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Completeness of Hilbert-style axiomatization for the extended computation tree logic ECTL

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Abstract

We give a complete Hilbert-style axiomatization for ECTL, which is an extension of the Computation Tree Logic (CTL) with a modal operator "infinitely often along some path".

1 Introduction

We treat extensions of the propositional Computation Tree Logic (CTL) (see, e,g, [5, 9] for general information on CTL and its neighbors). CTL has eight modal operators $\forall\mathsf{X}$, $\exists\mathsf{X}$, $\forall\mathsf{G}$, $\exists\mathsf{G}$, $\forall\mathsf{F}$, $\exists\mathsf{F}$, $\forall\mathsf{U}$, and $\exists\mathsf{U}$. For example, $\forall\mathsf{X}\alpha$ (or $\exists\mathsf{X}\alpha$), $\forall\mathsf{G}\beta$ (or $\exists\mathsf{F}\beta$), and $\gamma \,\forall\mathsf{U}\, \delta$ (or $\gamma \,\exists\mathsf{U}\, \delta$) represent "$\alpha$ holds for any (or some) next state", "β holds for any (or some) reachable state", and "along any (or some) path, γ holds until δ", respectively. There are a lot of extensions of CTL; among them, the logic CTL* is well studied. CTL* has six modal operators $\forall$, $\exists$, X, G, F, and U. For example, $\forall\exists\mathsf{FXGXF}\forall p$ is a CTL*-formula but not a CTL-formula. Note that, for example, "$\forall\mathsf{G}$" is a single operator in CTL while this represents successive applications of two operators G and $\forall$ in CTL*.

In this paper we treat the logic ECTL (by Emerson and Halpern [3]), which is a logic between CTL and CTL*. ECTL is obtained from CTL by adding two modal operators $\forall\mathsf{FG}$ and $\exists\mathsf{GF}$ where $\forall\mathsf{FG}\alpha$ and $\exists\mathsf{GF}\beta$ represent "along any path, there exists a state after which α always holds", and "there is a path along which β holds infinitely often" respectively (these two modalities are not expressible in CTL; see [3]). ECTL is a reasonable extension of CTL in the following sense: For any sequence $\vec{\mathbf{s}}$ of the unary modal operators $\forall$, $\exists$, X, G, and F where the first element of $\vec{\mathbf{s}}$ is $\forall$ or $\exists$, there is a sequence $\vec{\mathbf{s}'}$ of the unary modal operators $\forall\mathsf{X}$, $\exists\mathsf{X}$, $\forall\mathsf{G}$, $\exists\mathsf{G}$, $\forall\mathsf{F}$, $\exists\mathsf{F}$, $\forall\mathsf{FG}$, and $\exists\mathsf{GF}$ such that two formulas $\vec{\mathbf{s}}p$ and $\vec{\mathbf{s}'}p$ are equivalent (this will be shown in Section 2). For example, the CTL*-formula $\forall\exists\mathsf{FXGXF}\forall p$ is equivalent to the ECTL-formula $\exists\mathsf{X}\exists\mathsf{X}\exists\mathsf{GF}p$. A CTL*-formula whose outermost operator is $\forall$ or $\exists$ is called a *state formula*; hence the above property says that *each unary modality of state formulas of CTL* is expressible in ECTL.*

In general, to find a simple Hilbert-style axiomatization is a challenging problem in the study of non-classical logics. For example, its solutions for CTL* were published

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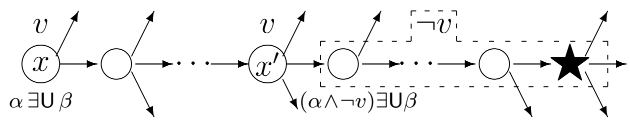
Figure 1: Property of models (1)



in the 2000s (Reynolds [7, 8]), while an axiomatization for CTL was given in the 1980s (Emerson and Halpern [2]). This paper gives a solution for ECTL — we prove that ECTL is axiomatized by adding the following schemata to CTL.

$$\forall\mathsf{G}(\alpha \to \beta) \to \exists\mathsf{GF}\alpha \to \exists\mathsf{GF}\beta$$
$$\exists\mathsf{GF}\alpha \leftrightarrow \exists\mathsf{X}\exists\mathsf{F}(\alpha \land \exists\mathsf{GF}\alpha)$$
$$\forall\mathsf{G}(\alpha \to \exists\mathsf{X}\exists\mathsf{F}\alpha) \to \alpha \to \exists\mathsf{GF}\alpha$$
$$\forall\mathsf{FG}\alpha \leftrightarrow \neg\exists\mathsf{GF}\neg\alpha$$

The first schema is a kind of "K-axiom" for $\exists\mathsf{GF}$, the second one says that $\exists\mathsf{GF}\varphi$ is a fixed point of $\exists\mathsf{X}\exists\mathsf{F}(\varphi \land \bullet)$, the third one is an induction axiom, and the forth one shows the duality between $\forall\mathsf{FG}$ and $\exists\mathsf{GF}$.

We show the completeness theorem: *If a formula is not provable in the above system of ECTL, then there exists a finite model in which the formula is false in some state.* As usual this is shown by constructing a model, of which each state is a kind of maximally consistent set; and in this construction, the following properties of models play a key role to define the accessibility relation. (For a formula ψ, the term "ψ-state" below denotes any state satisfying ψ.)

(1) Let v, α and β be formulas such that v implies both $\alpha \exists\mathsf{U} \beta$ and $\neg\beta$. If there is a v-state x, there is a path starting from x along which α holds until β. Then, on this path, there must be *the last v-state x' before the β-state*, and the next state of x' satisfies the formula $(\alpha \land \neg v) \exists\mathsf{U} \beta$ (see Figure 1 where ◯ is an α-state and ★ is a β-state).

(2) Let v, α and β be formulas such that v implies both $\alpha \forall\mathsf{U} \beta$ and $\neg\beta$. If there is a v-state x, then there must be *a last v-state x' before β-states*, and all the next states of x' satisfy the formula $(\alpha \land \neg v) \forall\mathsf{U} \beta$ (see Figure 2 where ◯ is an α-state and ★ is a β-state).

(3) Let v and φ be formulas such that v implies both $\forall\mathsf{FG}\neg\varphi$ and φ. If there is a v-state x, then there must be *a last v-state x'* ($\because$ otherwise we can construct a path along which infinitely many sates satisfy v and hence φ), and all the next states of x' satisfy the formula $\forall\mathsf{G}\neg v$ (see Figure 3).

Incidentally, the properties (1) and (2) were used by Lange and Stirling [6] for focus games and by Brünnler and Lange [1] and by Gaintzarain et al. [4] for sequent calculi.

The structure of this paper is as follows. In Section 2 we define models of ECTL and CTL*, and we show that each unary modality of state formulas of CTL* is expressible in ECTL. In Section 3 we introduce Hilbert-style axiomatization of ECTL, and we show derivability of certain formulas and of inference rules. In Section 4 we describe an outline of a standard completeness-proof for normal modal logics. In Section 5 we introduce

Figure 2: Property of models (2)

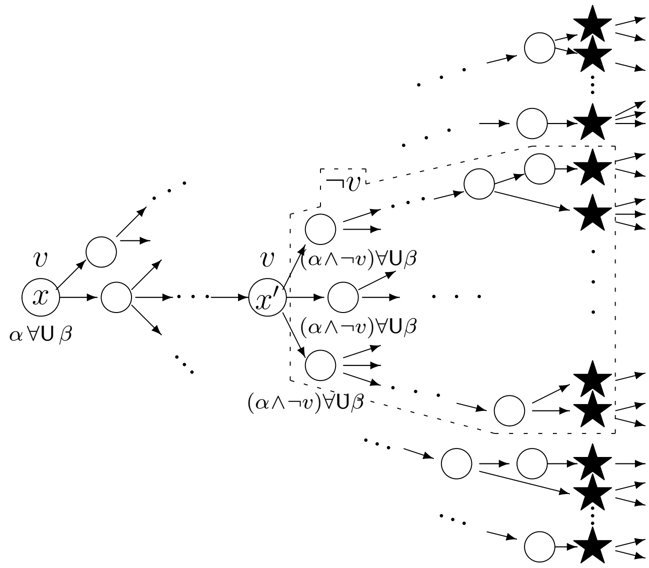
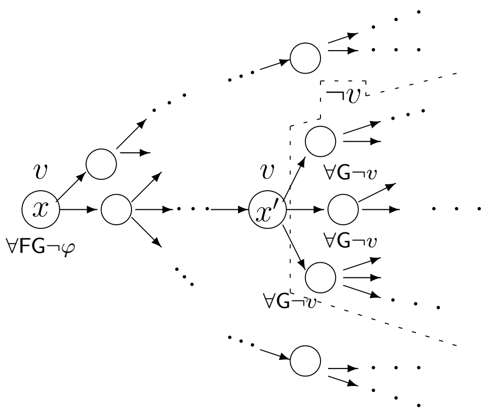


Figure 3: Property of models (3)



"consistent c-valuations", which will become the states of our model. In Section 6 we give an elaborate definition of the accessibility relation, and we show some lemmas on it. These definitions and lemmas are the main technical contribution of this paper. Finally in Section 7 we prove the completeness.

2 Semantics

In this section, we give a standard definitions of formulas and models for ECTL and CTL*.

ECTL-formulas are constructed from the following symbols: propositional variables and constants $\top$ and $\bot$; unary logical operator $\neg$; binary logical connectives $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$; unary modal operators $\forall\mathsf{X}$, $\exists\mathsf{X}$, $\forall\mathsf{G}$, $\exists\mathsf{G}$, $\forall\mathsf{F}$, $\exists\mathsf{F}$, $\forall\mathsf{FG}$ and $\exists\mathsf{GF}$; and binary modal connectives $\forall\mathsf{U}$ and $\exists\mathsf{U}$. *CTL*-formulas* are constructed from the following symbols: propositional variables/constants and unary/binary logical symbols as above; unary modal operators $\forall$, $\exists$, X, G, and F; and binary modal connective U. Propositional variables are denoted by $p, q, \ldots$, and formulas are denoted by $\alpha, \beta, \varphi, \psi, \ldots$. For example, $\mathsf{GF}\forall\mathsf{X}p$, $\exists\mathsf{GF}\forall\mathsf{X}p$, and $\exists\mathsf{GF}\forall\forall\mathsf{X}p$ are CTL*-formulas and the second one is also an ECTL-formula while the others are not. Note that the intended meaning of the ECTL-formula $p\,\forall\mathsf{U}\,q$ (or $p\,\exists\mathsf{U}\,q$) and CTL*-formula $\forall(p\,\mathsf{U}\,q)$ (or $\exists(p\,\mathsf{U}\,q)$, respectively) are equivalent. Parentheses are omitted by the convention that unary operators bind more stronger than binary connectives; $\wedge, \vee, \forall\mathsf{U}, \exists\mathsf{U}, \mathsf{U}$ bind more stronger than $\rightarrow$ and $\leftrightarrow$; and that $\alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_n$ is $\alpha_1 \rightarrow (\alpha_2 \rightarrow (\cdots \rightarrow (\alpha_{n-1} \rightarrow \alpha_n)\cdots))$. For example, $\forall\mathsf{G}(\alpha \rightarrow \beta) \rightarrow \exists\mathsf{GF}\alpha \rightarrow \exists\mathsf{GF}\beta$ (the first axiom of ECTL in the previous section) is $(\forall\mathsf{G}(\alpha \rightarrow \beta)) \rightarrow ((\exists\mathsf{GF}\alpha) \rightarrow (\exists\mathsf{GF}\beta))$.

By "*model*", we mean any triple $\langle S, R, V\rangle$ where S is a nonempty set, R is a binary relation on S satisfying $(\forall x \in S)(\exists y \in S)(xRy)$ (we call such a relation *serial*), and V is a mapping from $S \times \mathrm{PropVar}$ to $\{\mathtt{t}, \mathtt{f}\}$ where PropVar is the set of propositional variables. The elements of S are called *states*, and R is called the *accessibility relation*. A model is said to be *finite* if the set S of states is finite. A *path* is an infinite sequence $\langle x_0, x_1, x_2, \ldots\rangle$ of states such that $(\forall i \geq 0)(x_i R x_{i+1})$. If $\sigma = \langle x_0, x_1, x_2, \ldots\rangle$ is a path, then the state x_i is denoted by $\sigma(i)$, and the path $\langle x_n, x_{n+1}, x_{n+2}, \ldots\rangle$ is denoted by $\sigma\,|_n$, which is obtained from σ by deleting initial n elements. For any two paths σ and σ', we write "$\sigma =_0 \sigma'$" if and only if $\sigma(0) = \sigma'(0)$. We say that a path σ is an *x-path* if and only if $\sigma(0) = x$.

Truth values of ECTL-formulas are evaluated in each state. The notion "in a model $M = \langle S, R, V\rangle$, a state x satisfies an ECTL-formula φ", written by "$M, x \models \varphi$" (or "$x \models \varphi$" for short), is inductively defined as follows.

