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Completeness of Second Order Propositional Intuitionistic Logics

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Abstract

We give alternative proofs of the completeness of two second order propositional intuitionistic logics with respect to Kripke models. One is the logic having the full comprehension axiom, and the other has the constant domain axiom in addition. We also show that, for disjunction free fragment, the constant domain axiom is not needed for the completeness with respect to constant domain models. To show the completeness, we use "sequent tree calculi".

1 Introduction

Second order propositional intuitionistic logic (SOPIL) is obtained from the propositional intuitionistic logic by adding quantifiers which bind propositional variables. There are some variants of SOPIL. Among them, this paper mainly treats the logic which is axiomatized by the following inference rules in the sequent calculus:

$$\frac{A(B), \Gamma \Rightarrow C}{\forall x A(x), \Gamma \Rightarrow C}\ (\forall\ \text{left}) \qquad \frac{\Gamma \Rightarrow A(p)}{\Gamma \Rightarrow \forall x A(x)}\ (\forall\ \text{right})$$

$$\frac{A(p), \Gamma \Rightarrow C}{\exists x A(x), \Gamma \Rightarrow C}\ (\exists\ \text{left}) \qquad \frac{\Gamma \Rightarrow A(B)}{\Gamma \Rightarrow \exists x A(x)}\ (\exists\ \text{right})$$

where p does not occur in the lower sequent. We call this logic SOPIL_0. (SOPIL_0 is also characterized by the "full comprehension axiom": $\exists x(x \leftrightarrow A)$.)

Recall that a Kripke model for propositional intuitionistic logic is of the form $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle$ where $\langle \mathcal{W}, \mathcal{R} \rangle$ is a partially ordered set of "worlds" and $\mathcal{I}$ is an interpretation which assigns an $\mathcal{R}$-upper closed subset to each propositional variable; that is, $\mathcal{I} = \{\mathcal{I}_p \in \text{UP}(\mathcal{W}, \mathcal{R}) \mid p \text{ is a propositional variable}\}$ and $\text{UP}(\mathcal{W}, \mathcal{R}) = \{\alpha \subseteq \mathcal{W} \mid \forall a, b \in \mathcal{W}[\text{ if } a \in \alpha \text{ and } a\mathcal{R}b, \text{ then } b \in \alpha]\}$. Then, for a formula A and a world $a \in \mathcal{W}$, the relation $a \models A$ ("A holds at a") is inductively defined as follows.

$a \models p \Leftrightarrow a \in \mathcal{I}_p$. $a \not\models \bot$. $a \models A \rightarrow B \Leftrightarrow (b \not\models A)$ or $(b \models B)$ for all b such that $a\mathcal{R}b$.
$a \models A \wedge B \Leftrightarrow (a \models A)$ and $(a \models B)$. $a \models A \vee B \Leftrightarrow (a \models A)$ or $(a \models B)$.

A Kripke model for SOPIL_0 is obtained from the above by adding the "domain" $\mathcal{D} = \{\mathcal{D}_a \subseteq \text{UP}(\mathcal{W}, \mathcal{R}) \mid a \in \mathcal{W}\}$ that satisfies the *hereditary condition*: $a\mathcal{R}b \Rightarrow \mathcal{D}_a \subseteq \mathcal{D}_b$. The quantifiers are interpreted as follows.

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$a \models \forall x A(x) \;\Leftrightarrow\; b \models A(\alpha)$ for all b and all α such that $a\mathcal{R}b$ and $\alpha \in \mathcal{D}_a$.
$a \models \exists x A(x) \;\Leftrightarrow\; a \models A(\alpha)$ for some $\alpha \in \mathcal{D}_a$.

To describe the completeness theorem, we introduce one more condition — *fullness* — on the domain of quantifiers. Given a formula $A(p_1, p_2, \ldots, p_n)$, there is an n-array function f on $\mathrm{UP}(\mathcal{W}, \mathcal{R})$ such that $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \{a \in \mathcal{W} \mid a \models A(\alpha_1, \alpha_2, \ldots, \alpha_n)\}$. We say that this function f is *induced by* the formula A. For example, the formula $p \vee q$ induces the function $f(\alpha, \beta) = \alpha \cup \beta$. Then the domain $\mathcal{D}_a$ is said to be *full* if the set $\mathcal{D}_a$ is closed under any function that is induced by a formula, and the Kripke model is said to be *full* if each $\mathcal{D}_a$ is full. Note that these definitions are informal; precise definitions are given in Section 5. A formula A is said to be *valid* in the model if $a \models A$ for all $a \in \mathcal{W}$.

The completeness theorem is stated as follows: *A formula is provable in* SOPIL_0 *if and only if it is valid in any full model.* Moreover we have the constant domain case, like the first order intuitionistic logic. The axiom schema $\forall x(A \vee B(x)) \to A \vee \forall x B(x)$ is called **CD**. Then we have the completeness theorem: *A formula is provable in* $\mathrm{SOPIL}_0 + \mathbf{CD}$ *if and only if it is valid in any constant domain full model*, where the model is said to be *constant domain* if $\mathcal{D}_a = \mathcal{D}_b$ for any $a, b \in \mathcal{W}$. These completeness theorems or similar results have been proved in the literature [2, 3, 4, 5, 6, 7]. In this paper, we give alternative proofs of them using "*sequent tree calculi*" (or, often called "*nested sequent*"; see e.g. [1] for the history of nested sequent calculi).

The above theorems show that the two sets $\mathcal{J} = \{A \mid A \text{ is valid in any full model}\}$ and $\mathcal{J}_{\mathrm{cd}} = \{A \mid A \text{ is valid in any constant domain full model}\}$ are different; indeed, in Section 5, we give a full model in which an instance

$$\forall x\big(p \vee ((p \to x) \vee \neg x)\big) \to \big(p \vee \forall x((p \to x) \vee \neg x)\big)$$

of the axiom **CD** is not valid. On the other hand, we show the following:

The sets $\mathcal{J}$ and $\mathcal{J}_{\mathrm{cd}}$ coincide if the language does not contain the disjunction ($\vee$). (1)

One may doubt this result because, in SOPIL, disjunction can be represented by using other symbols as

$$A \vee B \;=\; \forall x((A \to x) \to (B \to x) \to x);$$

and the lack of disjunction seems to have no influence on the provability and validity. The following is the reason why the result depends on the existence of disjunction. The condition "$\mathcal{D}_a$ is closed under any function that is induced by a formula", which is the definition of the fullness, is weakened by the lack of disjunction, and disjunction-free formulas do not require that $\mathcal{D}_a$ is closed under $\cup$.

The structure of this paper is as follows. In Sections 2 and 3, we give the definitions of formulas and sequent calculi. In Section 4, we study the replacement of $A \vee B$ by $\forall x((A \to x) \to (B \to x) \to x)$ in detail. In Section 5, we give definitions of Kripke models; then the soundness of sequent calculi are shown in Section 6. In Section 7, we introduce a sequent tree calculus for SOPIL_0, and we prove its completeness. This argument is applied to $\mathrm{SOPIL}_0 + \mathbf{CD}$ in Section 8. So far, roughly speaking, we show (i) *if A is provable in the sequent calculus, then A is valid in Kripke models* (in Section 6); and (ii) *if A is valid in Kripke models, then A is provable in the sequent tree calculus* (in Sections 7 and 8). Then the final piece to complete is (iii) *if A is provable in the sequent tree calculus, then A is provable in the sequent calculus.* This is shown in Section 9. Moreover, for disjunction-free case, we show a stronger result in Section 10: (iii$^+$) *if A is provable in the disjunction-free sequent tree calculus for constant domain Kripke models, then A is cut-free provable in the sequent calculus for* SOPIL_0. This implies the above fact (1) and the cut-elimination theorem for disjunction-free SOPIL_0. Finally, in Section 11, we give precise statements of the main theorems of this paper; moreover, we make additional remarks and give problems for future study.

2 Formulas

We fix two countably infinite sets **PropPara** (of *propositional parameters*) and **PropVar** (of *propositional variables*) such that $\mathbf{PropPara} \cap \mathbf{PropVar} = \emptyset$. Both propositional parameters and propositional variables are called *propositional symbols*. Then *formulas* are inductively defined as follows.

> Propositional symbols and $\bot$ are formulas. If A and B are formulas and x is a propositional variable, then $(A{\to}B)$, $(A{\wedge}B)$, $(A{\vee}B)$, $(\forall xA)$, and $(\exists xA)$ are formulas.

Note that, for example, $\forall p(p{\to}p)$ is not a formula if p is a propositional parameter (only propositional variables can be bound by quantifiers). Propositional parameters are denoted by $p, q, \ldots$, propositional variables are denoted by $x, y, \ldots$, and formulas are denoted by $A, B, \ldots$. We consider the logical operators $\neg$ and $\leftrightarrow$ to be the abbreviations as follows. $\neg A = A{\to}\bot$, and $A{\leftrightarrow}B = (A{\to}B){\wedge}(B{\to}A)$. Parentheses are omitted by the convention that $\forall x$, $\exists x$ and $\neg$ bind more strongly than binary operators; $\wedge$ and $\vee$ bind more strongly than $\to$ and $\leftrightarrow$; and that $A_1{\to}A_2{\to}\cdots A_n$ is $(A_1{\to}(A_2{\to}\cdots(A_{n-1}{\to}A_n)\cdots))$. For example, $\forall xA{\to}B{\wedge}C{\to}D$ denotes the formula $(\forall xA){\to}((B{\wedge}C){\to}D)$.

For a formula A, the set of free propositional variables in A and the set of propositional parameters in A are denoted by $\mathrm{FV}(A)$ and $\mathrm{PP}(A)$ respectively. These are inductively defined as follows.

> $\mathrm{FV}(p) = \emptyset$ and $\mathrm{PP}(p) = \{p\}$ for $p \in \mathbf{PropPara}$. $\mathrm{FV}(x) = \{x\}$ and $\mathrm{PP}(x) = \emptyset$ for $x \in \mathbf{PropVar}$. $\mathrm{FV}(\bot) = \mathrm{PP}(\bot) = \emptyset$. $\mathrm{FV}(A \circ B) = \mathrm{FV}(A) \cup \mathrm{FV}(B)$ and $\mathrm{PP}(A \circ B) = \mathrm{PP}(A) \cup \mathrm{PP}(B)$ for $\circ \in \{\to, \wedge, \vee\}$. $\mathrm{FV}(\nabla xA) = \mathrm{FV}(A) - \{x\}$ and $\mathrm{PP}(\nabla xA) = \mathrm{PP}(A)$ for $\nabla \in \{\forall, \exists\}$.

We call a formula A is *closed* if $\mathrm{FV}(A) = \emptyset$.

If A and B are formulas and x is a propositional variable, then $A[x{:=}B]$ denotes the formula that is obtained from A by replacing all the free occurrences of x by B. This is inductively defined as follows.

> $x[x{:=}B] = B$. $a[x{:=}B] = a$ if $a \in \mathbf{PropPara}{\cup}\mathbf{PropVar}$ and $a \neq x$. $\bot[x{:=}B] = \bot$. $(A_1 \circ A_2)[x{:=}B] = (A_1[x{:=}B]) \circ (A_2[x{:=}B])$ for $\circ \in \{\to, \wedge, \vee\}$. $(\nabla xA)[x{:=}B] = \nabla xA$ for $\nabla \in \{\forall, \exists\}$. $(\nabla yA)[x{:=}B] = \nabla y(A[x{:=}B])$, if $y \neq x$ and $\nabla \in \{\forall, \exists\}$.

Note that if B is a closed formula, then this substitution does not cause undesired binding of a free-variable in B (such as $(\exists y(x{\wedge}y))[x{:=}\neg y]$) and we do not need the renaming of bound variables. This is the reason why we include propositional parameters in the definition of formulas. Note also that $A[x_1{:=}B_1][x_2{:=}B_2] = A[x_2{:=}B_2][x_1{:=}B_1]$ if B_1, B_2 are closed formulas and $x_1 \neq x_2$. This can be proved by induction on A, and we will tacitly use this fact.

3 Sequent calculi

If Γ is a finite set of *closed* formulas and A is a *closed* formula, then the expression $\Gamma \Rightarrow A$ is called a *sequent*. As usual, if $\Gamma = \{B_1, B_2, \ldots, B_n\}$, then $\Gamma \Rightarrow A$ is written as $B_1, B_2, \ldots, B_n \Rightarrow A$ instead of $\{B_1, B_2, \ldots, B_n\} \Rightarrow A$.

LJ2 is a system which derives sequents. Note that all the formulas occurring below must be closed since the definition of sequents. Axioms (initial sequents) of **LJ2** are of the forms

$$A \Rightarrow A \quad \text{and} \quad \bot \Rightarrow A.$$

Inference rules of **LJ2** are (cut), (weakening), ($\to$ left), ..., ($\exists$ right) as follows.

$$\frac{\Gamma \Rightarrow A \quad A,\Delta \Rightarrow B}{\Gamma,\Delta \Rightarrow B}\ (\text{cut}) \qquad \frac{\Gamma \Rightarrow A}{B,\Gamma \Rightarrow A}\ (\text{weakening})$$

$$\frac{\Gamma \Rightarrow A \quad B,\Delta \Rightarrow C}{A{\to}B,\Gamma,\Delta \Rightarrow C}\ (\to\ \text{left}) \qquad \frac{A,\Gamma \Rightarrow B}{\Gamma \Rightarrow A{\to}B}\ (\to\ \text{right})$$

$$\frac{A,\Gamma \Rightarrow C}{A{\wedge}B,\Gamma \Rightarrow C}\ (\wedge\ \text{left}) \quad \frac{B,\Gamma \Rightarrow C}{A{\wedge}B,\Gamma \Rightarrow C}\ (\wedge\ \text{left}) \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A{\wedge}B}\ (\wedge\ \text{right})$$

$$\frac{A,\Gamma \Rightarrow C \quad B,\Gamma \Rightarrow C}{A{\vee}B,\Gamma \Rightarrow C}\ (\vee\ \text{left}) \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A{\vee}B}\ (\vee\ \text{right}) \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A{\vee}B}\ (\vee\ \text{right})$$

$$\frac{A[x{:=}B],\Gamma \Rightarrow C}{\forall xA,\Gamma \Rightarrow C}\ (\forall\ \text{left})(\dagger 1) \qquad \frac{\Gamma \Rightarrow A[x{:=}p]}{\Gamma \Rightarrow \forall xA}\ (\forall\ \text{right})(\dagger 2)$$

$$\frac{A[x{:=}p],\Gamma \Rightarrow C}{\exists xA,\Gamma \Rightarrow C}\ (\exists\ \text{left})(\dagger 2) \qquad \frac{\Gamma \Rightarrow A[x{:=}B]}{\Gamma \Rightarrow \exists xA}\ (\exists\ \text{right})(\dagger 1)$$

(†1) Proviso: B is a closed formula.

(†2) Proviso: p is a propositional parameter which does not occur in the lower sequent.

As usual, the propositional parameter p in (∀ right)/(∃ left) rules is called an *eigenvariable*.[1] Note that the inference rules

$$\frac{A,A,\Gamma \Rightarrow B}{A,\Gamma \Rightarrow B}\ (\text{contraction}) \qquad \frac{\Gamma,A,B,\Delta \Rightarrow C}{\Gamma,B,A,\Delta \Rightarrow C}\ (\text{exchange})$$

are available because $\{A,A,\Gamma\} = \{A,\Gamma\}$ and $\{\Gamma,A,B,\Delta\} = \{\Gamma,B,A,\Delta\}$.

We say that *a formula A is provable in* **LJ2** (or, in another sequent calculus, say L) if the sequent ($\Rightarrow A$) is provable in **LJ2** (or, in L). We will use the symbol "$\vdash$" to mean the provability; that is, "$L \vdash \Gamma \Rightarrow A$" means the fact that the sequent $\Gamma \Rightarrow A$ is provable in the sequent calculus L. For example, $\exists x(x{\leftrightarrow}A)$ (full comprehension axiom) is provable in **LJ2** for any closed formula A:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A{\to}A}\ (\to\text{r.}) \quad \dfrac{A \Rightarrow A}{\Rightarrow A{\to}A}\ (\to\text{r.})}{\Rightarrow (A{\to}A){\wedge}(A{\to}A)}\ (\wedge\text{r.})}{\Rightarrow \exists x((x{\to}A){\wedge}(A{\to}x))}\ (\exists\text{r.})$$

Sequent calculus **LJ2CD** is obtained from **LJ2** by adding the "*constant domain axiom*" which is any sequent of the form

$$\forall x(A{\vee}B) \Rightarrow A{\vee}\forall xB.$$

Note that $x \notin \mathrm{FV}(A)$ since $A{\vee}\forall xB$ is a closed formula by the definition of sequent. The constant domain axiom is not provable in **LJ2**; this will be shown in Section 6.

The following lemma will be used in Section 9.

Lemma 1 *The following inference rules are derivable in* **LJ2CD***.*

$$\frac{A{\vee}\forall xB,\Gamma \Rightarrow C}{\forall x(A{\vee}B),\Gamma \Rightarrow C}\ (\forall\text{-}\vee\ \text{left}) \quad \textit{and} \quad \frac{A{\to}\forall xB,\Gamma \Rightarrow C}{\forall x(A{\to}B),\Gamma \Rightarrow C}\ (\forall\text{-}{\to}\ \text{left})$$

where A, $\forall xB$, C and all the formulas in Γ are closed formulas. ((∀-→ left) is derivable also in **LJ2***.)*

[1] p is not a propositional variable but a propositional parameter; however, we follow the convention.

