, we define a formula $\mathrm{Neg}(\mathcal{S})$ (called '*negation of* $\mathcal{S}$') inductively as follows. If a set

$$\{\varphi_1 : \mathsf{T}, \varphi_2 : \mathsf{T}, \ldots, \varphi_m : \mathsf{T},\ \psi_1 : \mathsf{F},\ \psi_2 : \mathsf{F}, \ldots, \psi_n : \mathsf{F}\}$$

is associatsed with the root of $\mathcal{S}$, and subdiagrams $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ are connected with the root by $\Box$-edges and $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_l$ are connected with the root by $\Box^+$-edges, then $\mathrm{Neg}(\mathcal{S})$ is the formula

$$\begin{aligned}&\bot \lor \neg\varphi_1 \lor \neg\varphi_2 \lor \cdots \lor \neg\varphi_m \lor \psi_1 \lor \psi_2 \lor \cdots \lor \psi_n \lor \\ &\Box(\mathrm{Neg}(\mathcal{S}_1)) \lor \Box(\mathrm{Neg}(\mathcal{S}_2)) \lor \cdots \lor \Box(\mathrm{Neg}(\mathcal{S}_k)) \lor \\ &\qquad\qquad\qquad \Box^+(\mathrm{Neg}(\mathcal{T}_1)) \lor \Box^+(\mathrm{Neg}(\mathcal{T}_2)) \lor \cdots \lor \Box^+(\mathrm{Neg}(\mathcal{T}_k)).\end{aligned}$$

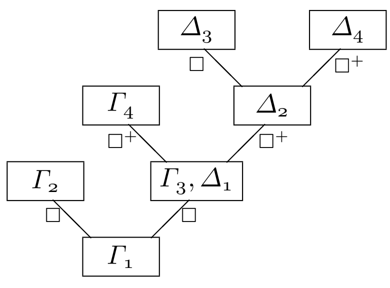
For example, the negation of the diagram of Figure 1 is provably equivalent to

$$\neg\alpha \lor \neg\beta \lor \gamma \lor \Box\Box^+(\neg\delta \lor \varepsilon) \lor \Box^+\neg\zeta \lor \Box^+\big(\eta \lor \Box^+\bot \lor \Box(\theta \lor \iota)\big).$$

A diagram $\mathcal{S}$ is said to be *consistent* if and only if $\nvdash \mathrm{Neg}(\mathcal{S})$.

Let $\mathcal{S}$ and $\mathcal{T}$ be semantic diagrams and a be a node of $\mathcal{S}$. By '$\mathcal{S} \overset{a}{+} \mathcal{T}$', we mean the diagram obtained by joining $\mathcal{S}$ and $\mathcal{T}$ in which a and the root of $\mathcal{T}$ are merged into one node. For example, if $\mathcal{S}$ and $\mathcal{T}$ are diagrams as Figure 2, then $\mathcal{S} \overset{a}{+} \mathcal{T}$ is the diagram as Figure 3.

Figure 3: Semantic diagram $\mathcal{S} \overset{a}{+} \mathcal{T}$.



Lemma 8 *Let $\mathcal{S}, \mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ be semantic diagrams ($n \geq 0$) and* a *be a node of* $\mathcal{S}$*. If*

$$\vdash \langle \text{ Neg}(\mathcal{T}_1), \text{ Neg}(\mathcal{T}_2), \ \ldots, \ \text{Neg}(\mathcal{T}_n) \ \rangle \Rightarrow \text{Neg}(\mathcal{T}),$$

then we have

$$\vdash \langle \text{ Neg}(\mathcal{S} \overset{a}{+} \mathcal{T}_1), \text{ Neg}(\mathcal{S} \overset{a}{+} \mathcal{T}_2), \ \ldots, \ \text{Neg}(\mathcal{S} \overset{a}{+} \mathcal{T}_n) \ \rangle \Rightarrow \text{Neg}(\mathcal{S} \overset{a}{+} \mathcal{T}).$$

Proof By Lemma 2 and the definition of Neg(). **QED**

Let $\mathbb{L}$ be a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ of formulas where λ_i and λ_j are distinct if $i \neq j$. We say that a set Λ of signed formulas is a *valuation of* $\mathbb{L}$ if Λ is $\{\lambda_1 : \bullet_1, \lambda_2 : \bullet_2, \ldots, \lambda_k : \bullet_k\}$ ($\bullet_i$ is T or F). There are 2^k distinct valuations of $\mathbb{L}$.