$x \models \top$. $x \not\models \bot$.

$x \models p \iff V(x, p) = \mathtt{t}$.

$x \models \neg\alpha \iff x \not\models \alpha$.

$x \models \alpha \wedge \beta \iff x \models \alpha$ and $x \models \beta$.

Logical connectives $\vee$, $\rightarrow$, $\leftrightarrow$ are evaluated similarly.

$x \models \forall\mathsf{X}\alpha \iff (\forall y)(xRy \Rightarrow y \models \alpha)$.

$x \models \exists\mathsf{X}\alpha \iff (\exists y)(xRy \;\&\; y \models \alpha)$.

$x \models \forall\mathsf{G}\alpha \iff (\forall\sigma : x\text{-path})(\forall n \geq 0)(\sigma(n) \models \alpha)$.

$x \models \exists\mathsf{G}\alpha \iff (\exists\sigma : x\text{-path})(\forall n \geq 0)(\sigma(n) \models \alpha)$.

$x \models \forall \mathsf{F}\alpha \iff (\forall\sigma : x\text{-path})(\exists n \geq 0)(\sigma(n) \models \alpha).$
$x \models \exists \mathsf{F}\alpha \iff (\exists\sigma : x\text{-path})(\exists n \geq 0)(\sigma(n) \models \alpha).$
$x \models \forall \mathsf{FG}\alpha \iff (\forall\sigma : x\text{-path})(\exists n \geq 0)(\forall m \geq n)(\sigma(m) \models \alpha).$
$x \models \exists \mathsf{GF}\alpha \iff (\exists\sigma : x\text{-path})(\forall n \geq 0)(\exists m \geq n)(\sigma(m) \models \alpha).$
$x \models \alpha \,\forall\mathsf{U}\, \beta \iff (\forall\sigma : x\text{-path})(\exists n \geq 0)\big(\sigma(n) \models \beta \;\&\; (\forall m < n)(\sigma(m) \models \alpha)\big).$
$x \models \alpha \,\exists\mathsf{U}\, \beta \iff (\exists\sigma : x\text{-path})(\exists n \geq 0)\big(\sigma(n) \models \beta \;\&\; (\forall m < n)(\sigma(m) \models \alpha)\big).$

In the last two clauses, the state $\sigma(n)$, which satisfies β, is called the *witness of* $\alpha \,\forall\mathsf{U}\, \beta$ (or $\alpha \,\exists\mathsf{U}\, \beta$).

Truth values of CTL*-formulas are evaluated in each path. The notion "in a model $M = \langle S, R, V\rangle$, a path σ satisfies a CTL*-formula φ", written by "$M, \sigma \models \varphi$" (or "$\sigma \models \varphi$" for short), is inductively defined as follows.

$\sigma \models \top$. $\sigma \not\models \bot$.
$\sigma \models p \iff V(\sigma(0), p) = \mathtt{t}$.
$\sigma \models \neg\alpha \iff \sigma \not\models \alpha$.
$\sigma \models \alpha \wedge \beta \iff \sigma \models \alpha$ and $\sigma \models \beta$.
Logical connectives $\vee$, $\to$, $\leftrightarrow$ are evaluated similarly.
$\sigma \models \forall\alpha \iff (\forall\sigma' =_0 \sigma)(\sigma' \models \alpha)$.
$\sigma \models \exists\alpha \iff (\exists\sigma' =_0 \sigma)(\sigma' \models \alpha)$.
$\sigma \models \mathsf{X}\alpha \iff \sigma|_1 \models \alpha$.
$\sigma \models \mathsf{G}\alpha \iff (\forall n \geq 0)(\sigma|_n \models \alpha)$.
$\sigma \models \mathsf{F}\alpha \iff (\exists n \geq 0)(\sigma|_n \models \alpha)$.
$\sigma \models \alpha \,\mathsf{U}\, \beta \iff (\exists n \geq 0)\big(\sigma|_n \models \beta \;\&\; (\forall m < n)(\sigma|_m \models \alpha)\big)$.

We say that an ECTL-formula (or CTL*-formula) φ is *valid* if and only if $M, x(\text{or } \sigma) \models \varphi$ for any model M and any state x (or any path σ). Moreover we say that two formulas φ and ψ are *equivalent*, written by "$\varphi \equiv \psi$", if and only if the formula $\varphi \leftrightarrow \psi$ is valid.

As is mentioned in the previous section, each unary modality of state formulas of CTL* is expressible in ECTL:

Theorem 1 *For any sequence* $\vec{\mathbf{s}}$ *of the unary modal operators* $\forall$, $\exists$, X, G, *and* F *of CTL* where the first element of* $\vec{s}$ *is* $\forall$ *or* $\exists$, *there is a sequence* $\vec{\mathbf{s}'}$ *of the unary modal operators* $\forall\mathsf{X}$, $\exists\mathsf{X}$, $\forall\mathsf{G}$, $\exists\mathsf{G}$, $\forall\mathsf{F}$, $\exists\mathsf{F}$, $\forall\mathsf{FG}$, *and* $\exists\mathsf{GF}$ *of ECTL such that* $\vec{\mathbf{s}}p \equiv \vec{\mathbf{s}'}p$.

Proof We have the following equations in CTL*.

$\forall\forall\varphi \equiv \forall\varphi$. $\exists\exists\varphi \equiv \exists\varphi$. $\forall\exists\varphi \equiv \exists\varphi$. $\exists\forall\varphi \equiv \forall\varphi$.
$\mathsf{GG}\varphi \equiv \mathsf{G}\varphi$. $\mathsf{FF}\varphi \equiv \mathsf{F}\varphi$. $\mathsf{GFG}\varphi \equiv \mathsf{FG}\varphi$. $\mathsf{FGF}\varphi \equiv \mathsf{GF}\varphi$.
$\mathsf{GX}\varphi \equiv \mathsf{XG}\varphi$. $\mathsf{FX}\varphi \equiv \mathsf{XF}\varphi$. $\forall\mathsf{X}\varphi \equiv \forall\mathsf{X}\forall\varphi$. $\exists\mathsf{X}\varphi \equiv \exists\mathsf{X}\exists\varphi$.
$\forall\mathsf{GF}\varphi \equiv \forall\mathsf{G}\forall\mathsf{F}\varphi$. $\exists\mathsf{FG}\varphi \equiv \exists\mathsf{F}\exists\mathsf{G}\varphi$.
$\forall p \equiv p$. $\exists p \equiv p$.

Using these, we can construct $\vec{\mathbf{s}'}$ from $\vec{\mathbf{s}}$. For example, suppose $\vec{\mathbf{s}} = \forall\forall\exists\mathsf{FXFGGXFGXG}\exists\forall$. We have (1) $\forall\forall\exists\varphi \equiv \exists\varphi$, (2) $\mathsf{FXFGGXFGXG}\varphi \equiv \mathsf{XXXFFGGFGG}\varphi \equiv \mathsf{XXXFG}\varphi$, and (3) $\exists\forall p \equiv p$. Therefore $\vec{\mathbf{s}}p = \forall\forall\exists\mathsf{FXFGGXFGXG}\exists\forall p \equiv \exists\mathsf{XXXFG}p \equiv \exists\mathsf{X}\exists\mathsf{X}\exists\mathsf{X}\exists\mathsf{FG}p \equiv \exists\mathsf{X}\exists\mathsf{X}\exists\mathsf{X}\exists\mathsf{F}\exists\mathsf{G}p$. **QED**

3 Axiomatization

The rest of this paper is devoted to the completeness of Hilbert-style axiomatization for ECTL; hence, from now on, "formula" will mean "ECTL-formula". To simplify the argument, we decrease the number of logical and modal symbols. We adopt $\top$, $\neg$, $\wedge$, $\forall\mathsf{X}$, $\forall\mathsf{U}$, $\exists\mathsf{U}$, and $\exists\mathsf{GF}$ as primitive symbols, and the others are considered to be the abbreviations:

$\bot = \neg\top$. $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$. $\rightarrow$ and $\leftrightarrow$ are defined as usual.
$\exists\mathsf{X}\varphi = \neg\forall\mathsf{X}\neg\varphi$.
$\forall\mathsf{F}\varphi = \top\,\forall\mathsf{U}\,\varphi$. $\exists\mathsf{G}\varphi = \neg\forall\mathsf{F}\neg\varphi = \neg(\top\,\forall\mathsf{U}\,\neg\varphi)$.
$\exists\mathsf{F}\varphi = \top\,\exists\mathsf{U}\,\varphi$. $\forall\mathsf{G}\varphi = \neg\exists\mathsf{F}\neg\varphi = \neg(\top\,\exists\mathsf{U}\,\neg\varphi)$.
$\forall\mathsf{FG}\varphi = \neg\exists\mathsf{GF}\neg\varphi$.

"Q" will be used as a variable on $\{\forall, \exists\}$. For example, "$\alpha\, Q\mathsf{U}\,\beta \leftrightarrow (\beta\vee(\alpha\wedge Q\mathsf{X}(\alpha\, Q\mathsf{U}\,\beta)))$" denotes two formulas "$\alpha\,\forall\mathsf{U}\,\beta \leftrightarrow (\beta \vee (\alpha \wedge \forall\mathsf{X}(\alpha\,\forall\mathsf{U}\,\beta)))$" and "$\alpha\,\exists\mathsf{U}\,\beta \leftrightarrow (\beta \vee (\alpha \wedge \exists\mathsf{X}(\alpha\,\exists\mathsf{U}\,\beta)))$".

We fix a Hilbert-style axiomatization (axiom schemata and inference rules) of CTL; for example, the following are due to Goldblatt [5]:

(Tautology)	Instances of classical tautologies.
($\mathrm{K}_{\forall\mathsf{X}}$)	$\forall\mathsf{X}(\alpha \rightarrow \beta) \rightarrow \forall\mathsf{X}\alpha \rightarrow \forall\mathsf{X}\beta$.
(D)	$\exists\mathsf{X}\top$.
($\forall\mathsf{U}$)	$\alpha\,\forall\mathsf{U}\,\beta \leftrightarrow (\beta \vee (\alpha \wedge \forall\mathsf{X}(\alpha\,\forall\mathsf{U}\,\beta)))$.
($\exists\mathsf{U}$)	$\alpha\,\exists\mathsf{U}\,\beta \leftrightarrow (\beta \vee (\alpha \wedge \exists\mathsf{X}(\alpha\,\exists\mathsf{U}\,\beta)))$.

$$\frac{\alpha \rightarrow \beta \quad \alpha}{\beta}\ \text{(modus ponens)} \qquad \frac{\alpha}{\forall\mathsf{X}\alpha}\ (\forall\mathsf{X}\text{-necessitaion})$$

$$\frac{\beta \vee (\alpha \wedge \forall\mathsf{X}\gamma) \rightarrow \gamma}{\alpha\,\forall\mathsf{U}\,\beta \rightarrow \gamma}\ (\forall\mathsf{U}\text{-induction}) \quad \frac{\beta \vee (\alpha \wedge \exists\mathsf{X}\gamma) \rightarrow \gamma}{\alpha\,\exists\mathsf{U}\,\beta \rightarrow \gamma}\ (\exists\mathsf{U}\text{-induction})$$

We call this system $\mathrm{H}_{\mathrm{CTL}}$. Then our main system $\mathrm{H}_{\mathrm{ECTL}}$ for ECTL is defined by adding the following axiom schemata to $\mathrm{H}_{\mathrm{CTL}}$.

($\mathrm{K}_{\exists\mathsf{GF}}$)	$\forall\mathsf{G}(\alpha \rightarrow \beta) \rightarrow \exists\mathsf{GF}\alpha \rightarrow \exists\mathsf{GF}\beta$.
($\exists\mathsf{GF}$)	$\exists\mathsf{GF}\alpha \leftrightarrow \exists\mathsf{X}\exists\mathsf{F}(\alpha \wedge \exists\mathsf{GF}\alpha)$.
($\exists\mathsf{GF}$-induction)	$\forall\mathsf{G}(\alpha \rightarrow \exists\mathsf{X}\exists\mathsf{F}\alpha) \rightarrow \alpha \rightarrow \exists\mathsf{GF}\alpha$.

Note that the forth axiom $\forall\mathsf{FG}\alpha \leftrightarrow \neg\exists\mathsf{GF}\neg\alpha$ in Section 1 is a tautology because of the abbreviation of $\forall\mathsf{FG}$.