Proof ($\forall$-$\vee$ left)

$$\dfrac{\overset{\text{(c.d.axiom)}}{\forall x(A\vee B) \Rightarrow A\vee\forall xB} \quad A\vee\forall xB, \varGamma \Rightarrow C}{\forall x(A\vee B), \varGamma \Rightarrow C}\ \text{(cut)}$$

($\forall$-$\rightarrow$ left) Let p be a propositional parameter such that $p \notin \mathrm{PP}(A, B)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{(axiom)}}{A \Rightarrow A} \quad \overset{\text{(axiom)}}{B[x{:=}p] \Rightarrow B[x{:=}p]}}{A\rightarrow B[x{:=}p], A \Rightarrow B[x{:=}p]}\ (\rightarrow\text{l.})}{\forall x(A\rightarrow B), A \Rightarrow B[x{:=}p]}\ (\forall\text{l.})}{\forall x(A\rightarrow B), A \Rightarrow \forall xB}\ (\forall\text{r.})}{\forall x(A\rightarrow B) \Rightarrow A\rightarrow\forall xB}\ (\rightarrow\text{r.}) \qquad A\rightarrow\forall xB, \varGamma \Rightarrow C}{\forall x(A\rightarrow B), \varGamma \Rightarrow C}\ \text{(cut)}$$

QED

4 Deleting disjunction

We say that a formula (or a sequent) is *disjunction free* if it contains no occurrence of $\vee$. Sequent calculus $\mathbf{LJ2}^{\mathrm{df}}$ is obtained from **LJ2** by imposing the restriction that all the sequents occurring in the proofs are disjunction free. Of course neither ($\vee$ left) nor ($\vee$ right) is available in $\mathbf{LJ2}^{\mathrm{df}}$.

It is natural to expect that: (†) *A disjunction free sequent $\varGamma \Rightarrow A$ is provable in* $\mathbf{LJ2}^{\mathrm{df}}$ *whenever it is provable in* **LJ2**. One may try to prove this by induction on the proof of $\varGamma \Rightarrow A$. However, such a simple induction does not work; for example, if $(\varGamma \Rightarrow A) = (\forall xB, \varGamma' \Rightarrow A)$ is inferred from $(B[x{:=}C], \varGamma' \Rightarrow A)$ by ($\forall$ left) rule and C contains $\vee$, then the induction hypothesis does not work. In this section, we prove the above (†) using the fact that disjunction can be expressed by universal quantification and implication (as mentioned in Section 1).

For a formula A, we define a set $\mathrm{DF}(A)$ of formulas as follows.

> $\mathrm{DF}(a) = \{a\}$ if a is a propositional symbol. $\mathrm{DF}(\bot) = \{\bot\}$.
> $\mathrm{DF}(A\vee B) = \{\forall x\big((A'\rightarrow x)\rightarrow(B'\rightarrow x)\rightarrow x\big) \mid A' \in \mathrm{DF}(A),\ B' \in \mathrm{DF}(B)$, and $x \notin \mathrm{FV}(A', B')\}$.
> $\mathrm{DF}(A \circ B) = \{A' \circ B' \mid A' \in \mathrm{DF}(A)$ and $B' \in \mathrm{DF}(B)\}$, for $\circ \in \{\rightarrow, \wedge\}$.
> $\mathrm{DF}(\nabla xA) = \{\nabla xA' \mid A' \in \mathrm{DF}(A)\}$, for $\nabla \in \{\forall, \exists\}$.

The following properties are easily shown:

- $\mathrm{DF}(A) \neq \emptyset$.
- All the elements in $\mathrm{DF}(A)$ are disjunction free formulas.
- If A is disjunction free, then $\mathrm{DF}(A) = \{A\}$.
- If $A' \in \mathrm{DF}(A)$, then $\mathrm{FV}(A') = \mathrm{FV}(A)$ and $\mathrm{PP}(A') = \mathrm{PP}(A)$.

We will tacitly use these. In the following, A^{df} will denote any (or some) formula in $\mathrm{DF}(A)$, and $A{+}B$ will denotes any (or some) formula of the form $\forall x\big((A\rightarrow x)\rightarrow(B\rightarrow x)\rightarrow x\big)$ provided that $x \notin \mathrm{FV}(A, B)$. Roughly speaking, A^{df} is a formula that is obtained from A by replacing each $P\vee Q$ by $P{+}Q$.

Lemma 2 *The following inference rules are derivable in* $\mathbf{LJ2}^{\mathrm{df}}$ *(and in* **LJ2***).*

$$\dfrac{A,\Gamma\Rightarrow C \quad B,\Gamma\Rightarrow C}{A{+}B,\Gamma\Rightarrow C}\ (\text{+left}) \qquad \dfrac{\Gamma\Rightarrow A}{\Gamma\Rightarrow A{+}B}\ (+\text{ right}) \quad \dfrac{\Gamma\Rightarrow B}{\Gamma\Rightarrow A{+}B}\ (+\text{ right})$$

Proof (+ left)

$$\dfrac{\dfrac{\dfrac{A,\Gamma\Rightarrow C}{\Gamma\Rightarrow A\to C}\ (\to\text{r.}) \quad \dfrac{\dfrac{B,\Gamma\Rightarrow C}{\Gamma\Rightarrow B\to C}\ (\to\text{r.}) \quad \overset{\text{(axiom)}}{C\Rightarrow C}}{(B\to C)\to C,\Gamma\Rightarrow C}\ (\to\text{l.})}{(A\to C)\to(B\to C)\to C,\Gamma\Rightarrow C}\ (\to\text{l.})}{\forall x((A\to x)\to(B\to x)\to x),\Gamma\Rightarrow C}\ (\forall\text{l.})$$

where $x\notin \mathrm{FV}(A,B)$.
(+ right)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\Rightarrow A \quad \overset{\text{(axiom)}}{p\Rightarrow p}}{A\to p,\Gamma\Rightarrow p}\ (\to\text{l.})}{B\to p,A\to p,\Gamma\Rightarrow p}\ (\text{weak.})}{A\to p,\Gamma\Rightarrow(B\to p)\to p}\ (\to\text{r.})}{\Gamma\Rightarrow(A\to p)\to(B\to p)\to p}\ (\to\text{r.})}{\Gamma\Rightarrow\forall x((A\to x)\to(B\to x)\to x)}\ (\forall\text{r.}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\Rightarrow B \quad \overset{\text{(axiom)}}{p\Rightarrow p}}{B\to p,\Gamma\Rightarrow p}\ (\to\text{l.})}{B\to p,A\to p,\Gamma\Rightarrow p}\ (\text{weak.})}{A\to p,\Gamma\Rightarrow(B\to p)\to p}\ (\to\text{r.})}{\Gamma\Rightarrow(A\to p)\to(B\to p)\to p}\ (\to\text{r.})}{\Gamma\Rightarrow\forall x((A\to x)\to(B\to x)\to x)}\ (\forall\text{r.})$$

where $x\notin \mathrm{FV}(A,B)$ and $p\notin \mathrm{PP}(\Gamma,A,B)$. **QED**

We say that a closed formula A is an *atomic instance of* a formula B if $A = B[x_1{:=}p_1]$ $[x_2{:=}p_2]\ \cdots\ [x_n{:=}p_n]$ for some propositional variables $x_1,x_2,\ldots,x_n$ and some propositional parameters $p_1,p_2,\ldots,p_n$. We define an equivalence relation $\Leftrightarrow_{\mathrm{LJ2}}$ between formulas as follows.

$A\Leftrightarrow_{\mathrm{LJ2}} B$ if and only if any atomic instance of $A\leftrightarrow B$ is provable in **LJ2**.

In other words, $A\Leftrightarrow_{\mathrm{LJ2}} B$ means that $\mathbf{LJ2}\vdash A\theta\Rightarrow B\theta$ and $\mathbf{LJ2}\vdash B\theta\Rightarrow A\theta$ for any substitution $\theta = [x_1{:=}p_1][x_2{:=}p_2]\cdots[x_n{:=}p_n]$. Similarly an equivalence relation $\Leftrightarrow_{\mathrm{LJ2^{df}}}$ is defined using $\mathbf{LJ2}^{\mathrm{df}}$ instead of **LJ2**.

Lemma 3 *If* $A'\in\mathrm{DF}(A)$ *and* $A''\in\mathrm{DF}(A)$*, then* $A\Leftrightarrow_{\mathrm{LJ2}} A'$ *and* $A'\Leftrightarrow_{\mathrm{LJ2^{df}}} A''$*.*

Proof By induction on A, we prove (1) $\mathbf{LJ2}\vdash A\theta\Rightarrow A'\theta$, (2) $\mathbf{LJ2}\vdash A'\theta\Rightarrow A\theta$, (3) $\mathbf{LJ2}^{\mathrm{df}}\vdash A'\theta\Rightarrow A''\theta$, and (4) $\mathbf{LJ2}^{\mathrm{df}}\vdash A''\theta\Rightarrow A'\theta$, where $\mathrm{FV}(A) = \{x_1,x_2,\ldots,x_n\}$, $\theta = [x_1{:=}p_1]$ $[x_2{:=}p_2]\ \cdots\ [x_n{:=}p_n]$, and $p_1,p_2,\cdots,p_n$ are arbitrary propositional parameters. Here we show a few cases.

If $A = B\vee C$, then $A' = B'{+}C'$ and $A'' = B''{+}C''$ for some $B',B''\in\mathrm{DF}(B)$ and $C',C''\in\mathrm{DF}(C)$. Then (3) is shown as follows.

$$\dfrac{\dfrac{\overset{\text{(i.h.)}}{B'\theta\Rightarrow B''\theta}}{B'\theta\Rightarrow B''\theta+C''\theta}\ (\text{+r.})(\text{Lem.2}) \quad \dfrac{\overset{\text{(i.h.)}}{C'\theta\Rightarrow C''\theta}}{C'\theta\Rightarrow B''\theta+C''\theta}\ (\text{+r.})(\text{Lem.2})}{B'\theta+C'\theta\Rightarrow B''\theta+C''\theta}\ (\text{+l.})(\text{Lem.2})$$

Note that $B'\theta{+}C'\theta = (B'{+}C')\theta$ because $B'{+}C' = \forall y((B'\to y)\to(C'\to y)\to y)$ and $y\notin\{x_1,x_2,\ldots,x_n\}$

If $A = \forall yB$, then $A' = \forall yB'$ for some $B'\in\mathrm{DF}(B)$. Then (1) is shown as follows.

$$\dfrac{\dfrac{\overset{\text{(i.h.)}}{B\theta[y{:=}p]\Rightarrow B'\theta[y{:=}p]}}{\forall y(B\theta)\Rightarrow B'\theta[y{:=}p]}\ (\forall\text{l.})}{\forall y(B\theta)\Rightarrow\forall y(B'\theta)}\ (\forall\text{r.})$$

where p is a fresh propositional parameter. Note that $\forall y(B\theta) = (\forall yB)\theta$ because $y \notin \{x_1, x_2, \ldots, x_n\}$.
QED

Lemma 4 *If $A \in \mathrm{DF}(B[x{:=}C])$ and C is a closed formula, then there are formulas $B^{\mathrm{df}} \in \mathrm{DF}(B)$ and $C^{\mathrm{df}} \in \mathrm{DF}(C)$ such that $A \Leftrightarrow_{\mathrm{LJ2^{df}}} B^{\mathrm{df}}[x{:=}C^{\mathrm{df}}]$.*

Proof By induction on B.

(Case 1: $B = x$) Define $B^{\mathrm{df}} = x$ and $C^{\mathrm{df}} = A$.

(Case 2: B is a propositional symbol other than x, or $B = \bot$) Define $B^{\mathrm{df}} = A = B$.

(Case 3: $B = B_1 \vee B_2$) Assume $A \in \mathrm{DF}((B_1 \vee B_2)[x{:=}C]) = \mathrm{DF}(B_1[x{:=}C] \vee B_2[x{:=}C])$. By the definition of DF(), $A = \forall y((D_1 \to y) \to (D_2 \to y) \to y)$ for some D_1, D_2, and y such that $D_1 \in \mathrm{DF}(B_1[x{:=}C])$, $D_2 \in \mathrm{DF}(B_2[x{:=}C])$, and $y \notin \mathrm{FV}(D_1, D_2)$. Then, by the induction hypotheses, there are formulas $B_1^{\mathrm{df}} \in \mathrm{DF}(B_1)$, $C_1^{\mathrm{df}} \in \mathrm{DF}(C)$, $B_2^{\mathrm{df}} \in \mathrm{DF}(B_2)$, $C_2^{\mathrm{df}} \in \mathrm{DF}(C)$, such that

$$D_1 \Leftrightarrow_{\mathrm{LJ2^{df}}} B_1^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}] \tag{2}$$

and

$$D_2 \Leftrightarrow_{\mathrm{LJ2^{df}}} B_2^{\mathrm{df}}[x{:=}C_2^{\mathrm{df}}]. \tag{3}$$

On the other hand, by Lemma 3, we have $C_1^{\mathrm{df}} \Leftrightarrow_{\mathrm{LJ2^{df}}} C_2^{\mathrm{df}}$ and therefore $B_2^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}] \Leftrightarrow_{\mathrm{LJ2^{df}}} B_2^{\mathrm{df}}[x{:=}C_2^{\mathrm{df}}]$; this and (3) imply

$$D_2 \Leftrightarrow_{\mathrm{LJ2^{df}}} B_2^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}]. \tag{4}$$

Now let z be a fresh propositional variable and define $B^{\mathrm{df}} = \forall z((B_1^{\mathrm{df}} \to z) \to (B_2^{\mathrm{df}} \to z) \to z)$. We have

$$\begin{aligned}
A &= \forall y((D_1 \to y) \to (D_2 \to y) \to y) \\
&\Leftrightarrow_{\mathrm{LJ2^{df}}} \forall z((D_1 \to z) \to (D_2 \to z) \to z) \\
&\Leftrightarrow_{\mathrm{LJ2^{df}}} \forall z\big((B_1^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}] \to z) \to (B_2^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}] \to z) \to z\big) && (\because (2) \text{ and } (4)) \\
&= \big(\forall z\big((B_1^{\mathrm{df}} \to z) \to (B_2^{\mathrm{df}} \to z) \to z\big)\big)[x{:=}C_1^{\mathrm{df}}] \\
&= B^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}].
\end{aligned}$$

(Case 4: $B = B_1 \circ B_2$ and $\circ \in \{\to, \wedge\}$) Assume $A \in \mathrm{DF}((B_1 \circ B_2)[x{:=}C]) = \mathrm{DF}(B_1[x{:=}C] \circ B_2[x{:=}C])$. By the definition of DF(), $A = D_1 \circ D_2$ for some D_1, D_2, and y such that $D_1 \in \mathrm{DF}(B_1[x{:=}C])$ and $D_2 \in \mathrm{DF}(B_2[x{:=}C])$. Then, by the induction hypotheses, there are formulas $B_1^{\mathrm{df}} \in \mathrm{DF}(B_1)$, $C_1^{\mathrm{df}} \in \mathrm{DF}(C)$, $B_2^{\mathrm{df}} \in \mathrm{DF}(B_2)$, $C_2^{\mathrm{df}} \in \mathrm{DF}(C)$, such that

$$D_1 \Leftrightarrow_{\mathrm{LJ2^{df}}} B_1^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}] \tag{5}$$

and

$$D_2 \Leftrightarrow_{\mathrm{LJ2^{df}}} B_2^{\mathrm{df}}[x{:=}C_2^{\mathrm{df}}]; \tag{6}$$

and we have

$$D_2 \Leftrightarrow_{\mathrm{LJ2^{df}}} B_2^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}] \tag{7}$$

by the similar argument to (4). Then define $B^{\mathrm{df}} = B_1^{\mathrm{df}} \circ B_2^{\mathrm{df}}$; we have

$$A = D_1 \circ D_2$$

$$\Leftrightarrow_{\mathrm{LJ2}^{\mathrm{df}}} B_1^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}] \circ B_2^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}] \qquad (\because (5) \text{ and } (7))$$
$$= B^{\mathrm{df}}[x{:=}C_1^{\mathrm{df}}].$$

(Case 5: $B = \nabla y B_1$ and $\nabla \in \{\forall, \exists\}$) If $y = x$, then define $B^{\mathrm{df}} = A$. In the following, we assume $y \neq x$ and $A \in \mathrm{DF}((\nabla y B_1)[x{:=}C]) = \mathrm{DF}(\nabla y(B_1[x{:=}C]))$. By the definition of DF(), $A = \nabla y D$ for some D such that $D \in \mathrm{DF}(B_1[x{:=}C])$. Then, by the induction hypothesis, there are formulas $B_1^{\mathrm{df}} \in \mathrm{DF}(B_1)$ and $C^{\mathrm{df}} \in \mathrm{DF}(C)$ such that

$$D \Leftrightarrow_{\mathrm{LJ2}^{\mathrm{df}}} B_1^{\mathrm{df}}[x{:=}C^{\mathrm{df}}]. \tag{8}$$

Then define $B^{\mathrm{df}} = \nabla y B_1^{\mathrm{df}}$; we have

$$A = \nabla y D$$
$$\Leftrightarrow_{\mathrm{LJ2}^{\mathrm{df}}} \nabla y(B_1^{\mathrm{df}}[x{:=}C^{\mathrm{df}}]) \qquad (\because (8))$$
$$= B^{\mathrm{df}}[x{:=}C^{\mathrm{df}}].$$

QED

Lemma 5 *If* **LJ2** $\vdash A_1, A_2, \ldots, A_n \Rightarrow A_0$, *then* $\mathbf{LJ2}^{\mathrm{df}} \vdash A_1^{\mathrm{df}}, A_2^{\mathrm{df}}, \ldots, A_n^{\mathrm{df}} \Rightarrow A_0^{\mathrm{df}}$, *for some* $A_0^{\mathrm{df}}, A_1^{\mathrm{df}}, \ldots, A_n^{\mathrm{df}}$ *such that* $A_i^{\mathrm{df}} \in \mathrm{DF}(A_i)$.