For a set Γ of signed formulas, we define a set $\Gamma^{\mathsf{T}}_{\Box}$ of signed formulas by $\Gamma^{\mathsf{T}}_{\Box} = \{\varphi : \mathsf{T} \mid (\Box\varphi : \mathsf{T}) \in \Gamma\}$. For example, $\{\Box\varphi_1 : \mathsf{T}, \ \Box\varphi_2 : \mathsf{F}, \ \Box^+\varphi_3 : \mathsf{T}, \ \neg\neg\Box\varphi_4 : \mathsf{T}, \ \Box\Box\varphi_5 : \mathsf{T}\}^{\mathsf{T}}_{\Box} = \{\varphi_1 : \mathsf{T}, \ \Box\varphi_5 : \mathsf{T}\}$.

Now we show a lemma, in which (3) is the novelty of our completeness proof.

Lemma 9 (Consistency preserving extensions of diagrams) *Let $\mathbb{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ ($k \geq 1$) be a finite set of formulas. We have the following.*

(1) If a semantic diagram $\mathcal{S}$ is consistent and a *is a node of it, then there exists a valuation Λ of $\mathbb{L}$ such that the diagram $\mathcal{S} \overset{\mathsf{a}}{+} \boxed{\Lambda}$ (i.e., the diagram obtained from $\mathcal{S}$ by adding Λ to the node* a*) is consistent. The process of making $\mathcal{S} \overset{\mathsf{a}}{+} \boxed{\Lambda}$ from $\mathcal{S}$ will be called "maximalization for* a *with respect to $\mathbb{L}$".*

(2) Let Γ be a valuation of $\mathbb{L}$ and $\mathcal{S}$ be a diagram as Figure 4; that is, $\text{Set}(\mathsf{a}) = \Gamma \cup \{\Box\varphi : \mathsf{F}\}$ *for some node* a *of $\mathcal{S}$. If $\mathcal{S}$ is consistent, then there exists a valuation Λ of $\mathbb{L}$ such that the diagram $\mathcal{T}$ of Figure 5 is consistent. The process of making $\mathcal{T}$ from $\mathcal{S}$ will be called "fulfillment of $\Box\varphi : \mathsf{F}$ for* a *with respext to $\mathbb{L}$".*

(3) Let Γ_1 be a valuation of $\mathbb{L}$ and $\mathcal{S}$ be a diagram as Figure 6; that is, $\text{Set}(\mathsf{a}) = \Gamma_1 \cup \{\Box^+\varphi : \mathsf{F}\}$ *for some node* a *of $\mathcal{S}$. If $\mathcal{S}$ is consistent, then one of the follwoing conditions holds.*

(I) There is a valuation Λ of $\mathbb{L}$ such that the diagram $\mathcal{T}$ of Figure 7 is consistent.

(II) There are valuations $\Gamma_2, \Gamma_3, \ldots, \Gamma_m$ and Λ of $\mathbb{L}$ such that $m \geq 2$ and the diagram $\mathcal{T}$ of Figure 8 is consistent.

The process of making $\mathcal{T}$ from $\mathcal{S}$ will be called "fulfillment of $\Box^+\varphi : \mathsf{F}$ for a *with respect to $\mathbb{L}$".*

Note that, in Figures 4–8, $\mathcal{U}$ and $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ are subdiagrams where $\mathcal{U}$ may be null (this means the node a *is the root) and $n \geq 0$.*

Proof

(1) Since $\vdash \langle \neg\lambda_i, \lambda_i \rangle \Rightarrow \bot$, one of the diagrams $\mathcal{S} \overset{\mathsf{a}}{+} \boxed{\lambda_i : \mathsf{T}}$ and $\mathcal{S} \overset{\mathsf{a}}{+} \boxed{\lambda_i : \mathsf{F}}$ is consistent (otherwise $\vdash \text{Neg}(\mathcal{S})$ by Lemma 8). Iterating this argument, we can chose $\bullet_1, \bullet_2, \ldots, \bullet_k$ ($\bullet_i \in \{\mathsf{T}, \mathsf{F}\}$) such that $\mathcal{S} \overset{\mathsf{a}}{+} \boxed{\lambda_1 : \bullet_1, \lambda_2 : \bullet_2, \ldots, \lambda_k : \bullet_k}$ is consistent.