By "$\vdash \varphi$", we mean "φ is provable in $\mathrm{H}_{\mathrm{ECTL}}$". The purpose of this paper is to show the soundness and completeness of $\mathrm{H}_{\mathrm{ECTL}}$ with respect to arbitrary and finite models:

Theorem 2 (Main Theorem) *The following three conditions are equivalent for any formula φ_0. (1) $\vdash \varphi_0$. (2) φ_0 is valid. (3) φ_0 is valid with respect to finite models, i.e., $M, x \models \varphi_0$ for any finite model M and any state x.*

Proof Soundness ($1 \Rightarrow 2 \Rightarrow 3$) is easily shown by verifying that each axiom is valid and that each rule preserves validity of formulas. Completeness ($3 \Rightarrow 1$) is hard as usual; the contraposition ($\neg 1 \Rightarrow \neg 3$) will be proved by Theorem 41 at the end of this paper. **QED**

In the rest of this section, we show some lemmas which give a list of provable formulas and derivable inferences of $\mathrm{H_{ECTL}}$. In the following, *finite* sets of formulas are denoted by $\Gamma, \Delta, \ldots$. If $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, then $\bigwedge \Gamma$ and $\bigvee \Gamma$ denote the formulas $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n$ (or $\top$ if $n = 0$) and $\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_n$ (or $\bot$ if $n = 0$) respectively; moreover if $\bullet$ is one of the unary operators, then $\bullet\Gamma$ denotes the set $\{\bullet\gamma_1, \bullet\gamma_2, \ldots, \bullet\gamma_n\}$. By "$\Gamma \vdash \varphi$", we mean $\vdash \bigwedge \Gamma \to \varphi$. As usual, for example, "$\Gamma, \alpha, \beta, \Delta \vdash \gamma$" means "$\Gamma \cup \{\alpha, \beta\} \cup \Delta \vdash \gamma$".

We say that an inference

$$\frac{\Gamma_1 \vdash \varphi_1 \quad \Gamma_2 \vdash \varphi_2 \quad \cdots \quad \Gamma_n \vdash \varphi_n}{\Delta \vdash \psi}$$

is *derivable* if and only if there is a derivation from n formulas $\bigwedge \Gamma_1 \to \varphi_1, \ldots, \bigwedge \Gamma_n \to \varphi_n$ to the formula $\bigwedge \Delta \to \psi$ in $\mathrm{H_{ECTL}}$. The inference rules of classical logic and of normal modal logic ($\forall\mathsf{X}$ is the modal operator) are available; for example:

$$\frac{\Gamma \vdash \varphi \vee \varphi' \quad \varphi', \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \vee \psi} \quad \frac{\Gamma \vdash \varphi}{\forall\mathsf{X}\Gamma \vdash \forall\mathsf{X}\varphi} \quad \frac{}{\vdash \forall\mathsf{X}(\alpha \wedge \beta) \leftrightarrow \forall\mathsf{X}\alpha \wedge \forall\mathsf{X}\beta}$$

We will tacitly use such inferences.

Lemma 3 (Property of $\forall\mathsf{G}$) *The inference rules*

$$\frac{\vdash \gamma \to \varphi \wedge \forall\mathsf{X}\gamma}{\vdash \gamma \to \forall\mathsf{G}\varphi}\ (\forall\mathsf{G}\text{-induction}) \qquad \frac{\vdash \varphi}{\vdash \forall\mathsf{G}\varphi}\ (\forall\mathsf{G}\text{-necessitaion})$$

$$\frac{\forall\mathsf{G}\Gamma \vdash \varphi}{\forall\mathsf{G}\Gamma \vdash \forall\mathsf{G}\varphi}\ (\forall\mathsf{G}\text{-R}) \qquad \frac{\varphi, \Gamma \vdash \psi}{\forall\mathsf{G}\varphi, \Gamma \vdash \psi}\ (\forall\mathsf{G}\text{-L})$$

are derivable, and the following schemata ($\forall\mathsf{G}$), ($\mathrm{K}_{\forall\mathsf{G}}$) and ($4_{\forall\mathsf{G}}$) are provable.

($\forall\mathsf{G}$) $\quad \forall\mathsf{G}\varphi \leftrightarrow \varphi \wedge \forall\mathsf{X}\forall\mathsf{G}\varphi$.
($\mathrm{K}_{\forall\mathsf{G}}$) $\quad \forall\mathsf{G}(\alpha \to \beta) \to \forall\mathsf{G}\alpha \to \forall\mathsf{G}\beta$.
($4_{\forall\mathsf{G}}$) $\quad \forall\mathsf{G}\varphi \to \forall\mathsf{G}\forall\mathsf{G}\varphi$.

Proof $\forall\mathsf{G}$-induction rule is equivalent to an instance of $\exists\mathsf{U}$-induction rule:

$$\frac{\neg\varphi \vee (\top \wedge \exists\mathsf{X}\neg\gamma) \to \neg\gamma}{\top\ \exists\mathsf{U}\ \neg\varphi \to \neg\gamma}\ (\exists\mathsf{U}\text{-ind.})$$

$\forall\mathsf{G}$-necessitation rule is obtained from $\forall\mathsf{G}$-induction rule by replacing γ by $\top$ using the fact $\vdash \forall\mathsf{X}\top$. The scheme ($\forall\mathsf{G}$) is provable from the axiom ($\exists\mathsf{U}$) where $\alpha = \top$, $\beta = \neg\varphi$. The scheme ($\mathrm{K}_{\forall\mathsf{G}}$) is provable as follows.

$$\dfrac{\dfrac{\dfrac{(\because \forall\mathsf{G})}{\forall\mathsf{G}(\alpha\to\beta) \to (\alpha\to\beta)} \quad \dfrac{(\because \forall\mathsf{G})}{\forall\mathsf{G}\alpha \to \alpha}}{\forall\mathsf{G}(\alpha\to\beta) \wedge \forall\mathsf{G}\alpha \ \to\ \beta} \quad \dfrac{\dfrac{(\because \forall\mathsf{G})}{\forall\mathsf{G}(\alpha\to\beta) \to \forall\mathsf{X}\forall\mathsf{G}(\alpha\to\beta)} \quad \dfrac{(\because \forall\mathsf{G})}{\forall\mathsf{G}\alpha \to \forall\mathsf{X}\forall\mathsf{G}\alpha}}{\forall\mathsf{G}(\alpha\to\beta) \wedge \forall\mathsf{G}\alpha \ \to\ \forall\mathsf{X}(\forall\mathsf{G}(\alpha\to\beta) \wedge \forall\mathsf{G}\alpha)}}{\dfrac{\forall\mathsf{G}(\alpha\to\beta) \wedge \forall\mathsf{G}\alpha \ \to\ \beta \wedge \forall\mathsf{X}(\forall\mathsf{G}(\alpha\to\beta) \wedge \forall\mathsf{G}\alpha)}{\forall\mathsf{G}(\alpha\to\beta) \wedge \forall\mathsf{G}\alpha \ \to\ \forall\mathsf{G}\beta}\ (\forall\mathsf{G}\text{-ind.})}$$

The schema ($4_{\forall\mathsf{G}}$) is provable using ($\forall\mathsf{G}$) and $\forall\mathsf{G}$-induction rule. Derivability of the rules ($\forall\mathsf{G}$-R/L) is easily shown (like the rules of the modal logic S4). **QED**

Lemma 4 *(1)* $\forall \mathsf{G}(\alpha \to \alpha'),\ \forall \mathsf{X}\alpha \ \vdash\ \forall \mathsf{X}\alpha'$.

(2) $\forall \mathsf{G}(\alpha \to \alpha'),\ \alpha\, Q\mathsf{U}\, \beta \ \vdash\ \alpha'\, Q\mathsf{U}\, \beta$.

(3) $\forall \mathsf{G}(\beta \to \beta'),\ \alpha\, Q\mathsf{U}\, \beta \ \vdash\ \alpha\, Q\mathsf{U}\, \beta'$.

(4) $\forall \mathsf{G}(\alpha \to \alpha'),\ \exists \mathsf{GF}\alpha \ \vdash\ \exists \mathsf{GF}\alpha'$.

Proof We show only an outline of (2).

$$\dfrac{\dfrac{(\because Q\mathsf{U})}{\beta \vee (\alpha' \wedge Q\mathsf{X}(\alpha'\, Q\mathsf{U}\, \beta)) \ \to\ \alpha'\, Q\mathsf{U}\, \beta} \\ \vdots \\ \beta \vee (\alpha \wedge Q\mathsf{X}(\forall \mathsf{G}(\alpha \to \alpha') \to \alpha'\, Q\mathsf{U}\, \beta)) \ \to\ \forall \mathsf{G}(\alpha \to \alpha') \to \alpha'\, Q\mathsf{U}\, \beta}{\alpha\, Q\mathsf{U}\, \beta \ \to\ \forall \mathsf{G}(\alpha \to \alpha') \to \alpha'\, Q\mathsf{U}\, \beta}\ (Q\mathsf{U}\text{-ind.})$$

QED

Note that Lemma 4(4) is the axiom ($\mathrm{K}_{\exists\mathsf{GF}}$), which will be used in not only the next lemma but also Lemma 29 in Section 6

Lemma 5 *The following inference rule is derivable.*

$$\dfrac{\alpha \leftrightarrow \alpha'}{\varphi[\alpha] \leftrightarrow \varphi[\alpha']}$$

where $\varphi[\alpha']$ is the formula that is obtained from the formula $\varphi[\alpha]$ by replacing one occurrence of subformula α by α'.

Proof By induction on φ, using Lemmas 3 and 4. **QED**

Lemma 5 guarantees that provability of a formula is preserved when we replace a subformula by another equivalent formula. We will tacitly use this property.

Lemma 6 (Property of $\forall\mathsf{U}$ and $\exists\mathsf{U}$) *(1)* $\beta \vdash \alpha\, Q\mathsf{U}\, \beta$.

(2) $\vdash \top\, Q\mathsf{U}\, \top$.

(3) $\alpha\, Q\mathsf{U}\, \beta \vdash \alpha \vee \beta$.

(4) $\alpha\, Q\mathsf{U}\, \beta \vdash \beta \vee Q\mathsf{X}(\alpha\, Q\mathsf{U}\, \beta)$.

(5) $\alpha, Q\mathsf{X}(\alpha\, Q\mathsf{U}\, \beta) \vdash \alpha\, Q\mathsf{U}\, \beta$.

Proof Use the axioms ($\forall\mathsf{U}$) and ($\exists\mathsf{U}$). **QED**

Lemma 7 *The following inference rule, which is a variant of $Q\mathsf{U}$-induction, is derivable.*

$$\dfrac{\forall \mathsf{G}\Delta,\ \beta \vee Q\mathsf{X}((\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta) \ \vdash\ \gamma}{\forall \mathsf{G}\Delta,\ \alpha\, Q\mathsf{U}\, \beta \ \vdash\ \gamma}$$

Proof First we consider the case that Δ is empty. We have the following derivation.

$$\beta \vee Q\mathsf{X}((\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta) \ \vdash\ \gamma \qquad \text{(assumption)} \qquad (3.1)$$
$$\beta \vee (\alpha \wedge Q\mathsf{X}((\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta)) \ \vdash\ \beta \vee Q\mathsf{X}((\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta) \qquad \text{(tautology)} \qquad (3.2)$$
$$\beta \vee ((\alpha \wedge \gamma) \wedge Q\mathsf{X}((\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta) \ \vdash\ (\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta \qquad \text{(axiom } (Q\mathsf{U})) \qquad (3.3)$$
$$\beta \vee (\alpha \wedge Q\mathsf{X}((\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta)) \ \vdash\ (\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta \qquad (\because 3.1, 3.2, 3.3) \qquad (3.4)$$
$$\alpha\, Q\mathsf{U}\, \beta \ \vdash\ (\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta \qquad (\because 3.4 \text{ and } Q\mathsf{U}\text{-ind.}) \qquad (3.5)$$
$$(\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta \ \vdash\ (\alpha \wedge \gamma) \vee \beta \qquad \text{(Lemma 6(3))} \qquad (3.6)$$
$$\alpha\, Q\mathsf{U}\, \beta \ \vdash\ \gamma \qquad (\because 3.1, 3.5, 3.6) \qquad (3.7)$$

For a general case, put $\delta = \bigwedge \forall\mathsf{G}\Delta$.

$$\delta,\ \beta \vee Q\mathsf{X}((\alpha \wedge \gamma)\, Q\mathsf{U}\, \beta) \ \vdash\ \gamma \qquad \text{(assumption)} \qquad (3.8)$$
$$\delta \ \vdash\ \forall\mathsf{G}((\delta \to \gamma) \to \gamma) \qquad \text{(from Lemma 3)} \qquad (3.9)$$
$$\forall\mathsf{G}((\delta \to \gamma) \to \gamma),\ \beta \vee Q\mathsf{X}(((\alpha \wedge (\delta \to \gamma))\, Q\mathsf{U} \beta) \ \vdash\ \beta \vee Q\mathsf{X}(((\alpha \wedge \gamma)\, Q\mathsf{U} \beta)$$
$$\text{(from Lemma 4)} \qquad (3.10)$$
$$\delta,\ \beta \vee Q\mathsf{X}((\alpha \wedge (\delta \to \gamma))\, Q\mathsf{U}\, \beta) \ \vdash\ \gamma \qquad (\because 3.8, 3.9, 3.10)$$
$$\beta \vee Q\mathsf{X}((\alpha \wedge (\delta \to \gamma))\, Q\mathsf{U}\, \beta) \ \vdash\ \delta \to \gamma$$

To this last formula, we apply the former derivation (from 3.1 to 3.7, where "γ" = "$\delta \to \gamma$"), and we get the formula $\alpha\, Q\mathsf{U}\, \beta \ \vdash\ \delta \to \gamma$, which is equivalent to the required formula $\delta, \alpha\, Q\mathsf{U}\, \beta \ \vdash\ \gamma$. **QED**

Lemma 8 (Property of $\exists\mathsf{GF}$) *(1)* $\exists\mathsf{GF}\varphi \vdash \exists\mathsf{X}\exists\mathsf{GF}\varphi$.