Proof By induction on the proof of $A_1, A_2, \ldots, A_n \Rightarrow A_0$ in **LJ2**. Here we show a few cases.

Suppose that $A_1 = B \vee C$ and that the last inference rule is

$$\frac{B, A_2, \ldots, A_n \Rightarrow A_0 \quad C, A_2, \ldots, A_n \Rightarrow A_0}{B \vee C,\ A_2, \ldots, A_n \Rightarrow A_0.} \ (\vee \text{ left})$$

Then we have

$$\cfrac{\overset{\text{(i.h.)}}{B^{\mathrm{df}}, A_2^{\mathrm{df}}, \ldots, A_n^{\mathrm{df}} \Rightarrow A_0^{\mathrm{df}}} \quad \begin{matrix}\text{(i.h.)} \\ C^{\mathrm{df}}, A_2^{\mathrm{df}\prime}, \ldots, A_n^{\mathrm{df}\prime} \Rightarrow A_0^{\mathrm{df}\prime} \\ \vdots\ (*) \\ C^{\mathrm{df}}, A_2^{\mathrm{df}}, \ldots, A_n^{\mathrm{df}} \Rightarrow A_0^{\mathrm{df}}\end{matrix}}{B^{\mathrm{df}}{+}C^{\mathrm{df}},\ A_2^{\mathrm{df}}, \ldots, A_n^{\mathrm{df}} \Rightarrow A_0^{\mathrm{df}}} \ (+ \text{ left}) \text{ by Lemma 2}$$

where $(*)$ is obtained by the fact $A_i^{\mathrm{df}} \Leftrightarrow_{\mathrm{LJ2}^{\mathrm{df}}} A_i^{\mathrm{df}\prime}$ (Lemma 3) and the cut rule.

Suppose that $A_1 = \forall x B$ and that the last inference rule is

$$\frac{B[x{:=}C], A_2, \ldots, A_n \Rightarrow A_0}{\forall x B, A_2, \ldots, A_n \Rightarrow A_0.} \ (\forall \text{ left})$$

Then we have

$$\cfrac{\cfrac{\overset{\text{(i.h.)}}{(B[x{:=}C])^{\mathrm{df}}, A_2^{\mathrm{df}}, \ldots, A_n^{\mathrm{df}} \Rightarrow A_0^{\mathrm{df}}}}{B^{\mathrm{df}}[x{:=}C^{\mathrm{df}}], A_2^{\mathrm{df}}, \ldots, A_n^{\mathrm{df}} \Rightarrow A_0^{\mathrm{df}}} \ (*)}{\forall x B^{\mathrm{df}}, A_2^{\mathrm{df}}, \ldots, A_n^{\mathrm{df}} \Rightarrow A_0^{\mathrm{df}}} \ (\forall \text{ left})$$

where $(*)$ is obtained by Lemma 4 and the cut rule. **QED**

Note that if both A' and A'' are elements of $\mathrm{DF}(A)$, then they are α-equivalent (that is, A'' is obtained from A' by renaming bound variables), and $A' \Leftrightarrow_{\mathrm{LJ2}^{\mathrm{df}}} A''$ (Lemma 3). Hence, from now on, we will identify A' with A''; in other words, given A, we consider A^{df} to be a unique formula. For example, the statement of Lemma 5 can be simplified as follows: If $\mathbf{LJ2} \vdash \Gamma \Rightarrow A$, then $\mathbf{LJ2}^{\mathrm{df}} \vdash \Gamma^{\mathrm{df}} \Rightarrow A^{\mathrm{df}}$, where $\Gamma^{\mathrm{df}} = \{A^{\mathrm{df}} \mid A \in \Gamma\}$.

Theorem 6 *For any sequent $\Gamma \Rightarrow A$, the following three conditions are equivalent.*

(1) $\mathbf{LJ2} \vdash \Gamma \Rightarrow A$.

(2) $\mathbf{LJ2} \vdash \Gamma^{\mathrm{df}} \Rightarrow A^{\mathrm{df}}$.

(3) $\mathbf{LJ2}^{\mathrm{df}} \vdash \Gamma^{\mathrm{df}} \Rightarrow A^{\mathrm{df}}$.

Proof $(1 \Rightarrow 3)$ is implied by Lemma 5. $(3 \Rightarrow 2)$ is trivial. $(2 \Rightarrow 1)$ is implied by Lemma 3. **QED**

Corollary 7 *For any disjunction free sequent $\Gamma \Rightarrow A$, the following two conditions are equivalent.*

(1) $\mathbf{LJ2} \vdash \Gamma \Rightarrow A$.

(2) $\mathbf{LJ2}^{\mathrm{df}} \vdash \Gamma \Rightarrow A$.

5 Kripke model

If $\langle \mathcal{W}, \mathcal{R} \rangle$ is a partially ordered set, then $\mathrm{UP}(\mathcal{W}, \mathcal{R})$ denotes the set of $\mathcal{R}$-upward closed subsets of $\mathcal{W}$; that is

$$\mathrm{UP}(\mathcal{W}, \mathcal{R}) = \{\alpha \subseteq \mathcal{W} \mid \forall a, b \in \mathcal{W}[\text{ if } a \in \alpha \text{ and } a\mathcal{R}b, \text{ then } b \in \alpha.]\}$$

We define a *Kripke model* (for second order propositional intuitionistic logic) to be a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ as follows.

- $\mathcal{W}$ is a non-empty set (of "*worlds*").
- $\mathcal{R}$ is a partial order ("*accessibility relation*") on $\mathcal{W}$.
- $\mathcal{D} = \{\mathcal{D}_a \mid a \in \mathcal{W}\}$ where each $\mathcal{D}_a$ is a non-empty subset of $\mathrm{UP}(\mathcal{W}, \mathcal{R})$ satisfying the condition: $a\mathcal{R}b \Rightarrow \mathcal{D}_a \subseteq \mathcal{D}_b$. ($\mathcal{D}_a$ is the *domain* of quantification at the world a).
- $\mathcal{I} = \{\mathcal{I}_p \mid p \in \mathbf{PropPara}\}$ where each $\mathcal{I}_p$ is an element of $\mathrm{UP}(\mathcal{W}, \mathcal{R})$. ($\mathcal{I}_p$ is the "interpretation" of the propositional parameter p.)

$\mathcal{M}$ is said to be *constant domain* if $\mathcal{D}_a = \mathcal{D}_b$ for all $a, b \in \mathcal{W}$.

Given a Kripke model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$, we extend the language by adding "propositional constant" $\overline{\alpha}$ for each $\alpha \in \mathrm{UP}(\mathcal{W}, \mathcal{R})$. We call $\overline{\alpha}$ the *name* of α, and define $\mathbf{Name}(\mathcal{W}, \mathcal{R}) = \{\overline{\alpha} \mid \alpha \in \mathrm{UP}(\mathcal{W}, \mathcal{R})\}$. The definition of formulas (Section 2) are extended by the clause: "Each name $\overline{\alpha}$ is a formula." We call a formula of this extended language an "$\mathcal{M}$-*formula*"; in other words, an $\mathcal{M}$-formula is a formula in which names of elements of $\mathrm{UP}(\mathcal{W}, \mathcal{R})$ can be used like propositional symbols.

For a Kripke model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$, $a \in \mathcal{M}$, and a closed $\mathcal{M}$-formula A, the relation $\mathcal{M}, a \models A$ ("A holds at a world a in $\mathcal{M}$") is defined inductively as follows ("$\mathcal{M}$" is often omitted).

$$a \models p \iff a \in \mathcal{I}_p, \text{ for } p \in \mathbf{PropPara}.$$

$$a \models \overline{\alpha} \iff a \in \alpha, \text{ for } \overline{\alpha} \in \mathbf{Name}(\mathcal{W}, \mathcal{R}).$$

$$a \not\models \bot.$$

$$a \models A \to B \iff [(b \not\models A) \text{ or } (b \models B)] \text{ for all } b \text{ such that } a\mathcal{R}b.$$

$a \models A \wedge B \iff (a \models A)$ and $(a \models B)$.
$a \models A \vee B \iff (a \models A)$ or $(a \models B)$.
$a \models \forall x A \iff b \models A[x := \overline{\alpha}]$ for all b such that $a\mathcal{R}b$ and all $\alpha \in \mathcal{D}_b$.
$a \models \exists x A \iff a \models A[x := \overline{\alpha}]$ for some $\alpha \in \mathcal{D}_a$.

We define $[\![A]\!]_{\mathcal{M}}$, which is a subset of $\mathcal{W}$, by

$$[\![A]\!]_{\mathcal{M}} = \{a \in \mathcal{W} \mid \mathcal{M}, a \models A\}.$$

The following equations are easily verified by the definition.

$[\![p]\!]_{\mathcal{M}} = \mathcal{I}_p$. $[\![\overline{\alpha}]\!]_{\mathcal{M}} = \alpha$. $[\![\bot]\!]_{\mathcal{M}} = \emptyset$.
$[\![A \wedge B]\!]_{\mathcal{M}} = [\![A]\!]_{\mathcal{M}} \cap [\![B]\!]_{\mathcal{M}}$. $[\![A \vee B]\!]_{\mathcal{M}} = [\![A]\!]_{\mathcal{M}} \cup [\![B]\!]_{\mathcal{M}}$.

Lemma 8 *If $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ is a Kripke model, then $[\![A]\!]_{\mathcal{M}} \in \mathrm{UP}(\mathcal{M}, \mathcal{R})$ for any closed $\mathcal{M}$-formula A.*

Proof By induction on the complexity (i.e., the number of occurrences of $\rightarrow$, $\wedge$, $\vee$, $\forall$, and $\exists$) of A, we show the following: If $a \models A$ and $a\mathcal{R}b$, then $b \models A$. Note that the condition "$a\mathcal{R}b \Rightarrow \mathcal{D}_a \subseteq \mathcal{D}_b$" is used in the case that $A = \exists x B$. **QED**

From now on, $\mathcal{M}$ always denotes a Kripke model $\langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$.

We say that a closed $\mathcal{M}$-formula A is *interpretable at* a world a if the following two conditions hold for all $p \in$ **PropPara** and all $\overline{\alpha} \in$ **Name**$(\mathcal{W}, \mathcal{R})$:

If p occurs in A, then $\mathcal{I}_p \in \mathcal{D}_a$. If $\overline{\alpha}$ occurs in A, then $\alpha \in \mathcal{D}_a$.

$\mathcal{M}$ is said to be *full* if the following condition holds for any closed $\mathcal{M}$-formula A and any world $a \in \mathcal{W}$.

(†) If A is interpretable at a, then $[\![A]\!]_{\mathcal{M}} \in \mathcal{D}_a$.

If $\mathcal{M}$ is full, then each domain $\mathcal{D}_a$ contains $\emptyset$ ($\because$ take $\bot$ as A in (†)), and $\mathcal{D}_a$ is closed under $\cap$ and $\cup$ ($\because$ take $\overline{\alpha} \wedge \overline{\beta}$ and $\overline{\alpha} \vee \overline{\beta}$ as A in (†)).

On the other hand, $\mathcal{M}$ is said to be *df-full* if the condition (†) holds for any *disjunction free* closed $\mathcal{M}$-formula A and any world a. Any full model is df-full; on the other hand, the converse is not true in general because a domain $\mathcal{D}_a$ might not be closed under $\cup$ in a df-full model.

(Remark) The term "full" is due to Sørensen and Urzyczyn [7]. However the definition is slightly different. $\mathcal{M}$ is said to be *full in the sense of Sørensen and Urzyczyn* if the following condition holds for any closed $\mathcal{M}$-formula A and any world $a \in \mathcal{W}$.

If A is interpretable at a, then there is an element $\alpha \in \mathcal{D}_a$ such that $[\![A]\!]_{\mathcal{M}}$ and α coincide above a; that is, $\{a' \mid a' \in [\![A]\!]_{\mathcal{M}}$ and $a\mathcal{R}a'\} = \{a' \mid a' \in \alpha$ and $a\mathcal{R}a'\}$.

Our definition of "full" is stronger and simpler than this.

(Example) Let $\alpha = \{a, b, c\}$, $\beta = \{b, c\}$, $\gamma = \{c\}$. We define a Kripke model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ as follows: $\mathcal{W} = \{a, b, c\}$. $a\mathcal{R}b\mathcal{R}c$. $\mathcal{D}_a = \{\alpha, \beta, \emptyset\}$. $\mathcal{D}_b = \mathcal{D}_c = \{\alpha, \beta, \gamma, \emptyset\}$. We can easily verify that this is a full Kripke model. Now consider two $\mathcal{M}$-formulas L and R as

$$L = \forall x \big(\overline{\beta} \vee ((\overline{\beta} \rightarrow x) \vee \neg x)\big), \quad R = \overline{\beta} \vee \forall x ((\overline{\beta} \rightarrow x) \vee \neg x),$$

which are instances of, respectively, the left-hand and right-hand sides of the constant domain axiom of **LJ2CD**. Then, $a \models L$ ($\because a \models \overline{\beta} \rightarrow \overline{\alpha}, a \models \overline{\beta} \rightarrow \overline{\beta}, a \models \neg \overline{\emptyset}, b \models \overline{\beta}, c \models \overline{\beta}$), and $a \not\models R$ ($\because a \not\models \overline{\beta}, b \not\models \overline{\beta} \rightarrow \overline{\gamma}, b \not\models \neg \overline{\gamma}$).

6 Soundness of sequent calculi

Lemma 9 *Let* $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ *be a Kripke model,* x *be a propositional variable,* A *be an* $\mathcal{M}$*-formula such that* $\mathrm{FV}(A) \subseteq \{x\}$*, and* B, C *be closed* $\mathcal{M}$*-formulas. If* $[\![B]\!]_{\mathcal{M}} = [\![C]\!]_{\mathcal{M}}$*, then* $[\![A[x:=B]]\!]_{\mathcal{M}} = [\![A[x:=C]]\!]_{\mathcal{M}}$.

Proof By induction on the complexity of A. **QED**

Lemma 10 *Suppose that two Kripke models* $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ *and* $\mathcal{M}' = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I}' \rangle$ *have* $\langle \mathcal{W}, \mathcal{R}, \mathcal{D} \rangle$ *in common.*

(1) *Let* A *be a closed* $\mathcal{M}$*-formula (and also a closed* $\mathcal{M}'$*-formula). If* $\mathcal{I}_p = \mathcal{I}'_p$ *for all* $p \in \mathrm{PP}(A)$*, then* $[\![A]\!]_{\mathcal{M}} = [\![A]\!]_{\mathcal{M}'}$.

(2) *If* $\mathcal{M}$ *is full, then so is* $\mathcal{M}'$.

Proof **(1)** By induction on the complexity of A.

(2) Assume that $\mathcal{M}'$ is not full; that is, there are a closed $\mathcal{M}'$-formula A and a world $a \in \mathcal{W}$ such that A is interpretable at a in $\mathcal{M}'$ and $[\![A]\!]_{\mathcal{M}'} \notin \mathcal{D}_a$. Let A° be a formula obtained from A by replacing each $p_i \in \mathrm{PP}(A)$ with the name $\overline{\mathcal{I}'_{p_i}}$. A° is a closed $\mathcal{M}$-formula (for $\overline{\mathcal{I}'_{p_i}}$ is also a name in $\mathcal{M}$), and we have $[\![A^\circ]\!]_{\mathcal{M}} = [\![A^\circ]\!]_{\mathcal{M}'} = [\![A]\!]_{\mathcal{M}'}$ by Lemma 10(1) and Lemma 9. This implies that $\mathcal{M}$ is not full, because A° is interpretable at a and $[\![A^\circ]\!]_{\mathcal{M}} \notin \mathcal{D}_a$. **QED**

We say that a sequent $B_1, B_2, \ldots, B_n \Rightarrow A$ is *valid* in a Kripke model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ if the following condition holds for any world $a \in \mathcal{W}$:

> If $B_1, B_2, \ldots, B_n$ and A are interpretable at a and if $\mathcal{M}, a \models B_1$, $\mathcal{M}, a \models B_2$, …, and $\mathcal{M}, a \models B_n$, then $\mathcal{M}, a \models A$.

Moreover, a closed formula A is said to be *valid* in $\mathcal{M}$ if the following condition holds for any world a:

> If A is interpretable at a, then $\mathcal{M}, a \models A$.

In other words, a formula A is valid if and only if the sequent $\Rightarrow A$ is valid.