(2) For any set Λ of signed formulas, we define the formula $\langle\!\langle \Lambda \rangle\!\rangle$ to be $\neg\text{Neg}(\boxed{\Lambda})$. For example, if $\Lambda = \{\Box\alpha : \mathsf{T}, \ \Box\beta : \mathsf{T}, \ \Box^+\gamma : \mathsf{T}, \ \Box^+\delta : \mathsf{F}\}$, then $\langle\!\langle \Lambda \rangle\!\rangle$ is provably equivalent to $\Box\alpha \wedge \Box\beta \wedge \Box^+\gamma \wedge \neg\Box^+\delta$ and $\langle\!\langle \Lambda^{\mathsf{T}}_{\Box} \rangle\!\rangle$ is provably equivalent to $\alpha \wedge \beta$. Note that

$$\vdash \langle\!\langle \Lambda \rangle\!\rangle \to \Box\langle\!\langle \Lambda^{\mathsf{T}}_{\Box} \rangle\!\rangle$$

because $\vdash \Box\alpha_1 \wedge \cdots \wedge \Box\alpha_x \to \Box(\alpha_1 \wedge \cdots \wedge \alpha_x)$. Now let $\mathcal{W}$ be a diagram as Figure 9. We have

$$\vdash \ \text{Neg}(\mathcal{W}) \to \text{Neg}(\boxed{\Gamma, \Box\varphi : \mathsf{F}}) \tag{4.1}$$

because $\text{Neg}(\mathcal{W})$ is provably equivalent to the formula $\langle\!\langle \Gamma \rangle\!\rangle \to \Box\varphi \vee \Box(\langle\!\langle \Gamma^{\mathsf{T}}_{\Box} \rangle\!\rangle \to \varphi)$, which implies (using the **K** axiom) $\langle\!\langle \Gamma \rangle\!\rangle \to \Box\varphi \vee (\Box\langle\!\langle \Gamma^{\mathsf{T}}_{\Box} \rangle\!\rangle \to \Box\varphi)$, and then $\langle\!\langle \Gamma \rangle\!\rangle \to \Box\varphi$. Now the consistent diagram $\mathcal{S}$ of Figure 4 is equivalent to $\mathcal{S} \overset{\mathsf{a}}{+} \boxed{\Gamma, \Box\varphi : \mathsf{F}}$. Then (4.1) and Lemma 8 imply that $\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{W}$ is consistent.

Figure 4: Diagram $\mathcal{S}$ of Lemma 9 (2).

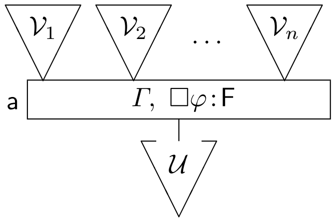


Figure 5: Diagram $\mathcal{T}$ of Lemma 9 (2).

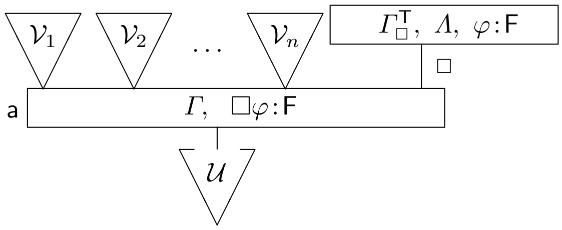


Figure 6: Diagram $\mathcal{S}$ of Lemma 9 (3)

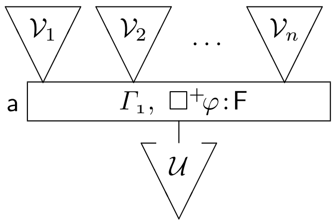


Figure 7: Diagram $\mathcal{T}$ of Lemma 9 (3-I).

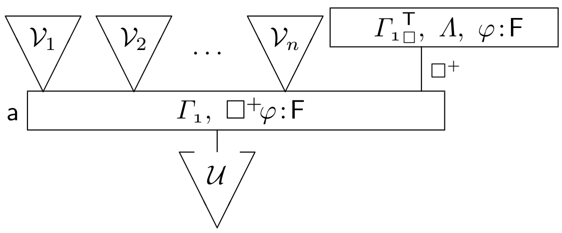


Figure 8: Diagram $\mathcal{T}$ of Lemma 9 (3-II).