(2) $\exists\mathsf{X}\exists\mathsf{GF}\varphi \vdash \exists\mathsf{GF}\varphi$.

(3) $\exists\mathsf{GF}\varphi \vdash \exists\mathsf{F}(\varphi \wedge \exists\mathsf{X}\exists\mathsf{GF}\varphi)$.

Proof We show an outline:

(1)
$$\begin{aligned} \exists\mathsf{GF}\varphi \ &\vdash\ \exists\mathsf{X}\exists\mathsf{F}(\varphi \wedge \exists\mathsf{GF}\varphi) && (\because \exists\mathsf{GF} \text{ axiom}) \\ &\vdash\ \exists\mathsf{X}\big((\varphi \wedge \exists\mathsf{GF}\varphi) \vee \exists\mathsf{X}\exists\mathsf{F}(\varphi \wedge \exists\mathsf{GF}\varphi)\big) && (\because \exists\mathsf{F}\psi \vdash \psi \vee \exists\mathsf{X}\exists\mathsf{F}\psi) \\ &\vdash\ \exists\mathsf{X}(\exists\mathsf{GF}\varphi \vee \exists\mathsf{GF}\varphi). && (\because \exists\mathsf{GF} \text{ axiom}) \end{aligned}$$

(2)
$$\begin{aligned} \exists\mathsf{X}\exists\mathsf{GF}\varphi \ &\vdash\ \exists\mathsf{X}\exists\mathsf{X}\exists\mathsf{F}(\varphi \wedge \exists\mathsf{GF}\varphi) && (\because \exists\mathsf{GF} \text{ axiom}) \\ &\vdash\ \exists\mathsf{X}\exists\mathsf{F}(\varphi \wedge \exists\mathsf{GF}\varphi) && (\because \exists\mathsf{X}\exists\mathsf{F}\psi \vdash \exists\mathsf{F}\psi) \\ &\vdash\ \exists\mathsf{GF}\varphi. && (\because \exists\mathsf{GF} \text{ axiom}) \end{aligned}$$

(3)
$$\begin{aligned} \exists\mathsf{GF}\varphi \ &\vdash\ \exists\mathsf{X}\exists\mathsf{F}(\varphi \wedge \exists\mathsf{GF}\varphi) && (\because \exists\mathsf{GF} \text{ axiom}) \\ &\vdash\ \exists\mathsf{F}(\varphi \wedge \exists\mathsf{X}\exists\mathsf{GF}\varphi). && (\because \exists\mathsf{X}\exists\mathsf{F}\psi \vdash \exists\mathsf{F}\psi \text{ and } (1)) \end{aligned}$$

QED

Note that the $\exists\mathsf{GF}$-induction axiom is not used in this section. It will be used in the proof of Lemma 30 in Section 6.

4 Completeness of K

It is well known that the smallest normal modal logic K is axiomatized by Tautology and $K_{\forall \mathsf{X}}$ axioms and modus ponens and $\forall\mathsf{X}$-necessitation rules, where $\forall\mathsf{X}$ is the only modal operator (usually written as $\Box$). K is complete with respect to finite Kripke models:

Proposition 9 (Completeness of K) *If φ is not provable in K, then there exists a finite Kripke model $M = \langle S, R, V\rangle$ (R may not be serial) such that $M, x \not\models \varphi$ for some $x \in S$.*

In this section, we show an outline of the standard proof of this completeness in order to utilize it as a base of our argument.

Definition 10 (valuation, $\bullet_{\mathtt{t}}$, $\bullet_{\mathtt{f}}$, $\bullet^*$) *Let Γ be a finite set of formulas. A* valuation *of Γ is a function from Γ into $\{\mathtt{t}, \mathtt{f}\}$. If v is a valuation of Γ, then $v_{\mathtt{t}}$ and $v_{\mathtt{f}}$ are sets of formulas and v^* is a formula as follows.*

$$v_{\mathtt{t}} = \{\varphi \mid \ \varphi \in \Gamma \text{ and } v(\varphi) = \mathtt{t}\}.$$
$$v_{\mathtt{f}} = \{\varphi \mid \ \varphi \in \Gamma \text{ and } v(\varphi) = \mathtt{f}\}.$$
$$v^* \ = \ \bigwedge v_{\mathtt{t}} \wedge \bigwedge(\neg v_{\mathtt{f}}).$$

Definition 11 ($\bullet_{\mathtt{t}/\forall\mathrm{x}}$, $\bullet \rhd \bullet$) *Let Γ be a finite set of formulas. For any valuation v of Γ, we define*

$$v_{\mathtt{t}/\forall\mathrm{x}} = \{\varphi \mid \ \forall\mathsf{X}\varphi \in \Gamma \text{ and } v(\forall\mathsf{X}\varphi) = \mathtt{t}\}.$$

Then a relation $\rhd$ between valuations of Γ is defined as follows.

$$v \rhd v' \iff v_{\mathtt{t}/\forall\mathrm{x}} \subseteq v'_{\mathtt{t}} \iff \big(v(\forall\mathsf{X}\varphi){=}\mathtt{t} \Rightarrow v'(\varphi){=}\mathtt{t}\big) \text{ for any } \forall\mathsf{X}\varphi \text{ in } \Gamma.$$

Definition 12 (K-consistent) *A valuation v is said to be* K-consistent *if and only if the formula $\neg(v^*)$ is not provable in K.*

Then the required counter-model $M = \langle S, R, V\rangle$ for φ is constructed as follows. S is the set of K-consistent valuations of $\mathrm{Sub}(\varphi)$ where $\mathrm{Sub}(\varphi)$ is the set of subformulas of φ. $R = \rhd$. $V(v, p) = v(p)$. The condition $(\exists x \in S)(M, x \not\models \varphi)$ is shown by the following propositions.

Proposition 13 *For any $\psi \in \mathrm{Sub}(\varphi)$ and any $v \in S$, we have the following. (1) If $v(\psi) = \mathtt{t}$, then $M, v \models \psi$. (2) If $v(\psi) = \mathtt{f}$, then $M, v \not\models \psi$.*

Proposition 14 *If φ is not provable in K, then there is a K-consistent valuation v of* $\mathrm{sub}(\varphi)$ *such that $v(\varphi) = \mathtt{f}$.*

Our completeness proof for ECTL is an elaborate extension of the above argument.

5 C-valuations

In our counter-model for $\mathrm{H}_{\mathrm{ECTL}}$, each state is not a valuation but a "valuation together with additional information" — we call this a *c-valuation* (*c* for "conditional" or "controlled"). The additional information is utilized to control the accessibility between states. In this section, we define c-valuations and we show some basic properties of them.

Definition 15 (c-valuation, designated formula) *Let $\mathbb{S}$ be a finite set of formulas that contains at least one until-formula, where an "until-formula" is a formula of the form $\alpha\, Q\mathsf{U}\, \beta$. A* c-valuation *of $\mathbb{S}$ is a 4-tuple $\langle \mathcal{F}, \mathcal{H}, U, v\rangle$ that satisfies the following three conditions.*

- *Both $\mathcal{F}$ and $\mathcal{H}$ are sets of valuations of $\mathbb{S}$. ($\mathcal{F}$ and $\mathcal{H}$ are finite because so is $\mathbb{S}$.)*
- *U is an until-formula in $\mathbb{S}$. (U is called the* designated formula *of this c-valuation.)*
- *v is a valuation of $\mathbb{S}$.*

Definition 16 (intended formula, consistent) *Let $\mathcal{F} = \{v_1^{\mathcal{F}}, v_2^{\mathcal{F}}, \ldots, v_m^{\mathcal{F}}\}$ and $\mathcal{H} = \{v_1^{\mathcal{H}}, v_2^{\mathcal{H}}, \ldots, v_n^{\mathcal{H}}\}$ be sets of valuations of a set $\mathbb{S}$. The* intended formula *of a c-valuation $\langle \mathcal{F}, \mathcal{H}, \alpha\, Q\mathsf{U}\, \beta, v\rangle$ is*

$$\forall\mathsf{G}\neg(v_1^{\mathcal{F}})^* \wedge \forall\mathsf{G}\neg(v_2^{\mathcal{F}})^* \wedge \cdots \wedge \forall\mathsf{G}\neg(v_m^{\mathcal{F}})^* \wedge \Big(\big(\alpha\wedge\neg(v_1^{\mathcal{H}})^*\wedge\neg(v_2^{\mathcal{H}})^*\wedge\cdots\wedge\neg(v_n^{\mathcal{H}})^*\big)\, Q\mathsf{U}\, \beta\Big) \wedge v^*.$$

(See Def.10 for "$*$".) We say that a c-valuation is consistent *if and only if the negation of its intended formula is not provable in $\mathrm{H}_{\mathrm{ECTL}}$.*

By the definitions, we have:

Proposition 17 *The following conditions are equivalent where $\forall\mathsf{G}\neg\mathcal{F}^* = \{\forall\mathsf{G}\neg(v^*) \mid v \in \mathcal{F}\}$ and $\neg\mathcal{H}^* = \{\neg(v^*) \mid v \in \mathcal{H}\}$.*

- *A c-valuation $\langle \mathcal{F}, \mathcal{H}, \alpha\, Q\mathsf{U}\, \beta, v\rangle$ is consistent.*
- $\forall\mathsf{G}\neg\mathcal{F}^*,\ \big(\alpha\wedge\bigwedge\neg\mathcal{H}^*\big)\, Q\mathsf{U}\, \beta \ \nvdash\ \neg(v^*)$.
- $\forall\mathsf{G}\neg\mathcal{F}^*,\ \big(\alpha\wedge\bigwedge\neg\mathcal{H}^*\big)\, Q\mathsf{U}\, \beta,\ v_{\mathtt{t}} \ \nvdash\ \bigvee v_{\mathtt{f}}$.
- $\forall\mathsf{G}\neg\mathcal{F}^*,\ \big(\alpha\wedge\bigwedge\neg\mathcal{H}^*\big)\, Q\mathsf{U}\, \beta,\ v^* \ \nvdash\ \bigvee v_{\mathtt{f}}$

For example, suppose that $\mathbb{S} = \{p\,\exists\mathsf{U}\, q, p, q\}$ and valuations v_1, v_2, v_3 are as follows.

$v_1(p\,\exists\mathsf{U}\, q) = v_1(p) = v_1(q) = \mathtt{t}$.
$v_2(p\,\exists\mathsf{U}\, q) = v_2(p) = \mathtt{t}, \quad v_2(q) = \mathtt{f}$.
$v_3(p\,\exists\mathsf{U}\, q) = \mathtt{t}, \quad v_3(p) = v_3(q) = \mathtt{f}$.

Then a c-valuation $\langle\{v_1, v_2\},\ \{v_2, v_3\},\ p\,\exists\mathsf{U}\, q,\ v_3\rangle$ is consistent if and only if

$$\forall\mathsf{G}\neg((p\exists\mathsf{U}q)\wedge p\wedge q),\ \forall\mathsf{G}\neg((p\exists\mathsf{U}q)\wedge p\wedge\neg q),$$
$$\big(p \wedge \neg((p\exists\mathsf{U}q)\wedge p\wedge\neg q) \wedge \neg((p\exists\mathsf{U}q)\wedge\neg p\wedge\neg q)\big)\, \exists\mathsf{U}\, q,\ p\,\exists\mathsf{U}\, q \ \nvdash\ p \vee q.$$

Definition 18 ($\mathbb{C}(\cdot)$) *For any finite set* $\mathbb{S}$ *of formulas,* $\mathbb{C}(\mathbb{S})$ *denotes the set of consistent c-valuations of* $\mathbb{S}$*.*

$\mathbb{C}(\mathbb{S})$ will be the very set of states in our counter-model. From now on, when we write "$\mathbb{C}(\mathbb{S})$", we assume that $\mathbb{S}$ is a finite set of formulas that contains at least one until-formula.