Theorem 11 (Soundness of LJ2) *If* **LJ2** $\vdash \Gamma \Rightarrow A$*, then this sequent is valid in any full Kripke model.*

Proof By induction on the proof of $\Gamma \Rightarrow A$ in **LJ2**. This theorem is trivial when $\Gamma \Rightarrow A$ is an axiom $A \Rightarrow A$ or $\bot \Rightarrow A$.

Then we show that the property "valid in any full Kripke model" is preserved through any inference rule R. To show this, we introduce a condition (♡) for a sequent, say $B_1, B_2, \ldots, B_n \Rightarrow A$, as follows.

> (♡) There is a full Kripke model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ and a world $a \in \mathcal{W}$ such that $B_1, B_2, \ldots, B_n$ and A are interpretable at a, $\mathcal{M}, a \models B_1$, $\mathcal{M}, a \models B_2$, …, $\mathcal{M}, a \models B_n$, and $\mathcal{M}, a \not\models A$.

This is the negation of the property in question. Now we will show the following: *If the conclusion of* R *satisfies the condition (♡), then so does one of the premises of* R.

(Case 1) $R = $ (cut):

$$\frac{\Delta \Rightarrow C \quad C, \Pi \Rightarrow A}{\Delta, \Pi \Rightarrow A}\ (\text{cut})$$

(Subcase 1-1: C is interpretable at a.) If $\mathcal{M}, a \models C$, then the right premise $(C, \Pi \Rightarrow A)$ satisfies the condition (♡); otherwise the left premise does.

(Subcase 1-2: C is not interpretable at a.) In this case, some propositional parameters, say $q_1, q_2, \ldots, q_k$, violate the interpretability of C. Then we modify the Kripke model $\mathcal{M}$ by replacing the interpretation of $q_1, q_2, \ldots, q_k$ to $\emptyset$ (or, to another element in $\mathcal{D}_a$). That is, we define $\mathcal{M}' = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I}' \rangle$ where

$$\mathcal{I}'_p = \begin{cases} \mathcal{I}_p & \text{if } p \notin \{q_1, q_2, \ldots, q_k\} \\ \emptyset & \text{if } p \in \{q_1, q_2, \ldots, q_k\} \end{cases}$$

This $\mathcal{M}'$ is full (by Lemma 10(2)) and all the formulas in (Δ, Π, A, C) are interpretable at a. Moreover, by Lemma 10(1), we have $[\![F]\!]_{\mathcal{M}'} = [\![F]\!]_{\mathcal{M}}$ for all F in (Γ, Π, A) since $q_i \notin \mathrm{PP}(F)$. Then, similarly to the subcase 1-1, either the left premise or the right premise satisfies the condition (♡).

(Case 2) $R = (\forall \text{ left})$:

$$\frac{B[x{:=}C], \Delta \Rightarrow A}{\forall x B, \Delta \Rightarrow A}\ (\forall \text{ left})$$

(Subcase 2-1: C is interpretable at a.) Since $\mathcal{M}$ is full, there is $\gamma \in \mathcal{D}_a$ such that $[\![C]\!]_{\mathcal{M}} = \gamma$. Then we have $\mathcal{M}, a \models \forall x B$ ($\because$ (♡) of $\forall x B, \Delta \Rightarrow A$), $\mathcal{M}, a \models B[x{:=}\overline{\gamma}]$ ($\because$ definition of Kripke model), and $\mathcal{M}, a \models B[x{:=}C]$ ($\because$ Lemma 9), which implies (♡) for $B[x{:=}C], \Delta \Rightarrow A$.

(Subcase 2-2: C is not interpretable at a.) Similarly to subcase 1-2, we modify $\mathcal{M}$ by replacing the interpretation of propositional parameters in C that violate the interpretability. Then, similarly to subcase 2-1, we get the condition (♡) for $B[x{:=}C], \Delta \Rightarrow A$.

(Case 3) $R = (\forall \text{ right})$:

$$\frac{\Gamma \Rightarrow A'[x{:=}p]}{\Gamma \Rightarrow \forall x A'}\ (\forall \text{ right})$$

By the condition (♡) of $\Gamma \Rightarrow \forall x A'$, we have $\mathcal{M}, a \not\models \forall x A'$; therefore there are $b \in \mathcal{W}$ and $\alpha \in \mathcal{D}_b$ such that $a \mathcal{R} b$ and $\mathcal{M}, b \not\models A'[x{:=}\overline{\alpha}]$ (note that $\mathcal{M}, b \models \Gamma$). We modify $\mathcal{M}$ by replacing the interpretation of p to α. The resulting model, called $\mathcal{M}'$, is full and we have $\mathcal{M}', b \models \Gamma$ and $\mathcal{M}', b \not\models A'[x{:=}p]$ by Lemmas 9 and 10. This is the condition (♡) of $\Gamma \Rightarrow A'[x{:=}p]$.

The other cases are similar. **QED**

Recall that $\mathbf{LJ2}^{\mathrm{df}}$ is the system in which all the sequents are disjunction free. The above proof holds for $\mathbf{LJ2}^{\mathrm{df}}$ and df-full models (note that, in case 2, the formula C is disjunction free). Therefore we have the following:

Theorem 12 (Soundness of $\mathbf{LJ2}^{\mathrm{df}}$) *If $\mathbf{LJ2}^{\mathrm{df}} \vdash \Gamma \Rightarrow A$, then this sequent is valid in any df-full Kripke model.*

By Theorem 11, we can show that **LJ2CD** is strictly stronger than **LJ2**:

Theorem 13 LJ2 *does not prove the constant domain axiom $\forall x(A \vee B) \Rightarrow A \vee \forall x B$ in general.*

Proof We have

$$\mathbf{LJ2} \not\vdash \forall x(p \vee ((p \to x) \vee \neg x) \Rightarrow p \vee \forall x((p \to x) \vee x)$$

by the example in the previous section and Theorem 11. **QED**

On the other hand, we can easily verify that the constant domain axiom is valid in any constant domain Kripke model, and the proof of Lemma 11 is available for constant domain Kripke models. Then we have:

Theorem 14 (Soundness of LJ2CD) *If* **LJ2CD** $\vdash \Gamma \Rightarrow A$, *then this sequent is valid in any constant domain full Kripke model.*

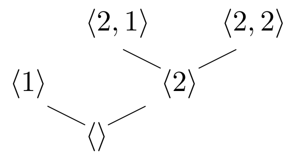

Figure 1: Label tree

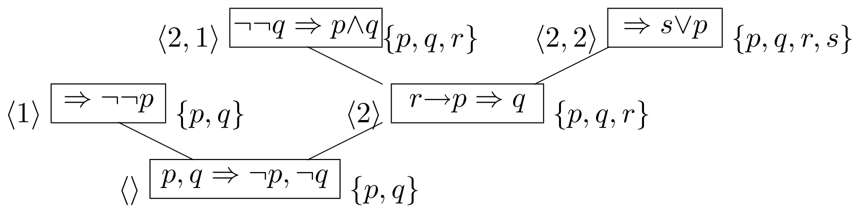

Figure 2: Sequent tree

7 Sequent tree calculus

In this section, we introduce a "sequent tree calculus", and prove its completeness.

If $n_1, n_2, \dots, n_k$ are natural numbers, then the finite sequence $\langle n_1, n_2, \dots, n_k\rangle$ is called a *label*. In the following, $\langle\vec{n}\rangle$ denotes a label. We define a partial order $\preceq$ between labels by $\langle\vec{m}\rangle \preceq \langle\vec{n}\rangle$ if and only if $\langle\vec{m}\rangle$ is a prefix of $\langle\vec{n}\rangle$, that is, $\langle\vec{n}\rangle = \langle\vec{m}, a_1, a_2, \dots, a_k\rangle$ for some $k \geq 0$ and some a_1, a_2,..., a_k. We say that a (finite or infinite) set T of labels is a *label tree* if T contains the empty label $\langle\rangle$ and if T is closed under prefix, that is, if $\langle\vec{m}\rangle \preceq \langle\vec{n}\rangle \in T$, then $\langle\vec{m}\rangle \in T$. For example, $\{\langle\rangle, \langle 1\rangle, \langle 2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$ is a label tree, where $\langle\rangle$ is the root node, nodes $\langle 1\rangle$ and $\langle 2\rangle$ are the children of the root node, and nodes $\langle 2,1\rangle$ and $\langle 2,2\rangle$ are the children of $\langle 2\rangle$, as illustrated in Figure 1.

If Γ is a set of closed formulas and Δ is a non-empty set of closed formulas, then the expression $\Gamma \Rightarrow \Delta$ is called a *c-sequent* (Γ and Δ may be infinite sets). A *sequent tree* is a label tree each node of which is associated with a pair of a c-sequent and a set of propositional parameters. By a *"finite sequent tree"*, we mean a sequent tree whose label tree is a finite set and all the c-sequents and the sets of propositional parameters are finite.

Figure 2 is an example of finite sequent tree, in which the root is associated with the c-sequent $(p, q \Rightarrow \neg p, \neg q)$ and the set $\{p, q\}$, and the inner node $\langle 2\rangle$ is associated with the c-sequent $(r{\rightarrow}p \Rightarrow q)$ and the set $\{p, q, r\}$.

A *labelled formula* is an expression

$$\langle\vec{n}\rangle : A$$

where $\langle\vec{n}\rangle$ is a label and A is a formula. In the following, $\Gamma, \Delta, \dots$ will denote (finite or infinite) sets of labelled formulas. Using labelled formulas, we will write any sequent tree in compact form as

$$\Gamma \overset{T,f}{\Rightarrow} \Delta$$

where T is its label tree, f is a function from T to $\mathfrak{P}(\mathbf{PropPara})$ such that $f(\langle\vec{n}\rangle)$ is the set of propositional parameters that is associated with the node $\langle\vec{n}\rangle$, and Γ and Δ are sets of labelled formulas such that $(\langle\vec{n}\rangle : A) \in \Gamma$ (or Δ) if and only if the formula A is in the left (or right, respectively) part of the c-sequent that is associated with the node $\langle\vec{n}\rangle$. For example, the sequent tree of Figure 2 is written as

$$\langle\rangle : p,\ \langle\rangle : q,\ \langle 2\rangle : r{\rightarrow}p,\ \langle 2,1\rangle : \neg\neg q \overset{T,f}{\Rightarrow} \langle\rangle : \neg p,\ \langle\rangle : \neg q,\ \langle 1\rangle : \neg\neg p,\ \langle 2\rangle : q,\ \langle 2,1\rangle : p{\wedge}q,\ \langle 2,2\rangle : s{\vee}p$$

where $T = \{\langle\rangle, \langle 1\rangle, \langle 2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$, $f(\langle\rangle) = f(\langle 1\rangle) = \{p,q\}$, $f(\langle 2\rangle) = f(\langle 2,1\rangle) = \{p,q,r\}$, and $f(\langle 2,2\rangle) = \{p,q,r,s\}$.

We say that a sequent tree $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is *legal* if is satisfies the following two conditions. (1) If $(\langle\vec{n}\rangle : A) \in \Gamma \cup \Delta$, then $\mathrm{PP}(A) \subseteq f(\langle\vec{n}\rangle)$; in other words, $f(\langle\vec{n}\rangle)$ denotes the set of propositional parameters that can be used in the formulas in the node $\langle\vec{n}\rangle$. (2) If $\langle\vec{m}\rangle \preceq \langle\vec{n}\rangle$, then $f(\langle\vec{m}\rangle) \subseteq f(\langle\vec{n}\rangle)$; therefore, available propositional parameters are inherited (sometimes new elements are added) by the children of the node. All the sequent tree that we treat are legal; so we will use the term "sequent tree" to mean "legal sequent tree" from now on.

We introduce a system **TLJ2**, which derives finite sequent trees. Axioms of **TLJ2** are finite sequent trees of the forms

$$\langle\vec{n}\rangle : A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A \qquad \text{and} \qquad \langle\vec{n}\rangle : \bot,\ \Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A.$$

Inference rules of **TLJ2** are the following eleven ((heredity), ($\wedge$ left), ..., ($\exists$ right)). Note that the cut rule is not there. In those rules, $\langle\vec{n}\rangle$ is called the *operated node.*

$$\frac{\langle\vec{n},a\rangle : A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{n}\rangle : A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}\ (\text{heredity})(\dagger 1)$$

($\dagger$1) Proviso: $\mathrm{PP}(A) \subseteq f(\langle\vec{n}\rangle)$ (therefore the lower sequent tree is legal).

$$\frac{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A \qquad \langle\vec{n}\rangle : B,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{n}\rangle : A \to B,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}\ (\to \text{left}) \qquad \frac{\langle\vec{n},a\rangle : A,\ \Gamma \overset{T^+,f^+}{\Rightarrow} \Delta,\ \langle\vec{n},a\rangle : B}{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A \to B}\ (\to \text{right})(\dagger 2)$$

($\dagger$2) Proviso: $\langle\vec{n},a\rangle$ is a leaf of T^+, the c-sequent associated with $\langle\vec{n},a\rangle$ is just $(A \Rightarrow B)$ (this node has no other formula than A, B), $T = T^+ \setminus \{\langle\vec{n},a\rangle\}$, $f^+(\langle\vec{n},a\rangle) = f^+(\langle\vec{n}\rangle)$, and $f = f^+ \restriction_T$.

$$\frac{\langle\vec{n}\rangle : A,\ \langle\vec{n}\rangle : B,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{n}\rangle : A \wedge B,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}\ (\wedge \text{ left}) \qquad \frac{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A \qquad \Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : B}{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A \wedge B}\ (\wedge \text{ right})$$

$$\frac{\langle\vec{n}\rangle : A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta \qquad \langle\vec{n}\rangle : B,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{n}\rangle : A \vee B,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}\ (\vee \text{ left}) \qquad \frac{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A,\ \langle\vec{n}\rangle : B}{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A \vee B}\ (\vee \text{ right})$$

$$\frac{\langle\vec{n}\rangle : A[x{:=}B],\ \Gamma \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{n}\rangle : \forall x A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}\ (\forall \text{ left})(\dagger 3) \qquad \frac{\Gamma \overset{T^+,f^+}{\Rightarrow} \Delta,\ \langle\vec{n},a\rangle : A[x{:=}p]}{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : \forall x A}\ (\forall \text{ right})(\dagger 4)$$

($\dagger$3) Proviso: B is a closed formula.
($\dagger$4) Proviso: p is a propositional parameter (called eigenvariable) which does not occur in the lower sequent tree, $\langle\vec{n},a\rangle$ is a leaf of T^+, the c-sequent associated with $\langle\vec{n},a\rangle$ is just $(\Rightarrow A[x{:=}p])$, $T = T^+ \setminus \{\langle\vec{n},a\rangle\}$, $f^+(\langle\vec{n},a\rangle) = f^+(\langle\vec{n}\rangle) \cup \{p\}$, and $f = f^+ \restriction_T$.

$$\frac{\langle\vec{n}\rangle : A[x{:=}p],\ \Gamma \overset{T,f^+}{\Rightarrow} \Delta}{\langle\vec{n}\rangle : \exists x A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}\ (\exists \text{ left})(\dagger 5) \qquad \frac{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A[x{:=}B]}{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : \exists x A}\ (\exists \text{ right})(\dagger 6)$$

(†5) Proviso: p is a propositional parameter (called eigenvariable) which does not occur in the lower sequent tree, and f^+ and f satisfy the following equation.

$$f^+(\langle\vec{m}\rangle) = \begin{cases} f(\langle\vec{m}\rangle) \cup \{p\} & \text{if } \langle\vec{m}\rangle \text{ is equal to or a descendant of } \langle\vec{n}\rangle, \\ f(\langle\vec{m}\rangle) & \text{otherwise.} \end{cases}$$

(†6) B is a closed formula.

For example, if we apply the rules ($\rightarrow$ right), ($\forall$ right), ($\forall$ left), and ($\exists$ left) to the sequent tree of Figure 2 (in which the operated node is $\langle 2\rangle$), the resulting sequent trees are Figures 3, 4, 5, and 6, respectively.

We say that a sequent tree $\Gamma \overset{T,f}{\Rightarrow} \Delta$ *contains* a sequent tree $\Gamma' \overset{T',f'}{\Rightarrow} \Delta'$, denoted by "$(\Gamma \overset{T,f}{\Rightarrow} \Delta) \supseteq (\Gamma' \overset{T',f'}{\Rightarrow} \Delta')$" or "$(\Gamma' \overset{T',f'}{\Rightarrow} \Delta') \subseteq (\Gamma \overset{T,f}{\Rightarrow} \Delta)$," if the following conditions hold: $T \supseteq T'$, $\Gamma \supseteq \Gamma'$, $\Delta \supseteq \Delta'$, and $f(\langle\vec{n}\rangle) \supseteq f'(\langle\vec{n}\rangle)$ for all $\langle\vec{n}\rangle \in T'$. Especially we say that $\Gamma \overset{T,f}{\Rightarrow} \Delta$ *contains a* **TLJ2**-*axiom* if it contains $(\langle\vec{n}\rangle\!:\!A, \Gamma' \overset{T',f'}{\Rightarrow} \Delta', \langle\vec{n}\rangle\!:\!A)$ or $(\langle\vec{n}\rangle\!:\!\bot, \Gamma' \overset{T',f'}{\Rightarrow} \Delta', \langle\vec{n}\rangle\!:\!A)$ for some $\vec{n}$, A, Γ', Δ', T', and f'. Note that $\Gamma \overset{T,f}{\Rightarrow} \Delta$ does not contain **TLJ2**-axioms if and only if the following two conditions hold for any label $\langle\vec{n}\rangle \in T$ and any closed formula A: (1) $(\langle\vec{n}\rangle\!:\!A \in \Delta)$ implies $(\langle\vec{n}\rangle\!:\!A \notin \Gamma)$. (2) $\langle\vec{n}\rangle\!:\!\bot \notin \Gamma$.