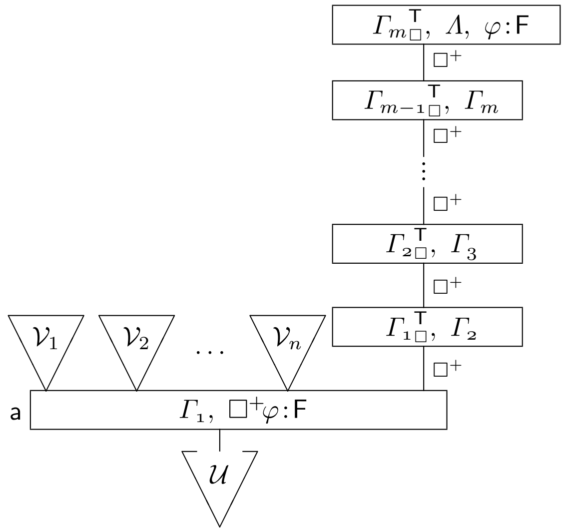


Figure 9: Diagram $\mathcal{W}$

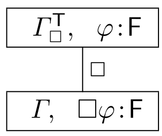


Figure 10: Special path from Γ_1 to $\varphi{:}\mathsf{F}$

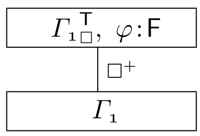
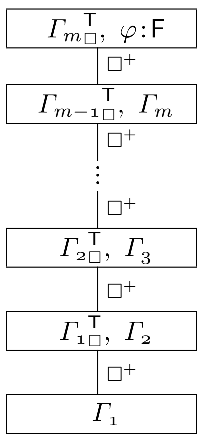


Figure 11: Special path from Γ_1 to $\varphi{:}\mathsf{F}$



Finally we apply the maximalization (i.e., Lemma 9 (1)) to the top node of $\mathcal{W}$ in $\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{W}$, and we get the required diagram $\mathcal{T}$ of Figure 5.

(3) We say that a diagram is *special path from* Γ_1 *to* $\varphi\,{:}\,\mathsf{F}$ if and only if it is of the form as Figure 10 or Figure 11 for some valuations $\Gamma_2, \ldots, \Gamma_m$ of $\mathbb{L}$ such that $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are mutually distinct. There are finitely many distinct valuations of $\mathbb{L}$, say $\Lambda_1, \Lambda_2 \ldots, \Lambda_N$ ($N = 2^k \geq 2$ because $k \geq 1$); therefore the number of all the special paths from Γ_1 to $\varphi\,{:}\,\mathsf{F}$ is also finite. Then let $\{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_P\}$ be the set of special paths. In the following, we will show

$$\vdash \langle\ \mathrm{Neg}(\mathcal{W}_1), \mathrm{Neg}(\mathcal{W}_2), \ldots, \mathrm{Neg}(\mathcal{W}_P)\ \rangle \Rightarrow \mathrm{Neg}(\boxed{\Gamma_1, \Box^+\varphi\,{:}\,\mathsf{F}}). \tag{4.2}$$

The negation of a special path is provably equivalent to the formla

$$\langle\!\langle \Gamma_1 \rangle\!\rangle \to \Box^+\Big(\langle\!\langle \Gamma_{1\Box}^{\mathsf{T}} \rangle\!\rangle \to \langle\!\langle \Gamma_2 \rangle\!\rangle \to \Box^+\Big(\cdots \to \Box^+\Big(\langle\!\langle \Gamma_{m-1\Box}^{\mathsf{T}} \rangle\!\rangle \to \langle\!\langle \Gamma_m \rangle\!\rangle \to \Box^+\Big(\langle\!\langle \Gamma_{m\Box}^{\mathsf{T}} \rangle\!\rangle \to \varphi\Big)\Big)\Big)\Big),$$

and the formula $\mathrm{Neg}(\boxed{\Gamma_1, \Box^+\varphi\,{:}\,\mathsf{F}})$ is provably equivalent to

$$\langle\!\langle \Gamma_1 \rangle\!\rangle \to \Box^+\varphi.$$

Moreover we have

$$\vdash \langle\!\langle \Lambda_1 \rangle\!\rangle \vee \langle\!\langle \Lambda_2 \rangle\!\rangle \vee \cdots \vee \langle\!\langle \Lambda_N \rangle\!\rangle$$