Lemma 19 $\mathbb{C}(\mathbb{S})$ *is a finite set.*

Proof $|\mathbb{C}(\mathbb{S})| \leq 2^m 2^m nm$ where m $(= 2^{|\mathbb{S}|})$ is the number of valuations of $\mathbb{S}$, and n is the number of until-formulas in $\mathbb{S}$. **QED**

Lemma 20 *If* $\langle \mathcal{F}, \mathcal{H}, \alpha\ Q\mathsf{U}\, \beta, v\rangle \in \mathbb{C}(\mathbb{S})$*, then we have the following.*

(1) $v \notin \mathcal{F}$*.*

(2) If $v(\beta) = \mathtt{f}$*, then* $v \notin \mathcal{H}$*.*

(3) $v(\alpha\ Q\mathsf{U}\, \beta) = \mathtt{t}$*.*

Proof (1) If $v \in \mathcal{F}$, then $\forall\mathsf{G}\neg\mathcal{F}^* \vdash \neg(v^*)$ by Lemma 3, and the c-valuation is inconsistent by Proposition 17. (2) If $v \in \mathcal{H}$ and $v(\beta) = \mathtt{f}$, then $\left(\alpha \wedge \bigwedge \neg\mathcal{H}^*\right) Q\mathsf{U}\, \beta \vdash \neg(v^*) \vee \bigvee v_{\mathtt{f}}$ by Lemma 6(3) ($\because \alpha \wedge \bigwedge \neg\mathcal{H}^* \vdash \neg(v^*)$ and $\beta \vdash \bigvee v_{\mathtt{f}}$), and the c-valuation is inconsistent by Proposition 17. (3) Similarly to (1) and (2), using the fact $(\alpha \wedge \bigwedge \neg\mathcal{H}^*)\ Q\mathsf{U}\, \beta \vdash \alpha\ Q\mathsf{U}\, \beta$ ($\because$ Lemma 4(2)). **QED**

Lemma 21 *Let* $\mathbb{T}_0$ *and* $\mathbb{F}_0$ *be disjoint subsets of a finite set* $\mathbb{S}$ *of formulas, and* Γ *be a finite set of formulas. If* $\Gamma, \mathbb{T}_0 \nvdash \bigvee \mathbb{F}_0$*, then there is a valuation* v *of* $\mathbb{S}$ *such that* $\mathbb{T}_0 \subseteq v_{\mathtt{t}}$*,* $\mathbb{F}_0 \subseteq v_{\mathtt{f}}$*, and* $\Gamma, v_{\mathtt{t}} \nvdash \bigvee v_{\mathtt{f}}$*.*

Proof By the standard argument as follows. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be an enumeration of the set $\mathbb{S} - (\mathbb{T}_0 \cup \mathbb{F}_0)$. We show that there are two disjoint sets $\mathbb{T}_n$ and $\mathbb{F}_n$ such that $\mathbb{T}_n \cup \mathbb{F}_n = \mathbb{S}$, $\mathbb{T}_0 \subseteq \mathbb{T}_n$, $\mathbb{F}_0 \subseteq \mathbb{F}_n$, and $\Gamma, \mathbb{T}_n \nvdash \bigvee \mathbb{F}_n$. We define $\mathbb{T}_i$ and $\mathbb{F}_i$, for $i = 1, \ldots, n$, as follows. Suppose $\mathbb{T}_{i-1}$ and $\mathbb{F}_{i-1}$ are already defined and $\Gamma, \mathbb{T}_{i-1} \nvdash \bigvee \mathbb{F}_{i-1}$, then at least one of the following holds: (1) $\Gamma, \mathbb{T}_{i-1}, \sigma_i \nvdash \bigvee \mathbb{F}_{i-1}$. (2) $\Gamma, \mathbb{T}_{i-1} \nvdash \bigvee \mathbb{F}_{i-1} \vee \sigma_i$. Then we define $\langle \mathbb{T}_i, \mathbb{F}_i\rangle = \langle \mathbb{T}_{i-1} \cup \{\sigma_i\}, \mathbb{F}_{i-1}\rangle$ if the condition (1) holds, otherwise $\langle \mathbb{T}_i, \mathbb{F}_i\rangle = \langle \mathbb{T}_{i-1}, \mathbb{F}_{i-1} \cup \{\sigma_i\}\rangle$. **QED**

6 Accessibility relation

From now on, we fix a formula φ_0 such that $\nvdash \varphi_0$. The goal of this paper is to show the existence of a finite counter-model for φ_0. For this purpose, the accessibility relation is defined in this section.

In the case of K, the set $\mathrm{Sub}(\varphi_0)$ is sufficient to construct a counter-model for φ_0 (see Section 4); however we need a larger set, called $\mathbb{S}_0$, for $\mathrm{H}_{\mathrm{ECTL}}$.

Definition 22 ($\mathbb{S}_0$) *A set $\mathbb{S}_0'$ of formulas is defined by*

$$\mathbb{S}_0' = \mathrm{Sub}(\varphi_0, \top \forall\mathsf{U}\, \top, \top \exists\mathsf{U}\, \top, \forall\mathsf{X}\neg\top)$$

where $\mathrm{Sub}(\Gamma)$ *is the set of subformulas of the formulas in* Γ*. Then a set* $\mathbb{S}_0$ *is defined by*

$$\mathbb{S}_0 = \mathrm{Sub}\Big(\{\forall\mathsf{X}(\alpha \forall\mathsf{U}\, \beta) \ \mid\ \alpha \forall\mathsf{U}\, \beta \in \mathbb{S}_0'\} \quad \cup \quad \{\forall\mathsf{X}\neg(\alpha \exists\mathsf{U}\, \beta) \ \mid\ \alpha \exists\mathsf{U}\, \beta \in \mathbb{S}_0'\} \quad \cup \quad \{\forall\mathsf{X}\neg(\top \exists\mathsf{U}(\alpha \wedge \neg\forall\mathsf{X}\neg\exists\mathsf{GF}\alpha)) \ \mid\ \exists\mathsf{GF}\alpha \in \mathbb{S}_0'\}\Big).$$

$\mathbb{S}_0$ is defined so as to satisfy the following property:

Lemma 23 *(1) $\mathbb{S}_0$ is a finite set including $\varphi_0, \top \forall\mathsf{U}\, \top, \top \exists\mathsf{U}\, \top,$ and $\forall\mathsf{X}\neg\top$.*

(2) $\mathbb{S}_0$ is closed under subformulas.

(3) If $\alpha \forall\mathsf{U}\, \beta \in \mathbb{S}_0$, then $\forall\mathsf{X}(\alpha \forall\mathsf{U}\, \beta) \in \mathbb{S}_0$.

(4) If $\alpha \exists\mathsf{U}\, \beta \in \mathbb{S}_0$, then $\forall\mathsf{X}\neg(\alpha \exists\mathsf{U}\, \beta) \in \mathbb{S}_0$.

(5) If $\exists\mathsf{GF}\alpha \in \mathbb{S}_0$, then $\top \exists\mathsf{U}(\alpha \wedge \neg\forall\mathsf{X}\neg\exists\mathsf{GF}\alpha) \in \mathbb{S}_0$.

Proof Easy. **QED**

The following definitions (especially Def. 26) are the core of our completeness proof.

Definition 24 (next, Next) *Let $\mathbb{U} = \{U_0, U_1, \ldots, U_{N-1}\}$ be the set of until-formulas in $\mathbb{S}_0$ where $U_i \neq U_j$ if $i \neq j$. We define a function* $\mathrm{next}(\cdot)$ *on* $\mathbb{U}$ *by*

$$\mathrm{next}(U_i) = U_{((i+1) \bmod N)}.$$

Then, for each valuation v of $\mathbb{S}_0$, we define a function $\mathrm{Next}_v(\cdot)$ *on* $\mathbb{U}$ *by*

$$\mathrm{Next}_v(U) = \mathrm{next}^m(U), \ \textit{where}\ m = \min\{m > 0 \mid v(\mathrm{next}^m(U)) = \mathtt{t}\}.$$

For example, if $\mathbb{U} = \{U_0, U_1, \ldots, U_4\}$, $v(U_0) = v(U_2) = \mathtt{t}$, and $v(U_1) = v(U_3) = v(U_4) = \mathtt{f}$, then $\mathrm{Next}_v(U_0) = U_2$ and $\mathrm{Next}_v(U_3) = U_0$. The formula $\mathrm{Next}_v(U)$ is defined only if there exists a formula U_i such that $v(U_i) = \mathtt{t}$.

Definition 25 ($\exists\mathsf{GF}$-condition, witness condition) *Two conditions on a c-valuation $\langle \mathcal{F}, \mathcal{H}, \alpha\, Q\mathsf{U}\, \beta, v\rangle$ of $\mathbb{S}_0$ are defined as follows.*

($\exists\mathsf{GF}$-condition) *If $v(\exists\mathsf{GF}\varphi) = \mathtt{f}$, then $v(\varphi) = \mathtt{f}$, for any $\exists\mathsf{GF}\varphi$ in $\mathbb{S}_0$.*

(witness condition) $v(\beta) = \mathtt{t}$.

Definition 26 ($\rightsquigarrow$) *We define a binary relation $\rightsquigarrow$ on $\mathbb{C}(\mathbb{S}_0)$ as follows. $\langle \mathcal{F}, \mathcal{H}, U, v\rangle \rightsquigarrow \langle \mathcal{F}', \mathcal{H}', U', v'\rangle$ if and only if all the conditions below are satisfied.*

(0) $\langle \mathcal{F}, \mathcal{H}, U, v\rangle, \langle \mathcal{F}', \mathcal{H}', U', v'\rangle \in \mathbb{C}(\mathbb{S}_0)$. (See Def. 18 for $\mathbb{C}(\mathbb{S}_0)$.)

(1) $v \rhd v'$. (See Def. 11 for $\rhd$.)

Table 1: Admissible next states of $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$.

When $U = (\cdots \forall \mathsf{U} \cdots)$		witness cond.	
		Yes	No
$\exists\mathsf{GF}$-cond.	Yes	$\heartsuit$	$\diamondsuit$
	No	$\spadesuit$	$\clubsuit$

When $U = (\cdots \exists \mathsf{U} \cdots)$		witness cond.	
		Yes	No
$\exists\mathsf{GF}$-cond.	Yes	$\heartsuit$	$\heartsuit, \diamondsuit$
	No	$\spadesuit$	$\spadesuit, \clubsuit$

(2) $\langle \mathcal{F}', \mathcal{H}', U', v' \rangle$ is one of the following forms

$$\langle \mathcal{F},\ \emptyset,\ \mathrm{Next}_{v'}(U),\ v' \rangle \tag{$\heartsuit$}$$
$$\langle \mathcal{F},\ \mathcal{H} \cup \{v\},\ U,\ v' \rangle \tag{$\diamondsuit$}$$
$$\langle \mathcal{F} \cup \{v\},\ \emptyset,\ \mathrm{Next}_{v'}(U),\ v' \rangle \tag{$\spadesuit$}$$
$$\langle \mathcal{F} \cup \{v\},\ \emptyset,\ U,\ v' \rangle \tag{$\clubsuit$}$$

where Table 1 specifies the suits ($\heartsuit$, $\diamondsuit$, $\spadesuit$, or $\clubsuit$) depending on the conditions of $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$.

For example, if U is an $\exists\mathsf{U}$-formula and $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ satisfies neither the $\exists\mathsf{GF}$-condition nor witness condition, then $\langle \mathcal{F}', \mathcal{H}', U', v' \rangle$ must be $\spadesuit$ or $\clubsuit$.

Our counter-model is $\mathcal{M}_0 = \langle \mathbb{C}(\mathbb{S}_0), \rightsquigarrow, V_0 \rangle$ where V_0 will be defined in the next section. $\mathcal{M}_0$ is expected to have a property that each state $\langle \mathcal{F}, \mathcal{H}, \alpha\ Q\mathsf{U}\ \beta, v \rangle$ satisfies its intended formula $\bigwedge\big(\forall\mathsf{G}\neg\mathcal{F}^*, (\alpha \wedge \bigwedge \neg\mathcal{H}^*)\ Q\mathsf{U}\ \beta, v^*\big)$. According to this expectation, the above Definition 26 can be intuitively explained as follows.

> Let $x = \langle \mathcal{F}, \mathcal{H}, U, v \rangle$ be a state of $\mathcal{M}_0$. For each until-formula $\alpha\ Q\mathsf{U}\ \beta$ in $\mathbb{S}_0$, if $v(\alpha\ Q\mathsf{U}\ \beta) = \mathtt{t}$ then we need a witness (or witnesses) (i.e., a state y such that $x \rightsquigarrow \cdots \rightsquigarrow y$ and y satisfies β). The designated formula represents top-priority until-formula of which we seek a witness (or witnesses).
>
> If x satisfies the witness condition, this means x itself is a witness of the designated formula, and then we shift the top-priority in the next states $\heartsuit$ and $\spadesuit$.
>
> If x fails in the witness condition and the designated formula is $\alpha\ Q\mathsf{U}\ \beta$, then x is a v^*-state and v^* implies both $\neg\beta$ and $\alpha\ Q\mathsf{U}\ \beta$. In this case, as is explained in Section 1, there is a last v^*-state x' before β-states. Then the state $\diamondsuit$ is intended to be a next state of not x but x'.
>
> If x fails in the $\exists\mathsf{GF}$-condition, then x is a v^*-state and v^* implies both φ and $\forall\mathsf{FG}\neg\varphi$ for some φ. In this case, as is explained in Section 1, there is a last v^*-state x'. Then the states $\spadesuit$ and $\clubsuit$ are intended to be next states of not x but x'.