We say that a sequent tree $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is *saturated* if it satisfies all the conditions (heredity), ($\rightarrow$ left), ..., ($\exists$ right) below, where $\langle\vec{n}\rangle$ is an arbitrary label, A, B are arbitrary closed formulas, x is an arbitrary propositional variable, and A' is an arbitrary formula that satisfies $\mathrm{FV}(A') \subseteq \{x\}$.

(heredity) If $\langle\vec{n}\rangle\!:\!A \in \Gamma$, then $\langle\vec{n}, a\rangle\!:\!A \in \Gamma$ for any child $\langle\vec{n}, a\rangle$ of $\langle\vec{n}\rangle$ in T.

($\rightarrow$ left) If $\langle\vec{n}\rangle\!:\!A{\rightarrow}B \in \Gamma$, then $\langle\vec{n}\rangle\!:\!A \in \Delta$ or $\langle\vec{n}\rangle\!:\!B \in \Gamma$.

($\rightarrow$ right) If $\langle\vec{n}\rangle : A{\rightarrow}B \in \Delta$, then there is a child $\langle\vec{n}, a\rangle$ of $\langle\vec{n}\rangle$ such that $\langle\vec{n}, a\rangle : A \in \Gamma$ and $\langle\vec{n}, a\rangle\!:\!B \in \Delta$.

($\wedge$ left) If $\langle\vec{n}\rangle\!:\!A{\wedge}B \in \Gamma$, then $\langle\vec{n}\rangle\!:\!A \in \Gamma$ and $\langle\vec{n}\rangle\!:\!B \in \Gamma$.

($\wedge$ right) If $\langle\vec{n}\rangle\!:\!A{\wedge}B \in \Delta$, then $\langle\vec{n}\rangle\!:\!A \in \Delta$ or $\langle\vec{n}\rangle\!:\!B \in \Delta$.

($\vee$ left) If $\langle\vec{n}\rangle\!:\!A{\vee}B \in \Gamma$, then $\langle\vec{n}\rangle\!:\!A \in \Gamma$ or $\langle\vec{n}\rangle\!:\!B \in \Gamma$.

($\vee$ right) If $\langle\vec{n}\rangle\!:\!A{\vee}B \in \Delta$, then $\langle\vec{n}\rangle\!:\!A \in \Delta$ and $\langle\vec{n}\rangle\!:\!B \in \Delta$.

($\forall$ left) If $\langle\vec{n}\rangle\!:\!\forall x A' \in \Gamma$ and $\mathrm{PP}(B) \subseteq f(\langle\vec{n}\rangle)$, then $\langle\vec{n}\rangle\!:\!A'[x{:=}B] \in \Gamma$.

($\forall$ right) If $\langle\vec{n}\rangle : \forall x A' \in \Delta$, then there are a child $\langle\vec{n}, a\rangle$ of $\langle\vec{n}\rangle$ and a closed formula C such that $\mathrm{PP}(C) \subseteq f(\langle\vec{n}, a\rangle)$ and $\langle\vec{n}, a\rangle\!:\!A'[x{:=}C] \in \Delta$.

($\exists$ left) If $\langle\vec{n}\rangle : \exists x A' \in \Gamma$, then there is a closed formula C such that $\mathrm{PP}(C) \subseteq f(\langle\vec{n}\rangle)$ and $\langle\vec{n}\rangle\!:\!A'[x{:=}C] \in \Gamma$.

($\exists$ right) If $\langle\vec{n}\rangle\!:\!\exists x A' \in \Delta$ and $\mathrm{PP}(B) \subseteq f(\langle\vec{n}\rangle)$, then $\langle\vec{n}\rangle\!:\!A'[x{:=}B] \in \Delta$.

Lemma 15 *Let $\Gamma \overset{T,f}{\Rightarrow} \Delta$ be a finite sequent tree. If* **TLJ2** $\nvdash \Gamma \overset{T,f}{\Rightarrow} \Delta$, *then there is a sequent tree $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$ which is saturated, contains $\Gamma \overset{T,f}{\Rightarrow} \Delta$, and does not contain* **TLJ2**-*axioms.*

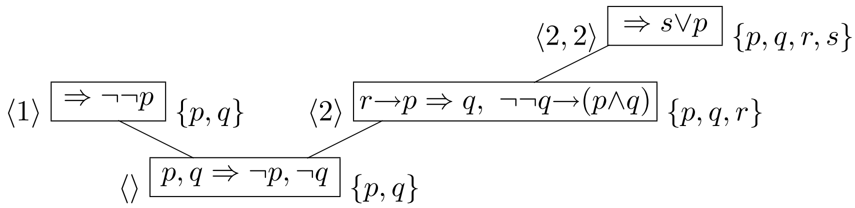

Figure 3: A conclusion of (→ right) for Fig.2.

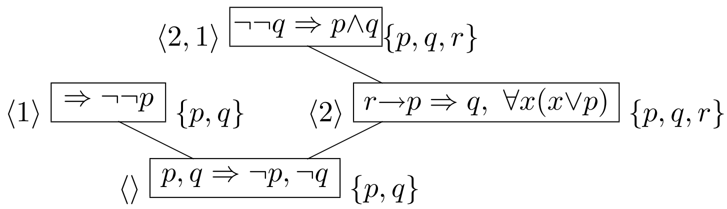

Figure 4: A conclusion of (∀ right) for Fig.2.

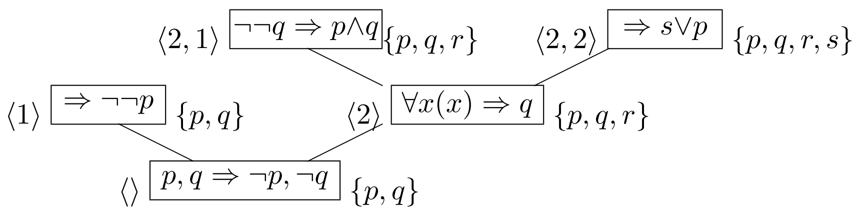

Figure 5: A conclusion of (∀ left) for Fig.2.

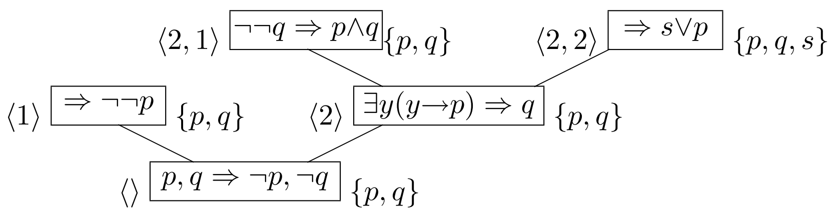

Figure 6: A consequence of (∃ left) for Fig.2.

Proof A tuple $\langle\langle\vec{n}\rangle, A, B, x, A'\rangle$ is called a *seed* if $\langle\vec{n}\rangle$ is a label, A and B are closed formulas, x is a propositional variable, and A' is a formula satisfying $\mathrm{FV}(A') \subseteq \{x\}$. Let SEED be the set of all seeds. Since SEED is a countable set, there is an infinite sequence

$$\langle\langle\vec{n_1}\rangle, A_1, B_1, x_1, A'_1\rangle,\ \langle\langle\vec{n_2}\rangle, A_2, B_2, x_2, A'_2\rangle,\ \ldots,\ \langle\langle\vec{n_i}\rangle, A_i, B_i, x_i, A'_i\rangle,\ \ldots$$

of seeds such that any seed occurs infinitely often in this sequence; in other words, for any seed $\langle\langle\vec{n}\rangle, A, B, x, A'\rangle$ and any natural number i, there is a natural number $j \geq i$ such that $\langle\langle\vec{n_j}\rangle, A_j, B_j, x_j, A'_j\rangle = \langle\langle\vec{n}\rangle, A, B, x, A'\rangle$.

Suppose $\mathbf{TLJ2} \nvdash \Gamma \overset{T,f}{\Rightarrow} \Delta$ where $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is a finite sequent tree. Then we construct an infinite sequence $(\Gamma_0 \overset{T_0,f_0}{\Rightarrow} \Delta_0),\ (\Gamma_1 \overset{T_1,f_1}{\Rightarrow} \Delta_1), \ldots$ of finite sequent trees by the following procedure.

(Step 0)
$(\Gamma_0 \overset{T_0,f_0}{\Rightarrow} \Delta_0) = (\Gamma \overset{T,f}{\Rightarrow} \Delta)$.

(Step $i+1$)
Suppose that $\Gamma_i \overset{T_i,f_i}{\Rightarrow} \Delta_i$ is already defined. Then we define eleven finite sequent trees $\Pi_j \overset{S_j,g_j}{\Rightarrow} \Sigma_j$ for $j = 1, 2, \ldots, 11$ successively by the procedure below; and finally we define

$$(\Gamma_{i+1} \overset{T_{i+1},f_{i+1}}{\Rightarrow} \Delta_{i+1}) = (\Pi_{11} \overset{S_{11},g_{11}}{\Rightarrow} \Sigma_{11}).$$

(1) for (heredity)

$$(\Pi_1 \overset{S_1,g_1}{\Rightarrow} \Sigma_1) = \begin{cases} (\Gamma_i \overset{T_i,f_i}{\Rightarrow} \Delta_i) & \text{if } \langle\vec{n_i}\rangle : A_i \notin \Gamma_i, \\ (\langle\vec{n_i}, a_1\rangle : A_i,\ \langle\vec{n_i}, a_2\rangle : A_i,\ \ldots, \langle\vec{n_i}, a_k\rangle : A_i,\ \Gamma_i \overset{T_i,f_i}{\Rightarrow} \Delta_i) & \text{if } \langle\vec{n_i}\rangle : A_i \in \Gamma_i, \end{cases}$$

where $\{\langle\vec{n_i}, a_1\rangle, \langle\vec{n_i}, a_2\rangle, \ldots, \langle\vec{n_i}, a_k\rangle\}$ is the set of all the children of $\langle\vec{n_i}\rangle$ in T_i.

(2) for ($\rightarrow$ left)

$$(\Pi_2 \overset{S_2,g_2}{\Rightarrow} \Sigma_2) = \begin{cases} (\Pi_1 \overset{S_1,g_1}{\Rightarrow} \Sigma_1) & \text{if } \langle\vec{n_i}\rangle : A_i \rightarrow B_i \notin \Pi_1, \\ (\Pi_1 \overset{S_1,g_1}{\Rightarrow} \Sigma_1,\ \langle\vec{n_i}\rangle : A_i) & \text{if } \langle\vec{n_i}\rangle : A_i \rightarrow B_i \in \Pi_1 \text{ and} \\ & \mathbf{TLJ2} \nvdash (\Pi_1 \overset{S_1,g_1}{\Rightarrow} \Sigma_1,\ \langle\vec{n_i}\rangle : A_i), \\ (\langle\vec{n_i}\rangle : B_i,\ \Pi_1 \overset{S_1,g_1}{\Rightarrow} \Sigma_1) & \text{otherwise.} \end{cases}$$

(3) for ($\rightarrow$ right)

$$(\Pi_3 \overset{S_3,g_3}{\Rightarrow} \Sigma_3) = \begin{cases} (\Pi_2 \overset{S_2,g_2}{\Rightarrow} \Sigma_2) & \text{if } \langle\vec{n_i}\rangle : A_i \rightarrow B_i \notin \Sigma_2, \\ (\langle\vec{n_i}, a\rangle : A_i,\ \Pi_2 \overset{S_2^+,g_2^+}{\Rightarrow} \Sigma_2,\ \langle\vec{n_i}, a\rangle : B_i) & \text{if } \langle\vec{n_i}\rangle : A_i \rightarrow B_i \in \Sigma_2, \end{cases}$$

where $S_2^+ = S_2 \cup \{\langle\vec{n_i}, a\rangle\}$ for some fresh number a, and

$$g_2^+(\langle\vec{m}\rangle) = \begin{cases} g_2(\langle\vec{n_i}\rangle) & \text{if } \langle\vec{m}\rangle = \langle\vec{n_i}, a\rangle, \\ g_2(\langle\vec{m}\rangle) & \text{otherwise.} \end{cases}$$

(4) for ($\wedge$ left)

$$(\Pi_4 \overset{S_4,g_4}{\Rightarrow} \Sigma_4) = \begin{cases} (\Pi_3 \overset{S_3,g_3}{\Rightarrow} \Sigma_3) & \text{if } \langle\vec{n_i}\rangle : A_i \wedge B_i \notin \Pi_3, \\ (\langle\vec{n_i}\rangle : A_i,\ \langle\vec{n_i}\rangle : B_i,\ \Pi_3 \overset{S_3,g_3}{\Rightarrow} \Sigma_3) & \text{if } \langle\vec{n_i}\rangle : A_i \wedge B_i \in \Pi_3. \end{cases}$$

(5) for ($\wedge$ right)

$$(\Pi_5 \overset{S_5,g_5}{\Rightarrow} \Sigma_5) = \begin{cases} (\Pi_4 \overset{S_4,g_4}{\Rightarrow} \Sigma_4) & \text{if } \langle\vec{n_i}\rangle : A_i \wedge B_i \notin \Sigma_4, \\ (\Pi_4 \overset{S_4,g_4}{\Rightarrow} \Sigma_4,\ \langle\vec{n_i}\rangle : A_i) & \text{if } \langle\vec{n_i}\rangle : A_i \wedge B_i \in \Sigma_4 \text{ and} \\ & \mathbf{TLJ2} \not\vdash (\Pi_4 \overset{S_4,g_4}{\Rightarrow} \Sigma_4,\ \langle\vec{n_i}\rangle : A_i), \\ (\Pi_4 \overset{S_4,g_4}{\Rightarrow} \Sigma_4,\ \langle\vec{n_i}\rangle : B_i) & \text{otherwise.} \end{cases}$$

(6) for ($\vee$ left) Symmetric form of (5).

(7) for ($\vee$ right) Symmetric form of (4).

(8) for ($\forall$ left)

$$(\Pi_8 \overset{S_8,g_8}{\Rightarrow} \Sigma_8) = \begin{cases} (\Pi_7 \overset{S_7,g_7}{\Rightarrow} \Sigma_7) & \text{if } \langle\vec{n_i}\rangle : \forall x_i A'_i \notin \Pi_7 \text{ or } \mathrm{PP}(B_i) \not\subseteq g_7(\langle\vec{n_i}\rangle), \\ (\langle\vec{n_i}\rangle : A'_i[x_i := B_i],\ \Pi_7 \overset{S_7,g_7}{\Rightarrow} \Sigma_7) & \text{if } \langle\vec{n_i}\rangle : \forall x_i A'_i \in \Pi_7 \text{ and } \mathrm{PP}(B_i) \subseteq g_7(\langle\vec{n_i}\rangle). \end{cases}$$

(9) for ($\forall$ right)

$$(\Pi_9 \overset{S_9,g_9}{\Rightarrow} \Sigma_9) = \begin{cases} (\Pi_8 \overset{S_8,g_8}{\Rightarrow} \Sigma_8) & \text{if } \langle\vec{n_i}\rangle : \forall x_i A'_i \notin \Sigma_8, \\ (\Pi_8 \overset{S_8^+,g_8^+}{\Rightarrow} \Sigma_8,\ \langle\vec{n_i}, a\rangle : A'_i[x_i := p]) & \text{if } \langle\vec{n_i}\rangle : \forall x_i A'_i \in \Sigma_8, \end{cases}$$

where p is a fresh propositional parameter, $S_8^+ = S_8 \cup \{\langle\vec{n_i}, a\rangle\}$ for some fresh number a, and

$$g_8^+(\langle\vec{m}\rangle) = \begin{cases} g_8(\langle\vec{n_i}\rangle) \cup \{p\} & \text{if } \langle\vec{m}\rangle = \langle\vec{n_i}, a\rangle, \\ g_8(\langle\vec{m}\rangle) & \text{otherwise.} \end{cases}$$

(10) for ($\exists$ left)

$$(\Pi_{10} \overset{S_{10},g_{10}}{\Rightarrow} \Sigma_{10}) = \begin{cases} (\Pi_9 \overset{S_9,g_9}{\Rightarrow} \Sigma_9) & \text{if } \langle\vec{n_i}\rangle : \exists x_i A'_i \notin \Pi_9, \\ (\langle\vec{n_i}\rangle : A'_i[x_i := p],\ \Pi_9 \overset{S_9,g_9^+}{\Rightarrow} \Sigma_9) & \text{if } \langle\vec{n_i}\rangle : \exists x_i A'_i \in \Pi_9, \end{cases}$$

where p is a fresh propositional parameter and

$$g_9^+(\langle\vec{m}\rangle) = \begin{cases} g_9(\langle\vec{n_i}\rangle) \cup \{p\} & \text{if } \langle\vec{m}\rangle \text{ is equal to or a descendant of } \langle\vec{n_i}\rangle, \\ g_9(\langle\vec{m}\rangle) & \text{otherwise.} \end{cases}$$

(11) for ($\exists$ right) Symmetric form of (8).