becasue this formula is a tautology. Therefore we can apply Theorem 5 ($\{\sigma_1, \sigma_2, \ldots, \sigma_N\} = \{\langle\!\langle \Lambda_1 \rangle\!\rangle, \langle\!\langle \Lambda_2 \rangle\!\rangle, \ldots, \langle\!\langle \Lambda_N \rangle\!\rangle\}$, $\sigma_1 = \langle\!\langle \Gamma_1 \rangle\!\rangle$, $\sigma_{f(i)} = \langle\!\langle \Gamma_i \rangle\!\rangle$, $\tau_{f(i)} = \langle\!\langle \Gamma_{i\Box}^{\mathsf{T}} \rangle\!\rangle$, $\omega = \varphi$), and we get (4.2). Now the consistent diagram $\mathcal{S}$ of Figure 6 is equivalent to $\mathcal{S} \overset{\mathsf{a}}{+} \boxed{\Gamma_1, \Box^+\varphi\,{:}\,\mathsf{F}}$. Then (4.2) and Lemma 8 imply that there is a special path $\mathcal{W}$ such that $\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{W}$ is consistent. Finally we apply the maximalization to the top node of $\mathcal{W}$ in $\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{W}$, and we get the required diagram $\mathcal{T}$ of Figure 7 or Figure 8. **QED**

We fix a formula α_0, and the set of subformulas of α_0 is called $\mathrm{Sub}(\alpha_0)$. We define some conditions on a node a of semantic diagrams as follows.

[$\mathrm{Sub}(\alpha_0)$**-maximality**] $\varphi \in \mathrm{Sub}(\alpha_0) \iff (\varphi \in_{\mathsf{T}} \mathsf{a}$ or $\varphi \in_{\mathsf{F}} \mathsf{a})$.

[$\Box$**-correctness**] If $\mathsf{a} < \mathsf{b}$ and $\Box\varphi \in_{\mathsf{T}} \mathsf{a}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$.

[$\Box$**-witness property**] If $\Box\varphi \in_{\mathsf{F}} \mathsf{a}$, then the following condition holds.

$$\exists \mathsf{b}\big(\mathsf{a} <^{\Box} \mathsf{b} \text{ and } \varphi \in_{\mathsf{F}} \mathsf{b}\big). \tag{$\spadesuit$}$$

[$\Box^+$**-witness property**] If $\Box^+\varphi \in_{\mathsf{F}} \mathsf{a}$, then the following condition holds.

$$\exists m \geq 1, \exists \mathsf{b}_1, \exists \mathsf{b}_2, \ldots, \exists \mathsf{b}_m \big(\mathsf{a} <^{\Box^+} \mathsf{b}_1 <^{\Box^+} \mathsf{b}_2 <^{\Box^+} \cdots <^{\Box^+} \mathsf{b}_m \text{ and } \varphi \in_{\mathsf{F}} \mathsf{b}_m\big). \tag{$\clubsuit$}$$

We say that a node x is *set-fresh* if and only if the condition ($\mathsf{y} \ll \mathsf{x} \Rightarrow \mathrm{Set}(\mathsf{y}) \neq \mathrm{Set}(\mathsf{x})$) holds for any node y. Then the following statement on a semantic diagram $\mathcal{T}$ is called *diagram-model condition with respect to* $\mathrm{Sub}(\alpha_0)$, which is the key notion for our completeness proof.

- $\mathcal{T}$ is consistent;
- all the nodes of $\mathcal{T}$ are $\mathrm{Sub}(\alpha_0)$-maximal and $\Box$-correct; and
- all the set-fresh nodes of $\mathcal{T}$ satisfy $\Box$-witness and $\Box^+$-witness properties.

Lemma 10 *If* $\nvdash \alpha_0$*, then there exists a semantic diagram* $\mathcal{T}$ *such that the diagram-model condition holds with respect to* $\mathrm{Sub}(\alpha_0)$ *and the root contains the signed formula* $\alpha_0\,{:}\,\mathsf{F}$.

Proof We define a procedure to construct semantic diagrams $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2 \ldots$, such that $\mathcal{T}_i$ is consistent and all the nodes of $\mathcal{T}_i$ are $\mathrm{Sub}(\alpha_0)$-maximal and $\Box$-correct.

[Construction of $\mathcal{T}_0$]
The one node diagram $\boxed{\alpha_0 : \mathsf{F}}$ is consistent because $\nvdash \alpha_0$. We apply the maximalization with respect to $\mathrm{Sub}(\alpha_0)$ (Lemma 9 (1)). Then we obtain a diagram whose only node is $\mathrm{Sub}(\alpha_0)$-maximal and contains $\alpha_0 : \mathsf{F}$. This is the diagram $\mathcal{T}_0$.