In the rest of this section, we show some important properties concerning the relation $\rightsquigarrow$. From now on, the expression $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i \rangle_{(i=0,1,2\ldots)} \rangle$ denotes an infinite $\rightsquigarrow$-sequence

$$\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0 \rangle \rightsquigarrow \langle \mathcal{F}_1, \mathcal{H}_1, U_1, v_1 \rangle \rightsquigarrow \langle \mathcal{F}_2, \mathcal{H}_2, U_2, v_2 \rangle \rightsquigarrow \cdots$$

in $\mathbb{C}(\mathbb{S}_0)$.

Lemma 27 *For any until-formula U in $\mathbb{S}_0$ and any c-valuation $\langle \mathcal{F}', \mathcal{H}', U', v' \rangle$ in $\mathbb{C}(\mathbb{S}_0)$, the until-formula $\mathrm{Next}_{v'}(U)$ is defined and it is different from U.*

Proof Lemmas 6(2) and 23(1), and consistency of $\langle \mathcal{F}', \mathcal{H}', U', v' \rangle$ guarantee that $v'(\top \forall \mathsf{U} \top) = v'(\top \exists \mathsf{U} \top) = \mathtt{t}$. This fact and the definition of $\mathrm{Next}_{v'}(U)$ imply this Lemma 27. **QED**

Lemma 28 *(1) There is no infinite $\rightsquigarrow$-sequence $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i \rangle_{(i=0,1,2\ldots)} \rangle$ such that all the designated formulas $U_0, U_1, U_2, \ldots$ are a same formula.*

(2) Suppose $\exists \mathsf{GF} \varphi \in \mathbb{S}_0$. For any infinite $\rightsquigarrow$-sequence $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i \rangle_{(i=0,1,2\ldots)} \rangle$, there is a number k such that $(\forall i \geq k)(v_i(\exists \mathsf{GF} \varphi) = \mathtt{f}$ implies $v_i(\varphi) = \mathtt{f})$. In other words, in any infinite $\rightsquigarrow$-sequence, the $\exists \mathsf{GF}$-condition (for $\exists \mathsf{GF} \varphi$) always holds after somewhere.

Proof (1) Assume that an infinite sequence $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i \rangle_{(i=0,1,2\ldots)} \rangle$ satisfies $U_i = U_{i+1}$ for all i. Lemma 27 and definition of $\rightsquigarrow$ show that each $\rightsquigarrow$-step is defined by $\diamondsuit$ or $\clubsuit$, and the witness condition always fails. Then Lemma 20 shows that either $\mathcal{F}_i \subsetneq \mathcal{F}_{i+1}$ (in $\clubsuit$) or ($\mathcal{F}_i = \mathcal{F}_{i+1}$ and $\mathcal{H}_i \subsetneq \mathcal{H}_{i+1}$) (in $\diamondsuit$) for each i. However, such an infinite $\rightsquigarrow$-sequence cannot exist because $\mathcal{F}_i$ and $\mathcal{H}_i$ are subsets of a finite set.

(2) Assume that an infinite sequence $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i \rangle_{(i=0,1,2\ldots)} \rangle$ contains infinitely many c-valuations that fail in the $\exists \mathsf{GF}$-condition for $\exists \mathsf{GF} \varphi$. Then it contains infinitely many $\spadesuit$ or $\clubsuit$. This means that $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ for all i, and $\mathcal{F}_i \subsetneq \mathcal{F}_{i+1}$ for infinitely many i; however this is impossible as (1). **QED**

Lemma 29 *If a c-valuation $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ does not satisfy the $\exists \mathsf{GF}$-condition, then $\exists \mathsf{GF} v^* \vdash \neg v^*$.*

Proof By the premise, there is a formula φ such that $\exists \mathsf{GF} \varphi \in v_{\mathtt{f}}$ and $\varphi \in v_{\mathtt{t}}$. Then we have (1) $\exists \mathsf{GF} \varphi \vdash \neg v^*$, and (2) $v^* \vdash \varphi$, which implies (2^+) $\exists \mathsf{GF} v^* \vdash \exists \mathsf{GF} \varphi$ using Lemma 4(4). The facts (1) and (2^+) imply $\exists \mathsf{GF} v^* \vdash \neg v^*$. **QED**

Lemma 30 *Let $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ be a c-valuation in $\mathbb{C}(\mathbb{S}_0)$ and $\forall \mathsf{X} \psi$ be a formula in $\mathbb{S}_0$. If $v(\forall \mathsf{X} \psi) = \mathtt{f}$, then we have the following.*

(1) There is a valuation v' such that $v \rhd v'$, $v'(\psi) = \mathtt{f}$, and the c-valuation $\heartsuit$ is consistent.

(2) If the designated formula U is an $\forall \mathsf{U}$-formula and $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ does not satisfy the witness condition, then there is a valuation v' such that $v \rhd v'$, $v'(\psi) = \mathtt{f}$, and the c-valuation $\diamondsuit$ is consistent.

(3) If $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ does not satisfy the $\exists \mathsf{GF}$-condition, then there is a valuation v' such that $v \rhd v'$, $v'(\psi) = \mathtt{f}$, and the c-valuation $\spadesuit$ is consistent.

(4) If the designated formula U is an $\forall \mathsf{U}$-formula and $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ satisfies neither the $\exists \mathsf{GF}$-condition nor the witness condition, then there is a valuation v' such that $v \rhd v'$, $v'(\psi) = \mathtt{f}$, and the c-valuation $\clubsuit$ is consistent.

Proof (1) First we show

$$\forall\mathsf{G}\neg\mathcal{F}^*,\ v_{\mathtt{t}/\forall\mathtt{X}}\ \nvdash\ \psi. \tag{6.1}$$

Assume otherwise, then we have

$$\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{X}}\ \vdash\ \forall\mathsf{X}\psi,$$

and the c-valuation $\langle\mathcal{F},\mathcal{H},U,v\rangle$ would be inconsistent (i.e., $\forall\mathsf{G}\neg\mathcal{F}^*, \big(\alpha\wedge\bigwedge\{\neg\mathcal{H}^*\}\big)\ Q\mathsf{U}\,\beta, v_{\mathtt{t}} \vdash \bigvee v_{\mathtt{f}}$ where $U = \alpha\ Q\mathsf{U}\,\beta$) because of the facts

$$\forall\mathsf{G}\neg({v_i}^*) \vdash \forall\mathsf{X}\forall\mathsf{G}\neg({v_i}^*) \text{ for all } v_i \in \mathcal{F} \qquad (\because \text{Lemma 3})$$

and

$$\forall\mathsf{X}\psi \in v_{\mathtt{f}} \text{ (premise of the lemma) and } \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{X}} \subseteq v_{\mathtt{t}}. \tag{6.2}$$

Now the fact (6.1) and Lemma 21 imply existence of the required valuation v' such that $v_{\mathtt{t}/\forall\mathtt{X}} \subseteq v'_{\mathtt{t}}$, $\psi \in v'_{\mathtt{f}}$, and $\forall\mathsf{G}\neg\mathcal{F}^*, v'_{\mathtt{t}} \nvdash \bigvee v'_{\mathtt{f}}$. (♡ is consistent because $\mathrm{Next}_{v'}(U) \in v'_{\mathtt{t}}$.)

(2) Let $U = \alpha\,\forall\mathsf{U}\,\beta$. We define a formula γ by $\gamma = \alpha \wedge \bigwedge\neg\mathcal{H}^*$, and we will show

$$\forall\mathsf{G}\neg\mathcal{F}^*,\ (\gamma\wedge\neg v^*)\,\forall\mathsf{U}\,\beta,\ v_{\mathtt{t}/\forall\mathtt{X}}\ \nvdash\ \psi, \tag{6.3}$$

which implies the existence of the required valuation v' as (1). Note that the failure of the witness condition means

$$\beta \in v_{\mathtt{f}}. \tag{6.4}$$

Now assume that the claim (6.3) does not hold, then we have the following derivation.

$$\begin{array}{lr}
\forall\mathsf{G}\neg\mathcal{F}^*,\ (\gamma\wedge\neg v^*)\,\forall\mathsf{U}\,\beta,\ v_{\mathtt{t}/\forall\mathtt{X}}\ \vdash\ \psi. & \text{(assumption)}\\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\big((\gamma\wedge\neg v^*)\,\forall\mathsf{U}\,\beta\big),\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{X}}\ \vdash\ \forall\mathsf{X}\psi. & \\
\forall\mathsf{G}\neg\mathcal{F}^*,\ \beta\vee\forall\mathsf{X}\big((\gamma\wedge\neg v^*)\,\forall\mathsf{U}\,\beta\big)\ \vdash\ \neg v^*. & (\because (6.2), (6.4), \text{ and Lemma 3})\\
\forall\mathsf{G}\neg\mathcal{F}^*,\ \gamma\,\forall\mathsf{U}\,\beta\ \vdash\ \neg v^*. & (\because \text{Lemma 7})
\end{array}$$

This contradicts the consistency of $\langle\mathcal{F},\mathcal{H},U,v\rangle$.

(3) As the proofs of (1) and (2), we show

$$\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{G}\neg v^*,\ v_{\mathtt{t}/\forall\mathtt{X}}\ \nvdash\ \psi.$$

Assume otherwise, then we have the following derivation.

$$\begin{array}{lrr}
\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{G}\neg v^*,\ v_{\mathtt{t}/\forall\mathtt{X}}\ \vdash\ \psi. & \text{(assumption)} & \\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\forall\mathsf{G}\neg v^*,\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{X}}\ \vdash\ \forall\mathsf{X}\psi. & & \\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\forall\mathsf{G}\neg v^*\ \vdash\ \neg v^*. & (\because (6.2)) & (\dagger)\\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*\ \vdash\ v^* \to \exists\mathsf{X}\exists\mathsf{F}v^*. & & \\
\forall\mathsf{G}\neg\mathcal{F}^*\ \vdash\ \forall\mathsf{G}(v^* \to \exists\mathsf{X}\exists\mathsf{F}v^*). & (\because \text{Lemma 3}) & \\
\forall\mathsf{G}\neg\mathcal{F}^*\ \vdash\ v^* \to \exists\mathsf{G}\mathsf{F}v^*. & (\because \exists\mathsf{GF}\text{-induction axiom}) & \\
\forall\mathsf{G}\neg\mathcal{F}^*\ \vdash\ \neg v^*. & (\because \text{Lemma 29}) &
\end{array}$$

This contradicts the consistency of $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$.

(4) Let $U = \alpha\,\forall\mathsf{U}\,\beta$. As the above proofs, we show

$$\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{G}\neg v^*,\ \alpha\,\forall\mathsf{U}\,\beta,\ v_{\mathtt{t}/\forall\mathtt{x}}\ \nvdash\ \psi.$$

Assume otherwise, then we have the following derivation.

$$\begin{array}{lr}
\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{G}\neg v^*,\ \alpha\,\forall\mathsf{U}\,\beta,\ v_{\mathtt{t}/\forall\mathtt{x}}\ \vdash\ \psi. & \text{(assumption)}\\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\forall\mathsf{G}\neg v^*,\ \forall\mathsf{X}(\alpha\,\forall\mathsf{U}\,\beta),\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{x}}\ \vdash\ \forall\mathsf{X}\psi. & \\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\forall\mathsf{G}\neg v^*,\ \alpha\,\forall\mathsf{U}\,\beta,\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{x}}\ \vdash\ \beta\vee\forall\mathsf{X}\psi. & (\because \text{Lemma } 6(4))\\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\forall\mathsf{G}\neg v^*\ \vdash\ \neg v^*. & (\because (6.2), (6.4) \text{ and Lemma } 20(3))
\end{array}$$

Here we reach the step (†) of the proof of (3), and the remaining steps are exactly same as (3). **QED**

Lemma 31 *Let $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ be a c-valuation in $\mathbb{C}(\mathbb{S}_0)$. If $U = \alpha\,\exists\mathsf{U}\,\beta$, then we have the following.*

(1) If $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ does not satisfy the witness condition, then there is a valuation v' such that $v \rhd v'$, $v'(\alpha\,\exists\mathsf{U}\,\beta) = \mathtt{t}$ and the c-valuation $\diamondsuit$ is consistent.

(2) If $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ satisfies neither the $\exists\mathsf{GF}$-condition nor the witness condition, then there is a valuation v' such that $v \rhd v'$, $v'(\alpha\,\exists\mathsf{U}\,\beta) = \mathtt{t}$ and the c-valuation $\clubsuit$ is consistent.