Now we define the required sequent tree by

$$(\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+) \ = \ \bigcup_{i \in \mathbb{N}} (\Gamma_i \overset{T_i,f_i}{\Rightarrow} \Delta_i).$$

In other words, $\Gamma^+ = \{(\langle\vec{n}\rangle : A) \mid (\exists i \in \mathbb{N})((\langle\vec{n}\rangle : A) \in \Gamma_i)\}$, $T^+ = \{\langle\vec{n}\rangle \mid (\exists i \in \mathbb{N})(\langle\vec{n}\rangle \in T_i)\}$, $f^+(\langle\vec{n}\rangle) = \{p \mid (\exists i \in \mathbb{N})(p \in f_i(\langle\vec{n}\rangle)\}$, and $\Delta^+ = \{(\langle\vec{n}\rangle : A) \mid (\exists i \in \mathbb{N})((\langle\vec{n}\rangle : A) \in \Delta_i)\}$. By the definition of the above procedure, we have $(\Pi_1 \overset{S_1,g_1}{\Rightarrow} \Sigma_1) \subseteq (\Pi_2 \overset{S_2,g_2}{\Rightarrow} \Sigma_2) \subseteq \cdots \subseteq (\Pi_{11} \overset{S_{11},g_{11}}{\Rightarrow} \Sigma_{11})$, and therefore $(\Gamma \overset{T,f}{\Rightarrow} \Delta) = (\Gamma_0 \overset{T_0,f_0}{\Rightarrow} \Delta_0) \subseteq (\Gamma_1 \overset{T_1,f_1}{\Rightarrow} \Delta_1) \subseteq \cdots \subseteq (\Gamma_i \overset{T_i,f_i}{\Rightarrow} \Delta_i) \subseteq \cdots$. Moreover the **TLJ2**-unprovability is preserved throughout the procedure; that is, "**TLJ2** $\not\vdash$

$(\Pi_i \overset{S_i,g_i}{\Rightarrow} \Sigma_i)$" implies "**TLJ2** $\nvdash (\Pi_{i+1} \overset{S_{i+1},g_{i+1}}{\Rightarrow} \Sigma_{i+1})$" for $i = 1, 2, \ldots, 10$, and therefore we have **TLJ2** $\nvdash (\Gamma_0 \overset{T_0,f_0}{\Rightarrow} \Delta_0)$, **TLJ2** $\nvdash (\Gamma_1 \overset{T_1,f_1}{\Rightarrow} \Delta_1)$, ..., **TLJ2** $\nvdash (\Gamma_i \overset{T_i,f_i}{\Rightarrow} \Delta_i)$, ... Using these facts, we can show that $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$ is saturated, contains $\Gamma \overset{T,f}{\Rightarrow} \Delta$, and does not contain **TLJ2**-axioms. **QED**

In the following, we fix a sequent tree $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$ which is saturated and does not contain **TLJ2**-axioms. For each closed formula A, we define $\mathrm{Left}(A)$ and $\overline{\mathrm{Right}}(A)$, which are subsets of T^+, as follows.

$\mathrm{Left}(A) = \{\langle\vec{n}\rangle \in T^+ \mid \langle\vec{n}\rangle : A \in \Gamma^+\}$.
$\overline{\mathrm{Right}}(A) = \{\langle\vec{n}\rangle \in T^+ \mid \langle\vec{n}\rangle : A \notin \Delta^+\}$.

We say that s subset α of T^+ is *approximated by A* if the following two conditions hold.

(1) $\alpha \in \mathrm{UP}(T^+, \preceq)$.

(2) $\mathrm{Left}(A) \subseteq \alpha \subseteq \overline{\mathrm{Right}}(A)$.

Note that the second condition is equivalent to

$$(\forall\langle\vec{n}\rangle \in T^+)\big[(\langle\vec{n}\rangle : A \in \Gamma^+ \Longrightarrow \langle\vec{n}\rangle \in \alpha) \text{ and } (\langle\vec{n}\rangle : A \in \Delta^+ \Longrightarrow \langle\vec{n}\rangle \notin \alpha)\big]. \tag{9}$$

Lemma 16 *For any closed formula A, we have* $\mathrm{Left}(A) \subseteq \overline{\mathrm{Right}}(A)$*, and* $\mathrm{Left}(A)$ *is approximated by A.*

Proof By the fact that $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$ does not contain **TLJ2**-axioms and the condition (heredity) of saturatedness of $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$. **QED**

Using $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$, we define a Kripke model $\mathcal{M}^+ = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ as follows.

$\mathcal{W} = T^+$.

$\mathcal{R} = \preceq$.

$\mathcal{D}_{\langle\vec{n}\rangle} = \{\alpha \subseteq T^+ \mid$ there is a closed formula A such that $\mathrm{PP}(A) \subseteq f^+(\langle\vec{n}\rangle)$ and α is approximated by A $\}$.

$\mathcal{I}_p = \mathrm{Left}(p)$, for $p \in$ **PropPara**.

We can easily verify that $\mathcal{M}^+$ is indeed a Kripke model: $\mathcal{W}$ is not empty because it contains $\langle\rangle$; $\mathcal{R}$ is reflexive and transitive; $\mathcal{D}(\langle\vec{n}\rangle)$ is not empty because it contains $\emptyset$ ($\because$ $\emptyset$ is approximated by the formula $\bot$); if $\langle\vec{n}\rangle\mathcal{R}\langle\vec{n'}\rangle$ then $\mathcal{D}(\langle\vec{n}\rangle) \subseteq \mathcal{D}(\langle\vec{n'}\rangle)$ ($\because$ $f^+(\langle\vec{n}\rangle) \subseteq f^+(\langle\vec{n'}\rangle)$); and $\mathcal{I}_p = \mathrm{Left}(p) \in \mathrm{UP}(T^+, \preceq)$ (cf. Lemma 16).

Lemma 17 *Let $\mathcal{M}^+$ be the Kripke model defined above, A be a formula, $x_1, x_2, \ldots, x_k$ be mutually distinct propositional variables such that* $\mathrm{FV}(A) \subseteq \{x_1, x_2, \ldots, x_k\}$*, $X_1, X_2, \ldots, X_k$ be closed formulas, and $\alpha_1, \alpha_2, \ldots, \alpha_k$ be elements of* $\mathrm{UP}(T^+, \preceq)$*. If α_i is approximated by X_i for any i, then* $[\![A[x_1 := \overline{\alpha_1}][x_2 := \overline{\alpha_2}] \cdots [x_k := \overline{\alpha_k}]]\!]_{\mathcal{M}^+}$ *is approximated by the closed formula* $A[x_1{:=}X_1][x_2{:=}X_2]\cdots[x_k{:=}X_k]$.

Proof We write $[\overline{\theta}]$ and $[\theta]$ to denote respectively the successive substitutions "$[x_1{:=}\overline{\alpha_1}][x_2{:=}\overline{\alpha_2}]\cdots[x_k{:=}\overline{\alpha_k}]$" and "$[x_1{:=}X_1][x_2{:=}X_2]\cdots[x_k{:=}X_k]$". We prove

$$\Big(\langle\vec{n}\rangle : A[\theta] \in \Gamma^+ \Longrightarrow \mathcal{M}^+, \langle\vec{n}\rangle \models A[\overline{\theta}]\Big) \text{ and } \Big(\langle\vec{n}\rangle : A[\theta] \in \Delta^+ \Longrightarrow \mathcal{M}^+, \langle\vec{n}\rangle \not\models A[\overline{\theta}]\Big)$$

(cf. the condition (9)) for any $\langle\vec{n}\rangle \in T^+$, any $[\theta]$, and any $[\overline{\theta}]$. The proof is done by induction on the complexity of A.

(Case 1: $A = p \in$ **PropPara**) In this case, $A[\theta] = A[\overline{\theta}] = p$, and $[\![p]\!]_{\mathcal{M}^+}$ (= Left(p)) is approximated by p ($\because$ Lemma 16).

(Case 2: $A = x_i$) Trivial, because α_i is approximated by X_i.

(Case 3: $A = \bot$) Trivial, because $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$ does not contain **TLJ2**-axioms.

In the following, (heredity), ($\rightarrow$ left), ..., ($\exists$ right) will mean those conditions of saturatedness of $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$.

(Case 4: $A = B{\rightarrow}C$)

$$\begin{aligned}
\langle\vec{n}\rangle\!:\!(B{\rightarrow}C)[\theta] \in \Gamma^+ \implies& \langle\vec{m}\rangle\!:\!(B[\theta]){\rightarrow}(C[\theta]) \in \Gamma^+, \text{ for any } \langle\vec{m}\rangle \text{ such that } \langle\vec{n}\rangle\mathcal{R}\langle\vec{m}\rangle\ (\because \text{heredity})\\
\implies& \langle\vec{m}\rangle\!:\!B[\theta] \in \Delta^+ \text{ or } \langle\vec{m}\rangle\!:\!C[\theta] \in \Gamma^+, \text{ for any } \langle\vec{m}\rangle \text{ such that } \langle\vec{n}\rangle\mathcal{R}\langle\vec{m}\rangle\ (\because \rightarrow \text{left})\\
\implies& \langle\vec{m}\rangle \not\models B[\overline{\theta}] \text{ or } \langle\vec{m}\rangle \models C[\overline{\theta}], \text{ for any } \langle\vec{m}\rangle \text{ such that } \langle\vec{n}\rangle\mathcal{R}\langle\vec{m}\rangle\ (\because \text{ind.hyp.})\\
\implies& \langle\vec{n}\rangle \models (B{\rightarrow}C)[\overline{\theta}].
\end{aligned}$$

$$\begin{aligned}
\langle\vec{n}\rangle\!:\!(B{\rightarrow}C)[\theta] \in \Delta^+ \implies& \langle\vec{n},a\rangle\!:\!B[\theta] \in \Gamma^+ \text{ and } \langle\vec{n},a\rangle\!:\!C[\theta] \in \Delta^+, \text{ for some child } \langle\vec{n},a\rangle \text{ of } \langle\vec{n}\rangle\ \ (\because \rightarrow \text{right})\\
\implies& \langle\vec{n},a\rangle \models B[\overline{\theta}] \text{ and } \langle\vec{n},a\rangle \not\models C[\overline{\theta}], \text{ for some } \langle\vec{n},a\rangle \text{ such that } \langle\vec{n}\rangle\mathcal{R}\langle\vec{n},a\rangle\ \ (\because \text{ind.hyp.})\\
\implies& \langle\vec{n}\rangle \not\models (B{\rightarrow}C)[\overline{\theta}].
\end{aligned}$$

(Case 5: $A = B{\wedge}C$)

$$\begin{aligned}
\langle\vec{n}\rangle\!:\!(B{\wedge}C)[\theta] \in \Gamma^+ \implies& \langle\vec{n}\rangle\!:\!B[\theta] \in \Gamma^+ \text{ and } \langle\vec{n}\rangle\!:\!C[\theta] \in \Gamma^+\ (\because \wedge \text{ left})\\
\implies& \langle\vec{n}\rangle \models B[\overline{\theta}] \text{ and } \langle\vec{n}\rangle \models C[\overline{\theta}]\ (\because \text{ind.hyp.})\\
\implies& \langle\vec{n}\rangle \models (B{\wedge}C)[\overline{\theta}].
\end{aligned}$$

$$\begin{aligned}
\langle\vec{n}\rangle\!:\!(B{\wedge}C)[\theta] \in \Delta^+ \implies& \langle\vec{n}\rangle\!:\!B[\theta] \in \Delta^+ \text{ or } \langle\vec{n}\rangle\!:\!C[\theta] \in \Delta^+\ (\because \wedge \text{ right})\\
\implies& \langle\vec{n}\rangle \not\models B[\overline{\theta}] \text{ or } \langle\vec{n}\rangle \not\models C[\overline{\theta}]\ (\because \text{ind.hyp.})\\
\implies& \langle\vec{n}\rangle \not\models (B{\wedge}C)[\overline{\theta}].
\end{aligned}$$

(Case 6: $A = B{\vee}C$) Similar to Case 5.

In the following, we assume that $x \neq x_i$ for all i. (The proof for the case of $x = x_i$ is similar.)

(Case 7: $A = \forall xB$)

$$\begin{aligned}
\langle\vec{n}\rangle\!:\!(\forall xB)[\theta] \in \Gamma^+ \implies& \langle\vec{n}\rangle\!:\!\forall x(B[\theta]) \in \Gamma^+\\
\implies& \text{if } \langle\vec{n}\rangle\mathcal{R}\langle\vec{m}\rangle, \text{ then } \langle\vec{m}\rangle\!:\!\forall x(B[\theta]) \in \Gamma^+, \text{ for any node } \langle\vec{m}\rangle, (\because \text{heredity})\\
\implies& \text{if } \langle\vec{n}\rangle\mathcal{R}\langle\vec{m}\rangle \text{ and PP}(C) \subseteq f^+(\langle\vec{m}\rangle), \text{ then } \langle\vec{m}\rangle\!:\!B[\theta][x\!:=\!C] \in \Gamma^+, \text{ for any node } \langle\vec{m}\rangle \text{ and any closed formula } C\ (\because \forall \text{ left})\\
\implies& \text{if } \langle\vec{n}\rangle\mathcal{R}\langle\vec{m}\rangle \text{ and } \gamma \in \mathcal{D}_{\langle\vec{m}\rangle}, \text{ then there exists a closed formula } C \text{ such that } \gamma \text{ is approximated by } C,\ \text{PP}(C) \subseteq f^+(\langle\vec{m}\rangle),\\
& \text{and } \langle\vec{m}\rangle\!:\!B[\theta][x\!:=\!C] \in \Gamma^+, \text{ for any node } \langle\vec{m}\rangle \text{ and any set } \gamma\ (\because \text{definition of } \mathcal{D}_{\langle\vec{m}\rangle})
\end{aligned}$$

$$\Longrightarrow \text{ if } \langle\vec{n}\rangle\mathcal{R}\langle\vec{m}\rangle \text{ and } \gamma \in \mathcal{D}_{\langle\vec{m}\rangle}, \text{ then } \langle\vec{m}\rangle \models B[\overline{\theta}][x:=\overline{\gamma}], \text{ for any node } \langle\vec{m}\rangle \text{ and any set } \gamma \ (\because \text{ ind.hyp.})$$
$$\Longrightarrow \langle\vec{n}\rangle \models \forall x(B[\overline{\theta}])$$
$$\Longrightarrow \langle\vec{n}\rangle \models (\forall xB)[\overline{\theta}].$$

$$\langle\vec{n}\rangle : (\forall xB)[\theta] \in \Delta^+ \Longrightarrow \langle\vec{n}\rangle : \forall x(B[\theta]) \in \Delta^+$$
$$\Longrightarrow \langle\vec{n}, a\rangle : B[\theta][x:=C] \in \Delta^+ \text{ for some child } \langle\vec{n}, a\rangle \text{ of } \langle\vec{n}\rangle \text{ and some closed formula } C \text{ such that } \mathrm{PP}(C) \subseteq f^+(\langle\vec{n}, a\rangle) \ (\because \forall \text{ right})$$
$$\Longrightarrow \langle\vec{n}, a\rangle \not\models B[\overline{\theta}][x:=\overline{\mathrm{Left}(C)}] \ (\because \text{ Lemma 16 and ind.hyp.})$$
$$\Longrightarrow \langle\vec{n}\rangle \not\models \forall x(B[\overline{\theta}]) \ (\because \mathrm{Left}(C) \in \mathcal{D}_{\langle\vec{n}, a\rangle})$$
$$\Longrightarrow \langle\vec{n}\rangle \not\models (\forall xB)[\overline{\theta}].$$

(Case 8: $A = \exists xB$)

$$\langle\vec{n}\rangle : (\exists xB)[\theta] \in \Gamma^+ \Longrightarrow \langle\vec{n}\rangle : \exists x(B[\theta]) \in \Gamma^+$$
$$\Longrightarrow \langle\vec{n}\rangle : B[\theta][x:=C] \in \Gamma^+ \text{ for some closed formula } C \text{ such that } \mathrm{PP}(C) \subseteq f^+(\langle\vec{n}\rangle) \ (\because \exists \text{ left})$$
$$\Longrightarrow \langle\vec{n}\rangle \models B[\overline{\theta}][x:=\overline{\mathrm{Left}(C)}] \ (\because \text{ Lemma 16 and ind.hyp.})$$
$$\Longrightarrow \langle\vec{n}\rangle \models \exists x(B[\overline{\theta}]) \ (\because \mathrm{Left}(C) \in \mathcal{D}_{\langle\vec{n}\rangle})$$
$$\Longrightarrow \langle\vec{n}\rangle \models (\exists xB)[\overline{\theta}].$$

$$\langle\vec{n}\rangle : (\exists xB)[\theta] \in \Delta^+ \Longrightarrow \langle\vec{n}\rangle : \exists x(B[\theta]) \in \Delta^+$$
$$\Longrightarrow \text{ if } \mathrm{PP}(C) \subseteq f^+(\langle\vec{n}\rangle), \text{ then } \langle\vec{n}\rangle : B[\theta][x:=C] \in \Delta^+, \text{ for any closed formula } C \ (\because \exists \text{ right})$$
$$\Longrightarrow \text{ if } \gamma \in \mathcal{D}_{\langle\vec{n}\rangle}, \text{ then there exists a closed formula } C \text{ such that } \gamma \text{ is approximated by } C, \ \mathrm{PP}(C) \subseteq f^+(\langle\vec{n}\rangle), \text{ and } \langle\vec{n}\rangle : B[\theta][x:=C] \in \Delta^+, \text{ for any set } \gamma \ (\because \text{ definition of } \mathcal{D}_{\langle\vec{n}\rangle})$$
$$\Longrightarrow \text{ if } \gamma \in \mathcal{D}_{\langle\vec{n}\rangle}, \text{ then } \langle\vec{n}\rangle \not\models B[\overline{\theta}][x:=\overline{\gamma}], \text{ for any set } \gamma \ (\because \text{ ind.hyp.})$$
$$\Longrightarrow \langle\vec{n}\rangle \not\models \exists x(B[\overline{\theta}])$$
$$\Longrightarrow \langle\vec{n}\rangle \not\models (\exists xB)[\overline{\theta}].$$

QED

Theorem 18 *The above Kripke model $\mathcal{M}^+$ is full.*

Proof Take an arbitrary closed $\mathcal{M}^+$-formula A and an arbitrary node $\langle\vec{n}\rangle \in \mathcal{W}$. Assume that A is interpretable at $\langle\vec{n}\rangle$. Our goal is to show $[\![A]\!]_{\mathcal{M}^+} \in \mathcal{D}_{\langle\vec{n}\rangle}$.