[Construction of $\mathcal{T}_{i+1}$ from $\mathcal{T}_i$]
If $\mathcal{T}_i$ satisfies the diagram-model condition with respect to $\mathrm{Sub}(\alpha_0)$, then we stop the procedure and we get the required diagram. Otherwise there is a node, say a, which is set-fresh and $\Box$-witness (or $\Box^+$-witness) property fails; that is, there is a formula $\Box\varphi$ (or $\Box^+\varphi$) $\in_{\mathsf{F}} \mathsf{a}$ such that the condition ♠ (or ♣) does not hold. Then we apply the fulfillment of $\Box\varphi : \mathsf{F}$ (or $\Box^+\varphi : \mathsf{F}$) for a with respect to $\mathrm{Sub}(\alpha_0)$ (Lemma 9 (2) or (3)), and the resulting diagram is $\mathcal{T}_{i+1}$. The node a will be called a *growing point*. Note that the fulfillment preserves $\mathrm{Sub}(\alpha_0)$-maximality and $\Box$-correctness; the latter is shown as follows. For example, if $\Box\psi : \mathsf{T}$ is an element of a node $\boxed{\Gamma_{j\Box}^{\mathsf{T}}, \Gamma_{j+1}}$ in the special path, then $(\Box\psi : \mathsf{T}) \in \Gamma_{j+1}$ (otherwise $(\Box\psi : \mathsf{F}) \in \Gamma_{j+1}$ and the diagram would be inconsistent—the negation of the diagram would be provable), and then $\psi : \mathsf{T}$ is an element of the next node $\boxed{\Gamma_{j+1\Box}^{\mathsf{T}}, \Gamma_{j+2}}$.

We show that the above procedure must terminate (hence we eventually get the required diagram). Otherwise an infinite sequence $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2 \dots$ is produced. Then consider the infinite diagram $\bigcup_{i=0}^{\infty} \mathcal{T}_i$. This infinite tree is finite branching because we can apply at most p times fulfillment for each growing point where p is the number of $\Box$- or $\Box^+$- formulas in $\mathrm{Sub}(\alpha_0)$. Therefore there is an infinite path which contains infinite many growing points; however this is impossible because each growing point must be set-fresh and the number of set-frech nodes in one path cannot be greater than $2^{|\mathrm{Sub}(\alpha_0)|}$. **QED**

Lemma 11 *If a semantic diagram $\mathcal{T}$ satisfies the diagram-model condition with respect to* $\mathrm{Sub}(\alpha_0)$*, then the following hold for any node* a *of* $\mathcal{T}$*.*

(1) If $\varphi \in_{\mathsf{F}} \mathsf{a}$, then $\varphi \notin_{\mathsf{T}} \mathsf{a}$.

(2) If $\varphi \wedge \psi \in_{\mathsf{T}} \mathsf{a}$, then $\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\psi \in_{\mathsf{T}} \mathsf{a}$.

(3) If $\varphi \wedge \psi \in_{\mathsf{F}} \mathsf{a}$, then $\varphi \in_{\mathsf{F}} \mathsf{a}$ or $\psi \in_{\mathsf{F}} \mathsf{a}$.

(4) If $\neg\varphi \in_{\mathsf{T}} \mathsf{a}$, then $\varphi \in_{\mathsf{F}} \mathsf{a}$.

(5) If $\neg\varphi \in_{\mathsf{F}} \mathsf{a}$, then $\varphi \in_{\mathsf{T}} \mathsf{a}$.

(6) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} < \mathsf{b}$, then $\Box^+\varphi \in_{\mathsf{T}} \mathsf{b}$ and $\varphi \in_{\mathsf{T}} \mathsf{b}$.