Proof (1) Put $\gamma = \alpha \wedge \bigwedge \neg\mathcal{H}^*$. Similarly to the proof of Lemma 30(2), we show

$$\forall\mathsf{G}\neg\mathcal{F}^*,\ (\gamma\wedge\neg v^*)\,\exists\mathsf{U}\,\beta,\ v_{\mathtt{t}/\forall\mathtt{x}},\ \alpha\,\exists\mathsf{U}\,\beta\ \nvdash\ \bot.$$

Assume otherwise, then we have the following derivation.

$$\begin{array}{lr}
\forall\mathsf{G}\neg\mathcal{F}^*,\ (\gamma\wedge\neg v^*)\,\exists\mathsf{U}\,\beta,\ v_{\mathtt{t}/\forall\mathtt{x}},\ \alpha\,\exists\mathsf{U}\,\beta\ \vdash\ \bot. & \text{(assumption)}\\
\forall\mathsf{G}\neg\mathcal{F}^*,\ v_{\mathtt{t}/\forall\mathtt{x}},\ (\gamma\wedge\neg v^*)\,\exists\mathsf{U}\,\beta\ \vdash\ \bot. & (\because (\gamma\wedge\neg v^*)\,\exists\mathsf{U}\,\beta \vdash \alpha\,\exists\mathsf{U}\,\beta, \text{ by Lemma } 4(2))\\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{x}},\ \exists\mathsf{X}\big((\gamma\wedge\neg v^*)\,\exists\mathsf{U}\,\beta\big)\ \vdash\ \bot. & \\
\forall\mathsf{G}\neg\mathcal{F}^*,\ \beta\vee\exists\mathsf{X}\big((\gamma\wedge\neg v^*)\,\exists\mathsf{U}\,\beta\big)\ \vdash\ \neg v^*. & (\because\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{x}} \subseteq v_{\mathtt{t}}, \beta\in v_{\mathtt{f}}, \text{ and Lemma } 3)\\
\forall\mathsf{G}\neg\mathcal{F}^*,\ \gamma\,\exists\mathsf{U}\,\beta\ \vdash\ \neg v^*. & (\because \text{Lemma } 7)
\end{array}$$

This contradicts the consistency of $\langle \mathcal{F}, \mathcal{H}, \alpha\,\exists\mathsf{U}\,\beta, v \rangle$.

(2) Similarly to the proof of Lemma 30(4), we show

$$\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{G}\neg v^*,\ \alpha\,\exists\mathsf{U}\,\beta,\ v_{\mathtt{t}/\forall\mathtt{x}},\ \alpha\,\exists\mathsf{U}\,\beta\ \nvdash\ \bot.$$

Assume otherwise, then we have the following derivation.

$$\begin{array}{lr}
\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{G}\neg v^*,\ v_{\mathtt{t}/\forall\mathtt{x}},\ \alpha\,\exists\mathsf{U}\,\beta\ \vdash\ \bot. & \text{(assumption)}\\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\forall\mathsf{G}\neg v^*,\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{x}},\ \exists\mathsf{X}(\alpha\,\exists\mathsf{U}\,\beta)\ \vdash\ \bot. & \\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\forall\mathsf{G}\neg v^*,\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{x}},\ \alpha\,\exists\mathsf{U}\,\beta\ \vdash\ \beta. & (\because \text{Lemma } 6(4))\\
\forall\mathsf{X}\forall\mathsf{G}\neg\mathcal{F}^*,\ \forall\mathsf{X}\forall\mathsf{G}\neg v^*\ \vdash\ \neg v^*. & (\because\ \forall\mathsf{X}v_{\mathtt{t}/\forall\mathtt{x}} \subseteq v_{\mathtt{t}}, \beta\in v_{\mathtt{f}}, \text{ and Lemma } 20(3))
\end{array}$$

Here we reach the step (†) of the proof of Lemma 30(3), and the remaining steps are same. **QED**

7 Proof of completeness

As is mentioned in the previous section, our counter-model $\mathcal{M}_0$ for φ_0 is $\langle \mathbb{C}(\mathbb{S}_0), \rightsquigarrow, V_0 \rangle$ where $\mathbb{C}(\mathbb{S}_0)$ and $\rightsquigarrow$ are already defined. Here we define the mapping $V_0 : \mathbb{C}(\mathbb{S}_0) \times \mathrm{PropVar} \to \{\mathtt{t}, \mathtt{f}\}$ as follows.

$$V_0(\langle \mathcal{F}, \mathcal{H}, U, v \rangle,\ p) = \begin{cases} v(p) & (p \in \mathbb{S}_0) \\ \text{arbitrary} & (p \notin \mathbb{S}_0) \end{cases}$$

Lemma 32 (Main Lemma) *The following hold for any formula φ in $\mathbb{S}_0$ and any c-valuation $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ in $\mathbb{C}(\mathbb{S}_0)$. (1) If $v(\varphi) = \mathtt{t}$, then $\mathcal{M}_0, \langle \mathcal{F}, \mathcal{H}, U, v \rangle \models \varphi$. (2) If $v(\varphi) = \mathtt{f}$, then $\mathcal{M}_0, \langle \mathcal{F}, \mathcal{H}, U, v \rangle \not\models \varphi$.*

Proof By induction on φ using the Lemmas 33, 34, 37, 39, and 40 below. **QED**

Lemma 33 (Truth condition for $\top, \neg, \wedge$) *Let $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ be a c-valuation in $\mathbb{C}(\mathbb{S}_0)$, and $\neg\psi$ and $\psi_1 \wedge \psi_2$ be formulas in $\mathbb{S}_0$.*

(1) $v(\top) = \mathtt{t}$.

(2) If $v(\neg\psi) = \mathtt{t}$, then $v(\psi) = \mathtt{f}$.

(3) If $v(\neg\psi) = \mathtt{f}$, then $v(\psi) = \mathtt{t}$.

(4) If $v(\psi_1 \wedge \psi_2) = \mathtt{t}$, then $v(\psi_1) = v(\psi_2) = \mathtt{t}$.

(5) If $v(\psi_1 \wedge \psi_2) = \mathtt{f}$, then $v(\psi_1) = \mathtt{f}$ or $v(\psi_2) = \mathtt{f}$.

Proof (1) If $v(\top) = \mathtt{f}$, then the c-valuation $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ would be inconsistent because of the fact $\vdash \top$ and the definition of the consistency (Prop. 17). Proofs of (2)–(5) are similar using the facts $(\neg\psi, \psi \vdash \bot)$, $(\vdash (\neg\psi) \vee \psi)$, $(\psi_1 \wedge \psi_2 \vdash \psi_1)$, $(\psi_1 \wedge \psi_2 \vdash \psi_2)$, and $(\psi_1, \psi_2 \vdash \psi_1 \wedge \psi_2)$. **QED**

Lemma 34 (Truth condition for $\forall\mathsf{X}$) *Let $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ be a c-valuation in $\mathbb{C}(\mathbb{S}_0)$ and $\forall\mathsf{X}\psi$ be a formula in $\mathbb{S}_0$.*

(1) If $v(\forall\mathsf{X}\psi) = \mathtt{t}$, then $v'(\psi) = \mathtt{t}$ for any c-valuation $\langle \mathcal{F}', \mathcal{H}', U', v' \rangle$ such that $\langle \mathcal{F}, \mathcal{H}, U, v \rangle \rightsquigarrow \langle \mathcal{F}', \mathcal{H}', U', v' \rangle$.

(2) If $v(\forall\mathsf{X}\psi) = \mathtt{f}$, then there is a c-valuation $\langle \mathcal{F}', \mathcal{H}', U', v' \rangle$ such that $\langle \mathcal{F}, \mathcal{H}, U, v \rangle \rightsquigarrow \langle \mathcal{F}', \mathcal{H}', U', v' \rangle$ and $v'(\psi) = \mathtt{f}$.

Proof By the definition of $\rightsquigarrow$ and Lemma 30 **QED**

Lemma 35 (Seriality) *The relation $\rightsquigarrow$ is serial; that is, for each c-valuation $\langle \mathcal{F}, \mathcal{H}, U, v \rangle$ in $\mathbb{C}(\mathbb{S}_0)$, there is a c-valuation $\langle \mathcal{F}', \mathcal{H}', U', v' \rangle$ such that $\langle \mathcal{F}, \mathcal{H}, U, v \rangle \rightsquigarrow \langle \mathcal{F}', \mathcal{H}', U', v' \rangle$.*

Proof We have $\forall\mathsf{X}\neg\top \in \mathbb{S}_0$ (Lemma 23) and $v(\forall\mathsf{X}\neg\top) = \mathtt{f}$ ($\because$ otherwise $\langle \mathcal{F}, \mathcal{H}, U, v\rangle$ is inconsistent by the axiom D). Then Lemma 34(2) implies the existence of $\langle \mathcal{F}', \mathcal{H}', U', v'\rangle$. **QED**

Lemma 36 *Suppose* $\langle \mathcal{F}, \mathcal{H}, U, v\rangle \in \mathbb{C}(\mathbb{S}_0)$ *and* $\alpha \,\forall\mathsf{U}\, \beta \in \mathbb{S}_0$. *If* $v(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{f}$ *and* $v(\alpha) = \mathtt{t}$, *then there is a c-valuation* $\langle \mathcal{F}', \mathcal{H}', U', v'\rangle$ *such that* $\langle \mathcal{F}, \mathcal{H}, U, v\rangle \rightsquigarrow \langle \mathcal{F}', \mathcal{H}', U', v'\rangle$ *and* $v'(\alpha \,\forall\mathsf{U}\, \beta) = v'(\beta) = \mathtt{f}$.

Proof By the definition of consistency and Lemmas 6(1), 6(5), and 34(2) (for $\psi = \alpha \,\forall\mathsf{U}\, \beta$). Note that $\forall\mathsf{X}(\alpha \,\forall\mathsf{U}\, \beta) \in \mathbb{S}_0$ by Lemma 23. **QED**

Lemma 37 (Truth condition for $\forall\mathsf{U}$) *Let* $\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0\rangle$ *be a c-valuation in* $\mathbb{C}(\mathbb{S}_0)$ *and* $\alpha \,\forall\mathsf{U}\, \beta$ *be a formula in* $\mathbb{S}_0$.

(1) If $v_0(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{t}$, *then for any infinite* $\rightsquigarrow$*-sequence* $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$ *in* $\mathbb{C}(\mathbb{S}_0)$ *there is a number* $k \geq 0$ *such that* $v_k(\beta) = \mathtt{t}$ *and* $(\forall i < k)(v_i(\alpha) = \mathtt{t})$.

(2) If $v_0(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{f}$, *then there is an infinite* $\rightsquigarrow$*-sequence* $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$ *in* $\mathbb{C}(\mathbb{S}_0)$ *that satisfies* $\big((v_k(\beta) = \mathtt{f})$ *or* $(\exists i < k)(v_i(\alpha) = \mathtt{f})\big)$ *for any* $k \geq 0$.

Proof (1) Given $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$, Lemmas 6(3) and 6(4), the definitions of consistency and $\rhd$ imply the fact:

$$(\forall i)\Big(\big(v_i(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{t} \text{ and } v_i(\beta) = \mathtt{f}\big) \;\Rightarrow\; \big(v_i(\alpha) = \mathtt{t},\; v_i(\forall\mathsf{X}(\alpha \,\forall\mathsf{U}\, \beta)) = \mathtt{t}, \text{ and } v_{i+1}(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{t}\big)\Big).$$

(Note that $\forall\mathsf{X}(\alpha \,\forall\mathsf{U}\, \beta) \in \mathbb{S}_0$ by Lemma 23.) We have $v_0(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{t}$ by the premise, then the above fact implies either $(\forall i)\big(v_i(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{t}$ and $v_i(\beta) = \mathtt{f}\big)$ or $(\exists k)\big(v_k(\beta) = \mathtt{t}$ and $(\forall i < k)(v_i(\alpha) = \mathtt{t})\big)$. We show that the former is impossible; this completes the proof of (1). Assume $(\forall i)\big(v_i(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{t}$ and $v_i(\beta) = \mathtt{f}\big)$, then Lemma 28(1) and the definitions of "Next" and "$\rightsquigarrow$" imply that there exists a c-valuation $\langle \mathcal{F}_k, \mathcal{H}_k, U_k, v_k\rangle$ whose designated formula U_k is $\alpha \,\forall\mathsf{U}\, \beta$. Because this c-valuation fails in the witness condition ($\because$ assumption), the next c-valuation $\langle \mathcal{F}_{k+1}, \mathcal{H}_{k+1}, U_{k+1}, v_{k+1}\rangle$ must be $\diamondsuit$ or $\clubsuit$, and U_{k+1} is still $\alpha \,\forall\mathsf{U}\, \beta$. Iterating this argument, we have $U_{k+x} = \alpha \,\forall\mathsf{U}\, \beta$ for all x; this contradicts Lemma 28(1).

(2) We show how to define an infinite $\rightsquigarrow$-sequence $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$ such that each c-valuation $\langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle$ satisfies at least one of the following conditions:

(I) $v_i(\alpha \,\forall\mathsf{U}\, \beta) = v_i(\beta) = \mathtt{f}$.

(II) $(\exists j < i)(v_j(\alpha) = \mathtt{f})$.