Let $\overline{\beta_1}, \overline{\beta_2}, \ldots, \overline{\beta_m}$ be all the names in A and $p_1, p_2, \ldots, p_k$ be all the propositional parameters in A. Using fresh propositional variables $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_k$, we define a closed formula A_0 by replacing each occurrences of $\overline{\beta_i}$ and p_i respectively with x_i and y_j. As a result, A_0 contains no propositional parameters, and

$$A = A_0[x_1:=\overline{\beta_1}][x_2:=\overline{\beta_2}]\cdots[x_m:=\overline{\beta_m}][y_1:=p_1][y_2:=p_2]\cdots[y_k:=p_k]. \tag{10}$$

Now we have $\beta_i \in \mathcal{D}_{\langle\vec{n}\rangle}$ for all i and $[\![p_j]\!]_{\mathcal{M}^+} \in \mathcal{D}_{\langle\vec{n}\rangle}$ for all j, by the assumption (A is interpretable at $\langle\vec{n}\rangle$). Therefore, by the definition of $\mathcal{D}$, there are closed formulas $B_1, B_2, \ldots, B_m$ and $C_1, C_2, \ldots, C_k$ such that $\mathrm{PP}(B_i) \subseteq f^+(\langle\vec{n}\rangle)$, β_i is approximated by B_i, $\mathrm{PP}(C_j) \subseteq f^+(\langle\vec{n}\rangle)$, and $[\![p_j]\!]_{\mathcal{M}^+}$ is approximated by C_j, for all i, j. Then, by Lemma 17, the set

$$\left[\!\!\left[A_0[x_1 := \overline{\beta_1}][x_2 := \overline{\beta_2}] \cdots [x_m := \overline{\beta_m}][y_1 := \overline{[\![p_1]\!]_{\mathcal{M}^+}}][y_2 := \overline{[\![p_2]\!]_{\mathcal{M}^+}}] \cdots [y_k := \overline{[\![p_k]\!]_{\mathcal{M}^+}}]\right]\!\!\right]_{\mathcal{M}^+} \quad (11)$$

is approximated by the closed formula

$$A_0[x_1 := B_1][x_2 := B_2] \cdots [x_m := B_m][y_1 := C_1][y_2 := C_2] \cdots [y_k := C_k]. \quad (12)$$

By (10) and Lemma 9, the set (11) is equal to $[\![A]\!]_{\mathcal{M}^+}$. On the other hand, the formula (12) consists of the only propositional parameters in $f^+(\langle\vec{n}\rangle)$. These implies $[\![A]\!]_{\mathcal{M}^+} \in \mathcal{D}_{\langle\vec{n}\rangle}$ by the definition of $\mathcal{D}_{\langle\vec{n}\rangle}$. **QED**

Theorem 19 (Completeness Theorem of TLJ2) *If a closed formula A is valid in any full Kripke model, then the sequent tree $(\overset{T,f}{\Rightarrow} \langle\rangle : A)$ is provable in* **TLJ2** *where $T = \{\langle\rangle\}$ and $f(\langle\rangle) = \mathrm{PP}(A)$.*

Proof We show the contraposition. Suppose **TLJ2** $\nvdash (\overset{T,f}{\Rightarrow} \langle\rangle : A)$. By Lemma 15, there is a saturated sequent tree $\Gamma^+ \overset{T^+,f^+}{\Rightarrow} \Delta^+$ which does not contain **TLJ2**-axioms and $\langle\rangle : A \in \Delta^+$. Using this, we define a Kripke model $\mathcal{M}^+$ as above, which is full by Theorem 18. Lemma 17 says that $[\![A]\!]_{\mathcal{M}^+}$ is approximated by A; therefore $\mathcal{M}^+, \langle\rangle \not\models A$ (see the condition (9)). **QED**

8 Sequent tree calculus for constant domains

In this section, we modify the arguments of the previous section for constant domain case and its disjunction free version.

We say that a sequent tree $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is *constant domain* if f is a constant function (that is, $f(\langle\vec{n}\rangle) = f(\langle\vec{m}\rangle)$ for all $\langle\vec{n}\rangle$, $\langle\vec{m}\rangle$). We introduce a system **TLJ2CD**, which deduces constant domain finite sequent trees. Axioms of **TLJ2CD** are the restriction of **TLJ2**-axioms to constant domain sequent trees. Inference rules of **TLJ2CD** are obtained from the rules of **TLJ2** by replacing ($\forall$ right) and ($\exists$ left) as follows.

$$\frac{\Gamma \overset{T,f^+}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : A[x := p]}{\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{n}\rangle : \forall x A}\ (\forall\ \mathrm{right_{cd}})(\dagger) \qquad \frac{\langle\vec{n}\rangle : A[x := p],\ \Gamma \overset{T,f^+}{\Rightarrow} \Delta}{\langle\vec{n}\rangle : \exists x A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta}\ (\exists\ \mathrm{left})(\dagger)$$

(†) Proviso: p is a propositional parameter (called eigenvariable) which does not occur in the lower sequent tree, and f^+ and f satisfy the equation $f^+(\langle\vec{m}\rangle) = f(\langle\vec{m}\rangle) \cup \{p\}$ for all $\vec{m}$.

We modify the notion of saturatedness by replacing the condition ($\forall$ right) as follows.

($\forall$ right) If $\langle\vec{n}\rangle : \forall x A' \in \Delta$, then there is a closed formula C such that $\mathrm{PP}(C) \subseteq f(\langle\vec{n}\rangle)$ and $\langle\vec{n}\rangle : A'[x := C] \in \Delta$.

This modified version of saturatedness is called "*cd-saturated*" ("cd" for "constant domain").

Then all the arguments of Lemmas 15, 16, 17, and Theorem 18 can be modified by replacing the terms "**TLJ2**", "sequent tree", and "saturated" by "**TLJ2CD**", "constant domain sequent tree", and "cd-saturated", respectively. In these modified arguments, the Kripke model $\mathcal{M}^+ = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{C} \rangle$ becomes constant domain because $\mathcal{D}_{\langle \vec{n} \rangle} = \{\alpha \subseteq T^+ \mid$ there is a closed formula A such that $\mathrm{PP}(A) \subseteq f^+(\langle \vec{n} \rangle)$ and α is approximated by A $\}$ and f^+ is a constant function. Consequently we have:

Theorem 20 (Completeness Theorem of TLJ2CD) *If a closed formula A is valid in any constant domain full Kripke model, then the sequent tree $(\overset{T,f}{\Rightarrow} \langle\rangle : A)$ is provable in* **TLJ2CD** *where $T = \{\langle\rangle\}$ and $f(\langle\rangle) = \mathrm{PP}(A)$.*

Moreover we modify the above arguments for disjunction free formulas.

TLJ2CD$^{\mathrm{df}}$ is obtained from **TLJ2CD** by imposing restriction that all the sequent tree are disjunction free. Of course, the rules ($\vee$ left) and ($\vee$ right) are deleted. We define the notion of "*cddf-saturated*" ("cddf" for "constant domain disjunction free") by deleting the two conditions ($\vee$ left) and ($\vee$ right) from the definition of cd-saturatedness. Then all the arguments of Lemmas 15, 16, 17, and Theorem 18 can be modified by replacing the terms "formula", "full" "**TLJ2**", "sequent tree", and "saturated" by "disjunction free formula", "df-full", "**TLJ2CD**$^{\mathrm{df}}$", "constant domain disjunction free sequent tree", and "cddf-saturated", respectively. Consequently we have:

Theorem 21 (Completeness Theorem of TLJ2CD$^{\mathrm{df}}$) *If a disjunction free closed formula A is valid in any constant domain df-full Kripke model, then the sequent tree $(\overset{T,f}{\Rightarrow} \langle\rangle : A)$ is provable in* **TLJ2CD**$^{\mathrm{df}}$ *where $T = \{\langle\rangle\}$ and $f(\langle\rangle) = \mathrm{PP}(A)$.*

9 From sequent trees to formulas

In this section, we show the following:

> If a sequent tree $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is provable in **TLJ2** (or **TLJ2CD**), then a formula $\#(\Gamma \overset{T,f}{\Rightarrow} \Delta)$, which is a translation of $\Gamma \overset{T,f}{\Rightarrow} \Delta$, is provable in **LJ2** (or **LJ2CD**, respectively).

As a corollary, we have the following.

> If a sequent tree $\overset{\{\langle\rangle\},f}{\Rightarrow} \langle\rangle : A$ is provable in **TLJ2** (or **TLJ2CD**), then the formula A is provable in **LJ2** (or **LJ2CD**, respectively).

Before defining the translation $\#(\Gamma \overset{T,f}{\Rightarrow} \Delta)$, we make a remake on formulas. As usual, if Γ is a finite set of formulas, then $\bigwedge \Gamma$ and $\bigvee \Gamma$ will denotes respectively the conjunction and disjunction of all the formulas in Γ. There are many formulas that are regarded as $\bigwedge \Gamma$ (or $\bigvee \Gamma$) for a fixed Γ. For example, if $\Gamma = \{A, x, y\}$, then both $((x\wedge y)\wedge A)$ and $((A\wedge x)\wedge(A\wedge y))$ are $\bigwedge \Gamma$. We do not have to distinguish these formulas because they are equivalent with respect to **LJ2**-provability. For example, two conditions **LJ2** $\vdash \big(\forall x \exists y(((x\wedge y)\wedge A)\to B), \Pi \Rightarrow \Sigma\big)$ and **LJ2** $\vdash \big(\forall x \exists y(((A\wedge x)\wedge(A\wedge y))\to B), \Pi \Rightarrow \Sigma\big)$ are equivalent because the former implies the latter:

$$\dfrac{\begin{array}{c}\vdots\\ (p\wedge q)\wedge A \Rightarrow (A\wedge p)\wedge(A\wedge q)\\ \vdots\\ \forall x\exists y(((A\wedge x)\wedge(A\wedge y))\to B) \Rightarrow \forall x\exists y(((x\wedge y)\wedge A)\to B)\end{array} \quad \begin{array}{c}(\text{the former})\\ \forall x\exists y(((x\wedge y)\wedge A)\to B), \Pi \Rightarrow \Sigma\end{array}}{\forall x\exists y(((A\wedge x)\wedge(A\wedge y))\to B), \Pi \Rightarrow \Sigma}\ (\text{cut})$$

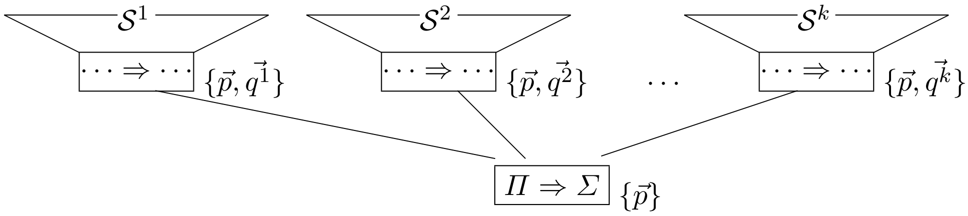

Figure 7: Sequent tree $\mathcal{S}$.

and the converse is similarly proved.

Moreover, if $x_1, x_2, \dots, x_k$ are mutually distinct propositional variables, and if $\vec{x} = \{x_1, x_2, \dots, x_k\}$, then we write $\overrightarrow{\forall x}$ to represent $\forall x_1 \forall x_2 \cdots \forall x_k$. In this case, we do not have to mind the order of x_i because, for example, $\forall x \forall y \forall z(\cdots)$ and $\forall z \forall y \forall x(\cdots)$ are equivalent with respect to **LJ2**-provability.

Now the formula $\#(\Gamma \overset{T,f}{\Rightarrow} \Delta)$ for sequent tree $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is defined by induction on the height of the tree T, as follows.

In general, a subtree $\mathcal{S}$ of a finite sequent tree is illustrated as Figure 7, where $\mathcal{S}^1$, $\mathcal{S}^2$, $\dots$, $\mathcal{S}^k$ are subtrees of $\mathcal{S}$ ($k \geq 0$), and $\{\vec{p}\}$ and $\{\vec{p}, \vec{q^i}\}$ (where $\{\vec{p}\} \cap \{\vec{q^i}\} = \emptyset$) are respectively the sets of propositional parameters that are associated with the nodes. For each $i \in \{1, 2, \dots, k\}$, we define a number n_i by $n_i = |\{\vec{q^i}\}|$ and we define a set $\vec{x^i}$ of n_i fresh propositional variables by $\vec{x^i} = \{x^i_1, x^i_2, \dots, x^i_{n_i}\}$. Suppose that the formulas $\#(\mathcal{S}^1)$, $\#(\mathcal{S}^2)$, $\dots$, $\#(\mathcal{S}^k)$, for subtrees $\mathcal{S}^1$, $\mathcal{S}^2$, $\dots$, $\mathcal{S}^k$, are already defined. Then the formula $\#(\mathcal{S})$ is defined as follows.

$$\#(\mathcal{S}) = \begin{cases} (\bigwedge \Pi) \rightarrow \big((\bigvee \Sigma) \vee A_1 \vee A_2 \vee \cdots \vee A_k\big) & \text{if } \Pi \text{ is not empty,} \\ (\bigvee \Sigma) \vee A_1 \vee A_2 \vee \cdots \vee A_k & \text{if } \Pi \text{ is empty} \end{cases}$$

where

$$A_i \; = \; \overrightarrow{\forall x^i}\big(\#(\mathcal{S}^i)[\langle \vec{q^i} \rangle := \langle \vec{x^i} \rangle]\big) \quad (i = 1, 2, \dots, k)$$

and $[\langle \vec{q^i} \rangle := \langle \vec{x^i} \rangle]$ is the substitution to replace each occurrence of q^i_j by x^i_j for $j = 1, 2, \dots, n_i$. For example, if $\mathcal{S}$ is the sequent tree in Figure 2, then $\#(\mathcal{S})$ is the formula

$$(p \wedge q) \rightarrow \Big[\neg p \vee \neg q \vee \neg\neg p \vee \forall x\Big((x \rightarrow p) \rightarrow \big(q \vee (\neg\neg q \rightarrow p \wedge q) \vee \forall y(y \vee p)\big)\Big)\Big].$$

Lemma 22 *Let*

$$\frac{\mathcal{S}}{\mathcal{T}}\ (\rho) \quad \text{or} \quad \frac{\mathcal{S}_1 \quad \mathcal{S}_2}{\mathcal{T}}\ (\rho)$$

be an instance of an inference rule of **TLJ2** *in which the root node is operated. If ρ is a one-premise rule except ($\exists$ left), then we have* **LJ2** $\vdash (\#(\mathcal{S}) \Rightarrow \#(\mathcal{T}))$. *If ρ is a two-premises rule, then we have* **LJ2** $\vdash (\#(\mathcal{S}_1),\ \#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T}))$. *If ρ = ($\exists$ left), then we have* **LJ2** $\vdash$ $(\forall y(\#(\mathcal{S})[p := y]) \Rightarrow \#(\mathcal{T}))$ *where p is the eigenvariable and y is a fresh propositional variable.*

Proof

(Case 1: ρ = (heredity)) Suppose $\mathcal{S} = \big(\langle a \rangle : A, \Gamma \overset{T,f}{\Rightarrow} \Delta\big)$, $\mathcal{T} = \big(\langle\rangle : A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta\big)$, $\#(\mathcal{S}) = \big(C \rightarrow (D \vee \overrightarrow{\forall x}((A \wedge E) \rightarrow F)\big)$, and $\#(\mathcal{T}) = \big((A \wedge C) \rightarrow (D \vee \overrightarrow{\forall x}(E \rightarrow F)\big)$, where $\overrightarrow{\forall x} = (\forall x_1 \forall x_2 \cdots \forall x_k)$

and $x_1, x_2, \dots x_k \notin \mathrm{FV}(A)$. Then a proof sketch for $\mathbf{LJ2} \vdash (\#(\mathcal{S}) \Rightarrow \#(\mathcal{T}))$ is as follows.