Proof
(1) If $\varphi \in_{\mathsf{F}} \mathsf{a}$ and $\varphi \in_{\mathsf{T}} \mathsf{a}$, then $\mathcal{T}$ would be inconsistent.
(2) If $\varphi \wedge \psi \in_{\mathsf{T}} \mathsf{a}$ and $\varphi \notin_{\mathsf{T}} \mathsf{a}$ (or $\psi \notin_{\mathsf{T}} \mathsf{a}$), then $\varphi \in_{\mathsf{F}} \mathsf{a}$ (or $\psi \in_{\mathsf{F}} \mathsf{a}$) by $\mathrm{Sub}(\varphi_0)$-maximality, and then $\mathcal{T}$ would be inconsistent because $\vdash \neg(\varphi \wedge \psi) \vee \varphi$ (or $\vdash \neg(\varphi \wedge \psi) \vee \psi$).
(3),(4),(5): Similar to (1) and (2).
(6) This is divided into the following four: (6-1) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} <^{\Box} \mathsf{b}$, then $\Box^+\varphi \in_{\mathsf{T}} \mathsf{b}$. (6-2) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} <^{\Box} \mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$. (6-3) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} <^{\Box^+} \mathsf{b}$, then $\Box^+\varphi \in_{\mathsf{T}} \mathsf{b}$. (6-4) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} <^{\Box^+} \mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$. The claim (6-1) is verified as follows. If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$, $\mathsf{a} <^{\Box} \mathsf{b}$, and $\Box^+\varphi \notin_{\mathsf{T}} \mathsf{b}$, then $\Box^+\varphi \in_{\mathsf{F}} \mathsf{b}$ by $\mathrm{Sub}(\varphi_0)$-maximality, and then $\mathcal{T}$ would be inconsistent because $\vdash \neg\Box^+\varphi \vee \Box(\Box^+\varphi \vee \cdots)$ ($\because \vdash \Box^+\varphi \to \Box\Box^+\varphi$). The claims (6-2), (6-2) and (6-4) are similary shown using the facts $\vdash \neg\Box^+\varphi \vee \Box(\varphi \vee \cdots)$ ($\because \vdash \Box^+\varphi \to \Box\varphi$), $\vdash \neg\Box^+\varphi \vee \Box^+(\Box^+\varphi \vee \cdots)$ ($\because \vdash \Box^+\varphi \to \Box^+\Box^+\varphi$), and $\vdash \neg\Box^+\varphi \vee \Box^+(\varphi \vee \cdots)$ ($\because \vdash \Box^+\varphi \to \Box^+\varphi$). **QED**

Theorem 12 (Completeness) *If α_0 is valid, then $\vdash \alpha_0$.*

Proof Suppose $\nvdash \alpha_0$. We will show that $\mathcal{M}, w \nvDash \alpha_0$ for some Kripke model $\mathcal{M}$ and some wourld w in $\mathcal{M}$. By Lemma 10, there is a semantic diagram $\mathcal{T}$ such that the diagram-model condition holds with respect to $\mathrm{Sub}(\alpha_0)$ and the root contains the signed formula $\alpha_0 : \mathsf{F}$. We define $\mathcal{M} = \langle W, R, V \rangle$ as follows.

- W is the set of nodes in $\mathcal{T}$.

- $\mathsf{a}R\mathsf{b} \iff \mathsf{a} < \mathsf{b}$ or $\exists \mathsf{a}_0 \big(\mathsf{a}_0 \ll \mathsf{a},\ \mathrm{Set}(\mathsf{a}_0) = \mathrm{Set}(\mathsf{a}),\ \text{and } \mathsf{a}_0 < \mathsf{b} \big)$.
- $V(\mathsf{a}, p) = \mathsf{T} \iff p \in_{\mathsf{T}} \mathsf{a}$.

Using the diagram-model condition of $\mathcal{T}$ and (6) of Lemma 11, we can show the following:

(i) If $\Box\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a}R\mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$.

(ii) If $\Box\varphi \in_{\mathsf{F}} \mathsf{a}$, then there is a node b such that $\mathsf{a}R\mathsf{b}$ and $\varphi \in_{\mathsf{F}} \mathsf{b}$.

(iii) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a}R^+\mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$.

(iv) If $\Box^+\varphi \in_{\mathsf{F}} \mathsf{a}$, then there is a node b such that $\mathsf{a}R^+\mathsf{b}$ and $\varphi \in_{\mathsf{F}} \mathsf{b}$.

Then we have:

(1) If $\varphi \in_{\mathsf{T}} \mathsf{a}$, then $\mathcal{M}, \mathsf{a} \models \varphi$.

(2) If $\varphi \in_{\mathsf{F}} \mathsf{a}$, then $\mathcal{M}, \mathsf{a} \not\models \varphi$.

These are proved simultaneously by induction on φ, using (1)–(5) of Lemma 11 and (i)–(iv) above. The claim (2) implies that $\mathcal{M}$ is the required model because the root of $\mathcal{T}$ contains $\alpha_0 : \mathsf{F}$. **QED**

References

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