The first c-valuation $\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0\rangle$ satisfies the condition (I) because of $v_0(\alpha \,\forall\mathsf{U}\, \beta) = \mathtt{f}$ (premise), Lemma 6(1), and the definition of consistency. Suppose a sequence $\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0\rangle \rightsquigarrow \cdots \rightsquigarrow \langle \mathcal{F}_n, \mathcal{H}_n, U_n, v_n\rangle$ is already defined; then we define the next c-valuation $\langle \mathcal{F}_{n+1}, \mathcal{H}_{n+1}, U_{n+1}, v_{n+1}\rangle$ as follows: If $v_j(\alpha) = \mathtt{f}$ for some $j \leq n$, then the next node is an arbitrary c-valuation obtained by Lemma 35; otherwise, $\langle \mathcal{F}_n, \mathcal{H}_n, U_n, v_n\rangle$ satisfies the conditions "$v_n(\alpha) = \mathtt{t}$" and (I), and the next node is obtained by Lemma 36. **QED**

Lemma 38 *Let* $\langle \mathcal{F}, \mathcal{H}, U, v\rangle$ *be a c-valuation in* $\mathbb{C}(\mathbb{S}_0)$ *and* $\alpha \exists \mathsf{U}\, \beta$ *be a formula in* $\mathbb{S}_0$*.*

(1) If $v(\alpha \exists \mathsf{U}\, \beta) = \mathtt{t}$ *and* $v(\beta) = \mathtt{f}$*, then there is a c-valuation* $\langle \mathcal{F}', \mathcal{H}', U', v'\rangle \in \mathbb{C}(\mathbb{S}_0)$ *such that* $\langle \mathcal{F}, \mathcal{H}, U, v\rangle \rightsquigarrow \langle \mathcal{F}', \mathcal{H}', U', v'\rangle$ *and* $v'(\alpha \exists \mathsf{U}\, \beta) = \mathtt{t}$*.*

(2) If the designated formula U *is* $\alpha \exists \mathsf{U}\, \beta$ *and* $v(\beta) = \mathtt{f}$*, then there is a c-valuation* $\langle \mathcal{F}', \mathcal{H}', \alpha \exists \mathsf{U}\, \beta, v'\rangle \in \mathbb{C}(\mathbb{S}_0)$ *such that* $\langle \mathcal{F}, \mathcal{H}, \alpha \exists \mathsf{U}\, \beta, v\rangle \rightsquigarrow \langle \mathcal{F}', \mathcal{H}', \alpha \exists \mathsf{U}\, \beta, v'\rangle$ *and* $v'(\alpha \exists \mathsf{U}\, \beta) = \mathtt{t}$*.*

Proof (1) $\forall \mathsf{X} \neg(\alpha \exists \mathsf{U}\, \beta) \in \mathbb{S}_0$ by Lemma 23; then existence of the required c-valuation is guaranteed by the definition of consistency and Lemmas 6(4), 33(3), and 34(2). Note that $\exists \mathsf{X}(\alpha \exists \mathsf{U}\, \beta) = \neg \forall \mathsf{X} \neg(\alpha \exists \mathsf{U}\, \beta)$. (2) By Lemma 31. **QED**

Lemma 39 (Truth condition for $\exists\mathsf{U}$**)** *Let* $\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0\rangle$ *be a c-valuation in* $\mathbb{C}(\mathbb{S}_0)$ *and* $\alpha \exists \mathsf{U}\, \beta$ *be a formula in* $\mathbb{S}_0$*.*

(1) If $v_0(\alpha \exists \mathsf{U}\, \beta) = \mathtt{t}$*, then there is an infinite* $\rightsquigarrow$*-sequence* $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$ *in* $\mathbb{C}(\mathbb{S}_0)$ *that satisfies* $v_k(\beta) = \mathtt{t}$ *and* $(\forall i < k)(v_i(\alpha) = \mathtt{t})$ *for some* $k \geq 0$*.*

(2) If $v_0(\alpha \exists \mathsf{U}\, \beta) = \mathtt{f}$*, then for any infinite* $\rightsquigarrow$*-sequence* $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$ *in* $\mathbb{C}(\mathbb{S}_0)$ *and any* $k \geq 0$*, we have* $v_k(\beta) = \mathtt{f}$ *or* $(\exists i < k)(v_i(\alpha) = \mathtt{f})$*.*

Proof (1) We define $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$ consisting of three parts. The first part is $\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0\rangle \rightsquigarrow \langle \mathcal{F}_1, \mathcal{H}_1, U_1, v_1\rangle \rightsquigarrow \cdots \rightsquigarrow \langle \mathcal{F}_a, \mathcal{H}_a, U_a, v_a\rangle$ where $(\forall i < a)\big(U_i \neq \alpha \exists \mathsf{U}\, \beta$, $v_i(\alpha \exists \mathsf{U}\, \beta) = \mathtt{t}$, and $v_i(\beta) = \mathtt{f}\big)$, $v_a(\alpha \exists \mathsf{U}\, \beta) = \mathtt{t}$ and $\big(U_a = \alpha \exists \mathsf{U}\, \beta$ or $v_a(\beta) = \mathtt{t}\big)$. This part is constructed from $\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0\rangle$ by iterated applications of Lemma 38(1). The existence of such a number a is guaranteed by Lemma 28(1) and the definition of "Next". The second part is $\langle \mathcal{F}_a, \mathcal{H}_a, U_a, v_a\rangle \rightsquigarrow \langle \mathcal{F}_{a+1}, \mathcal{H}_{a+1}, U_{a+1}, v_{a+1}\rangle \rightsquigarrow \cdots \rightsquigarrow \langle \mathcal{F}_k, \mathcal{H}_k, U_k, v_k\rangle$ where $(a \leq \forall i < k)\big(U_i = \alpha \exists \mathsf{U}\, \beta$, $v_i(\alpha \exists \mathsf{U}\, \beta) = \mathtt{t}$, and $v_i(\beta) = \mathtt{f}\big)$ and $v_k(\beta) = \mathtt{t}$. This part is constructed from $\langle \mathcal{F}_a, \mathcal{H}_a, U_a, v_a\rangle$ by iterated applications of Lemma 38(2). The existence of such a number k is guaranteed by Lemma 28(1). The third part $\langle \mathcal{F}_k, \mathcal{H}_k, U_k, v_k\rangle \rightsquigarrow \langle \mathcal{F}_{k+1}, \mathcal{H}_{k+1}, U_{k+1}, v_{k+1}\rangle \rightsquigarrow \cdots$ is constructed by infinite iteration of Lemma 35. The condition $(\forall i < k)(v_i(\alpha) = \mathtt{t})$ is guaranteed by the definition of consistency, the fact $(\forall i < k)\big(v_i(\alpha \exists \mathsf{U}\, \beta) = \mathtt{t}$ and $v_i(\beta) = \mathtt{f}\big)$, and Lemma 6(3).

(2) Given $\langle \langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$, Lemmas 6(1) and 6(5) and the definition of consistency imply the fact:

$$(\forall i)\Big(v_i(\alpha \exists \mathsf{U}\, \beta) = \mathtt{f} \;\Rightarrow\; \Big(v_i(\beta) = \mathtt{f} \text{ and } \big(v_i(\alpha) = \mathtt{f} \text{ or } v_i(\forall \mathsf{X} \neg(\alpha \exists \mathsf{U}\, \beta)) = \mathtt{t}\big)\Big)\Big).$$

(Note that $\forall \mathsf{X} \neg(\alpha \exists \mathsf{U}\, \beta) \in \mathbb{S}_0$ by Lemma 23 and that $\exists \mathsf{X}(\alpha \exists \mathsf{U}\, \beta) = \neg \forall \mathsf{X} \neg(\alpha \exists \mathsf{U}\, \beta)$.) We have $v_0(\alpha \exists \mathsf{U}\, \beta) = \mathtt{f}$ by the premise, hence the required condition $(\forall i)\big(v_i(\beta) = \mathtt{f}$ or $(\exists j < i)(v_i(\alpha) = \mathtt{f})\big)$ holds by the above fact and "$v_i(\forall \mathsf{X} \neg(\alpha \exists \mathsf{U}\, \beta)) = \mathtt{t} \Rightarrow v_{i+1}(\alpha \exists \mathsf{U}\, \beta) = \mathtt{f}$" ($\because$ the definition of $\rhd$ and Lemma 33(2)).

QED

Lemma 40 (Truth condition for $\exists\mathsf{GF}$**)** *Let* $\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0\rangle$ *be a c-valuation in* $\mathbb{C}(\mathbb{S}_0)$ *and* $\exists \mathsf{GF} \psi$ *be a formula in* $\mathbb{S}_0$*.*

(1) If $v_0(\exists\mathsf{GF}\psi) = \mathtt{t}$*, then there is an infinite* $\rightsquigarrow$*-sequence* $\langle\langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$ *in* $\mathbb{C}(\mathbb{S}_0)$ *such that* $(\forall i)(\exists j \geq i)(v_j(\psi) = \mathtt{t})$*.*

(2) If $v_0(\exists\mathsf{GF}\psi) = \mathtt{f}$*, then for any infinite* $\rightsquigarrow$*-sequence* $\langle\langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$ *in* $\mathbb{C}(\mathbb{S}_0)$ *there is a number* i *such that* $(\forall j \geq i)(v_j(\psi) = \mathtt{f})$*.*

Proof (1) The formula $\exists\mathsf{F}(\psi \wedge \exists\mathsf{X}\exists\mathsf{GF}\psi)$, which is equal to $\top\,\exists\mathsf{U}(\psi \wedge \neg\forall\mathsf{X}\neg\exists\mathsf{GF}\psi)$, is in $\mathbb{S}_0$ by Lemma 23. Hence the definition of consistency, the fact $v_0(\exists\mathsf{GF}\psi) = \mathtt{t}$ (premise), and Lemma 8(3) imply $v_0(\top\,\exists\mathsf{U}(\psi \wedge \neg\forall\mathsf{X}\neg\exists\mathsf{GF}\psi)) = \mathtt{t}$. We apply Lemma 39(1) and we get a finite sequence $\langle \mathcal{F}_0, \mathcal{H}_0, U_0, v_0\rangle \rightsquigarrow \cdots \rightsquigarrow \langle \mathcal{F}', \mathcal{H}', U', v'\rangle$ (the "first and second parts" in the proof of Lemma 39(1)) such that $v'(\psi \wedge \neg\forall\mathsf{X}\neg\exists\mathsf{GF}\psi) = \mathtt{t}$. Then Lemmas 33 and 34(2) imply that $v'(\psi) = \mathtt{t}$ and that there is a c-valuation $\langle \mathcal{F}'', \mathcal{H}'', U'', v''\rangle$ such that $\langle \mathcal{F}', \mathcal{H}', U', v'\rangle \rightsquigarrow \langle \mathcal{F}'', \mathcal{H}'', U'', v''\rangle$ and $v''(\exists\mathsf{GF}\psi) = \mathtt{t}$. Iterating this argument, we gat the required infinite sequence $\langle\langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$.

(2) Given $\langle\langle \mathcal{F}_i, \mathcal{H}_i, U_i, v_i\rangle_{(i=0,1,2\ldots)}\rangle$, Lemmas 8(2) and 33 and the definitions of consistency and $\rhd$ imply the fact:

$$(\forall i)\Big(v_i(\exists\mathsf{GF}\psi) = \mathtt{f} \;\Rightarrow\; \big(v_i(\neg\forall\mathsf{X}\neg\exists\mathsf{GF}\psi) = \mathtt{f} \text{ and } v_{i+1}(\exists\mathsf{GF}\psi) = \mathtt{f}\big)\Big)$$

(Note that $\exists\mathsf{X}\exists\mathsf{GF}\psi$ is equal to $\neg\forall\mathsf{X}\neg\exists\mathsf{GF}\psi$ and is in $\mathbb{S}_0$ by Lemma 23.) This implies $(\forall i)(v_i(\exists\mathsf{GF}\psi) = \mathtt{f})$ because of the premise $v_0(\exists\mathsf{GF}\psi) = \mathtt{f}$. Then the existence of the required number i is guaranteed by Lemma 28(2). **QED**

Finally the main result of this paper is proved:

Theorem 41 (Completeness of $\mathrm{H}_{\mathrm{ECTL}}$**)** $\mathcal{M}_0$ *is a finite model, and* $\mathcal{M}_0, x \not\models \varphi_0$ *for some state* x*. (*φ_0 *is a formula, fixed at the beginning of Section 6, such that* $\not\vdash \varphi_0$*, and* $\mathcal{M}_0$ *was defined at the beginning of this section.)*

Proof Lemma 21 shows that there is a valuation v of $\mathbb{S}_0$ such that $v_{\mathtt{t}} \not\vdash \bigvee v_{\mathtt{f}}$ and $v(\varphi_0) = \mathtt{f}$. Then put $x = \langle \emptyset, \emptyset, \top\forall\mathsf{U}\top, v\rangle$; x is consistent by the definition of consistency and Lemmas 6(2) and 23(1), and we have $\mathcal{M}_0, x \not\models \varphi_0$ by the Main Lemma 32(2). Finiteness and seriality of $\mathcal{M}_0$ is guaranteed by Lemmas 19 and 35. **QED**

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