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{(A\wedge E')\to F', A \Rightarrow E'\to F'}}{\overrightarrow{\forall x}((A\wedge E)\to F), A \Rightarrow \overrightarrow{\forall x}(E\to F)}}{D\vee\overrightarrow{\forall x}((A\wedge E)\to F), A \Rightarrow D\vee\overrightarrow{\forall x}(E\to F)}}{\cfrac{C\to(D\vee\overrightarrow{\forall x}((A\wedge E)\to F)), A, C \Rightarrow D\vee\overrightarrow{\forall x}(E\to F)}{C\to(D\vee\overrightarrow{\forall x}((A\wedge E)\to F)) \Rightarrow (A\wedge C)\to(D\vee\overrightarrow{\forall x}(E\to F))}}$$

(Case 2: $\rho = (\to \text{ left})$) Suppose $\mathcal{S}_1 = \big(\varGamma \overset{T,f}{\Rightarrow} \varDelta,\ \langle\rangle : A\big)$, $\mathcal{S}_2 = \big(\langle\rangle : B,\ \varGamma \overset{T,f}{\Rightarrow} \varDelta\big)$, $\mathcal{T} = \big(\langle\rangle : A\to B,\ \varGamma \overset{T,f}{\Rightarrow} \varDelta\big)$, $\#(\mathcal{S}_1) = C\to(D\vee A)$, $\#(\mathcal{S}_2) = (B\wedge C)\to D$, and $\#(\mathcal{T}) = ((A\to B)\wedge C)\to D$. Then a proof sketch for $\mathbf{LJ2} \vdash (\#(\mathcal{S}_1), \#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T}))$ is as follows.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{(B\wedge C)\to D, B, C \Rightarrow D}}{A, (B\wedge C)\to D, A\to B, C \Rightarrow D}}{D\vee A, (B\wedge C)\to D, A\to B, C \Rightarrow D}}{C\to(D\vee A), (B\wedge C)\to D, A\to B, C \Rightarrow D}}{C\to(D\vee A), (B\wedge C)\to D \Rightarrow ((A\to B)\wedge C)\to D}$$

(Case 3: $\rho = (\to \text{ right})$) In this case, two formulas $\#(\mathcal{S})$ and $\#(\mathcal{T})$ are equivalent.

(Case 4: $\rho = (\wedge \text{ left})$) In this case, two formulas $\#(\mathcal{S})$ and $\#(\mathcal{T})$ are equivalent.

(Case 5: $\rho = (\wedge \text{ right})$) Suppose $\mathcal{S}_1 = \big(\varGamma \overset{T,f}{\Rightarrow} \varDelta,\ \langle\rangle : A\big)$, $\mathcal{S}_2 = \big(\varGamma \overset{T,f}{\Rightarrow} \varDelta,\ \langle\rangle : B\big)$, $\mathcal{T} = \big(\varGamma \overset{T,f}{\Rightarrow} \varDelta,\ \langle\rangle : A\wedge B\big)$, $\#(\mathcal{S}_1) = C\to(D\vee A)$, $\#(\mathcal{S}_2) = C\to(D\vee B)$, and $\#(\mathcal{T}) = C\to(D\vee(A\wedge B))$. Then a proof sketch for $\mathbf{LJ2} \vdash (\#(\mathcal{S}_1), \#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T}))$ is as follows.

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{A, B \Rightarrow D\vee(A\wedge B)}}{D\vee A, D\vee B \Rightarrow D\vee(A\wedge B)}}{C\to(D\vee A), C\to(D\vee B), C \Rightarrow D\vee(A\wedge B)}}{C\to(D\vee A), C\to(D\vee B) \Rightarrow C\to(D\vee(A\wedge B))}$$

(Case 6: $\rho = (\vee \text{ left})$) Suppose $\mathcal{S}_1 = \big(\langle\rangle : A,\ \varGamma \overset{T,f}{\Rightarrow} \varDelta\big)$, $\mathcal{S}_2 = \big(\langle\rangle : B,\ \varGamma \overset{T,f}{\Rightarrow} \varDelta\big)$, $\mathcal{T} = \big(\langle\rangle : A\vee B,\ \varGamma \overset{T,f}{\Rightarrow} \varDelta\big)$, $\#(\mathcal{S}_1) = (A\wedge C)\to D$, $\#(\mathcal{S}_2) = (B\wedge C)\to D$, and $\#(\mathcal{T}) = ((A\vee B)\wedge C)\to D$. Then a proof sketch for $\mathbf{LJ2} \vdash (\#(\mathcal{S}_1), \#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T}))$ is as follows.

$$\cfrac{\cfrac{\cfrac{\vdots}{(A\wedge C)\to D, A, C \Rightarrow D} \quad \cfrac{\vdots}{(B\wedge C)\to D, B, C \Rightarrow D}}{(A\wedge C)\to D, (B\wedge C)\to D, A\vee B, C \Rightarrow D}}{(A\wedge C)\to D, (B\wedge C)\to D \Rightarrow ((A\vee B)\wedge C)\to D}$$

(Case 7: $\rho = (\vee \text{ right})$) In this case, two formulas $\#(\mathcal{S})$ and $\#(\mathcal{T})$ are equivalent.

(Case 8: $\rho = (\forall \text{ left})$) Suppose $\mathcal{S} = \big(\langle\rangle : A[x{:=}B],\ \varGamma \overset{T,f}{\Rightarrow} \varDelta\big)$, $\mathcal{T} = \big(\langle\rangle : \forall x A,\ \varGamma \overset{T,f}{\Rightarrow} \varDelta\big)$, $\#(\mathcal{S}) = \big((A[x{:=}B]\wedge C)\to D\big)$, and $\#(\mathcal{T}) = \big(((\forall x A)\wedge C)\to D\big)$. Then a proof sketch for $\mathbf{LJ2} \vdash (\#(\mathcal{S}) \Rightarrow \#(\mathcal{T}))$ is as follows.

$$\cfrac{\cfrac{\cfrac{\vdots}{\forall x A, C \Rightarrow A[x{:=}B]\wedge C}}{(A[x{:=}B]\wedge C)\to D, \forall x A, C \Rightarrow D}}{(A[x{:=}B]\wedge C)\to D \Rightarrow ((\forall x A)\wedge C)\to D}$$

(Case 9: $\rho = (\forall \text{ right})$) In this case, two formulas $\#(\mathcal{S})$ and $\#(\mathcal{T})$ are equivalent.

(Case 10: $\rho = (\exists \text{ left})$) Suppose $\mathcal{S} = (\langle\rangle : A[x{:=}p],\ \Gamma \overset{T,f^+}{\Rightarrow} \Delta)$, $\mathcal{T} = (\langle\rangle : \exists x A,\ \Gamma \overset{T,f}{\Rightarrow} \Delta)$, $\#(\mathcal{S}) = \big((A[x{:=}p]\wedge C)\to D\big)$, and $\#(\mathcal{T}) = \big(((\exists x A)\wedge C)\to D\big)$, where $p \notin \mathrm{PP}(((\exists x A)\wedge C)\to D)$ and $((\exists x A)\wedge C)\to D$ is a closed formula. Then a proof sketch for $\mathbf{LJ2} \vdash (\forall y(\#(\mathcal{S})[p{:=}y]) \Rightarrow \#(\mathcal{T}))$ is as follows (y is a fresh propositional variable).

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ (A[x{:=}p]\wedge C)\to D, A[x{:=}p], C \Rightarrow D\end{array}}{\forall y((A[x{:=}y]\wedge C)\to D), \exists x A, C \Rightarrow D}}{\forall y((A[x{:=}y]\wedge C)\to D) \Rightarrow ((\exists x A)\wedge C)\to D}$$

(Case 11: $\rho = (\exists \text{ right})$) Suppose $\mathcal{S} = (\Gamma \overset{T,f^+}{\Rightarrow} \Delta,\ \langle\rangle : A[x{:=}B])$, $\mathcal{T} = (\Gamma \overset{T,f}{\Rightarrow} \Delta,\ \langle\rangle : \exists x A)$, $\#(\mathcal{S}) = \big(C\to(D\vee A[x{:=}B])\big)$, and $\#(\mathcal{T}) = \big(C\to(D\vee\exists x A)\big)$. Then a proof sketch for $\mathbf{LJ2} \vdash (\#(\mathcal{S}) \Rightarrow \#(\mathcal{T}))$ is as follows.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ D\vee A[x{:=}B] \Rightarrow D\vee\exists x A\end{array}}{C\to(D\vee A[x{:=}B]), C \Rightarrow D\vee\exists x A}}{C\to(D\vee A[x{:=}B]) \Rightarrow C\to(D\vee\exists x A)}$$

QED

Lemma 23 *Let*

$$\frac{\mathcal{S}}{\mathcal{T}}\ (\rho) \quad \text{or} \quad \frac{\mathcal{S}_1 \quad \mathcal{S}_2}{\mathcal{T}}\ (\rho)$$

be an instance of an inference rule of **TLJ2** *in which the root node is not the operated node. If ρ is a one-premise rule, then we have* $\mathbf{LJ2} \vdash (\#(\mathcal{S}) \Rightarrow \#(\mathcal{T}))$. *If ρ is a two-premises rule, then we have* $\mathbf{LJ2} \vdash (\#(\mathcal{S}_1),\ \#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T}))$.

Proof We give a proof for the case that ρ is a two-premise rule and $\langle a\rangle$ (a child of the root node) is the operated node. The other cases can be similarly proved by the repetition of the following argument.

Let $\mathcal{S}_1^{\langle a\rangle}$, $\mathcal{S}_2^{\langle a\rangle}$ and $\mathcal{T}^{\langle a\rangle}$ be the subtrees of respectively $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{T}$ whose root is $\langle a\rangle$. Then the figure

$$\frac{\mathcal{S}_1^{\langle a\rangle} \quad \mathcal{S}_2^{\langle a\rangle}}{\mathcal{T}^{\langle a\rangle}}\ (\rho)$$

is an instance of the inference rule ρ in which the root node is operated. Therefore we can apply the previous Lemma 22 and we have (†) $\mathbf{LJ2} \vdash (\#(\mathcal{S}_1^{\langle a\rangle}),\ \#(\mathcal{S}_2^{\langle a\rangle}) \Rightarrow \#(\mathcal{T}^{\langle a\rangle}))$.

By the definition of the translation $\#()$, there are closed formulas C and D, propositional parameters $\vec{p}$ $(= \langle p_1, p_2, \ldots, p_k\rangle)$, propositional variables $\vec{x}$ $(= \langle x_1, x_2, \ldots, x_k\rangle)$ $(k \geq 0)$ such that $\#(\mathcal{S}_1) = C\to\big(D\vee\overrightarrow{\forall x}\big(\#(\mathcal{S}_1^{\langle a\rangle})[\langle\vec{p}\rangle := \langle\vec{x}\rangle]\big)\big)$, $\#(\mathcal{S}_2) = C\to\big(D\vee\overrightarrow{\forall x}\big(\#(\mathcal{S}_2^{\langle a\rangle})[\langle\vec{p}\rangle := \langle\vec{x}\rangle]\big)\big)$, and $\#(\mathcal{T}) = C\to\big(D\vee\overrightarrow{\forall x}\big(\#(\mathcal{T}^{\langle a\rangle})[\langle\vec{p}\rangle := \langle\vec{x}\rangle]\big)\big)$. Define the formulas A_1, A_2, and B by $A_1 = \#(\mathcal{S}_1^{\langle a\rangle})[\langle\vec{p}\rangle := \langle\vec{x}\rangle]$, $A_2 = \#(\mathcal{S}_2^{\langle a\rangle})[\langle\vec{p}\rangle := \langle\vec{x}\rangle]$, and $B = \#(\mathcal{T}^{\langle a\rangle})[\langle\vec{p}\rangle := \langle\vec{x}\rangle]$. Then a proof sketch for **LJ2** $\vdash (\#(\mathcal{S}_1), \#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T}))$ is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ (\dagger)}{A_1[\langle\vec{x}\rangle := \langle\vec{p}\rangle],\ A_2[\langle\vec{x}\rangle := \langle\vec{p}\rangle] \Rightarrow B[\langle\vec{x}\rangle := \langle\vec{p}\rangle]}}{\forall\overrightarrow{x}A_1,\ \forall\overrightarrow{x}A_2 \Rightarrow \forall\overrightarrow{x}B}}{D\vee\forall\overrightarrow{x}A_1,\ D\vee\forall\overrightarrow{x}A_2 \Rightarrow D\vee\forall\overrightarrow{x}B}}{\dfrac{C\to(D\vee\forall\overrightarrow{x}A_1),\ C\to(D\vee\forall\overrightarrow{x}A_2),\ C \Rightarrow D\vee\forall\overrightarrow{x}B}{C\to(D\vee\forall\overrightarrow{x}A_1),\ C\to(D\vee\forall\overrightarrow{x}A_2) \Rightarrow C\to(D\vee\forall\overrightarrow{x}B)}}$$

Note that, for example, $A_1[\langle\vec{x}\rangle := \langle\vec{p}\rangle] = \#(\mathcal{S}_1^{\langle a\rangle})$ because the propositional variables $\vec{x}$ do not occur elsewhere. **QED**

Theorem 24 *Let $\mathcal{T}$ be a sequent tree. If* **TLJ2** $\vdash \mathcal{T}$*, then* **LJ2** $\vdash\ \Rightarrow \#(\mathcal{T})$.

Proof By induction on the proof of $\mathcal{T}$ in **TLJ2**. If $\mathcal{T}$ is an axiom, it is easy to get **LJ2** $\vdash\ \Rightarrow \#(\mathcal{T})$. For the other cases, suppose that $\mathcal{T}$ is inferred by a rule

$$\frac{\mathcal{S}}{\mathcal{T}}\ (\rho) \quad \text{or} \quad \frac{\mathcal{S}_1 \quad \mathcal{S}_2}{\mathcal{T}.}\ (\rho)$$

If $\rho = (\exists\ \text{left})$ and the root node is operated, then $\Rightarrow \#(\mathcal{T})$ is provable in **LJ2** as follows.

$$\dfrac{\dfrac{\begin{matrix}\text{ind. hyp.}\\ \Rightarrow \#(\mathcal{S})\end{matrix}}{\Rightarrow \forall y(\#(\mathcal{S})[p:=y])}\ (\forall\ \text{right}) \qquad \begin{matrix}\text{Lemma 22}\\ \forall y(\#(\mathcal{S})[p:=y]) \Rightarrow \#(\mathcal{T})\end{matrix}}{\Rightarrow \#(\mathcal{T})}\ (\text{cut})$$

where p is the eigenvariable. In the other cases, we have

$$\dfrac{\begin{matrix}\text{ind. hyp.}\\ \Rightarrow \#(\mathcal{S})\end{matrix} \qquad \begin{matrix}\text{Lemma 22 or 23}\\ \#(\mathcal{S}) \Rightarrow \#(\mathcal{T})\end{matrix}}{\Rightarrow \#(\mathcal{T})}\ (\text{cut}) \quad \text{or} \quad \dfrac{\begin{matrix}\text{ind. hyp.}\\ \Rightarrow \#(\mathcal{S}_2)\end{matrix} \qquad \dfrac{\begin{matrix}\text{ind. hyp.}\\ \Rightarrow \#(\mathcal{S}_1)\end{matrix} \qquad \begin{matrix}\text{Lemma 22 or 23}\\ \#(\mathcal{S}_1), \#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T})\end{matrix}}{\#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T})}\ (\text{cut})}{\Rightarrow \#(\mathcal{T}).}\ (\text{cut})$$

QED

Corollary 25 *Let A be a closed formula. If* **TLJ2** $\vdash \overset{\{\langle\rangle\},f}{\Rightarrow} \langle\rangle : A$ *(where $f(\langle\rangle) \supseteq \mathrm{PP}(A)$), then* **LJ2** $\vdash\ \Rightarrow A$.

Proof By the previous theorem, since $\#(\overset{\{\langle\rangle\},f}{\Rightarrow} \langle\rangle : A) = A$. **QED**

In the rest of this section, we modify the above argument for the constant domain case.

Recall that $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is a constant domain sequent tree if and only if f is a constant function. If the sequent tree in Figure 7 is constant domain, then all of $\vec{q}^1$, $\vec{q}^2$, ..., $\vec{q}^k$ are empty, and the translation $\#(\mathcal{S})$ is simply defined as follows.

$$\#(\mathcal{S}) = \begin{cases} (\bigwedge \Pi)\to\big((\bigvee \Sigma)\vee\#(\mathcal{S}^1)\vee\#(\mathcal{S}^2)\vee\cdots\vee\#(\mathcal{S}^k)\big) & \text{if } \Pi \text{ is not empty,} \\ (\bigvee \Sigma)\vee\#(\mathcal{S}^1)\vee\#(\mathcal{S}^2)\vee\cdots\vee\#(\mathcal{S}^k) & \text{if } \Pi \text{ is empty.} \end{cases}$$

Lemma 26 *Let*

$$\frac{\mathcal{S}}{\mathcal{T}}\ (\rho) \quad or \quad \frac{\mathcal{S}_1 \quad \mathcal{S}_2}{\mathcal{T}}\ (\rho)$$

be an instance of an inference rule of **TLJ2CD** *in which the root node is operated. If* ρ *is a one-premise rule except (*$\forall$ *right) and (*$\exists$ *left), then we have* **LJ2CD** $\vdash (\#(\mathcal{S}) \Rightarrow \#(\mathcal{T}))$*. If* ρ *is a two-premises rule, then we have* **LJ2CD** $\vdash (\#(\mathcal{S}_1),\ \#(\mathcal{S}_2) \Rightarrow \#(\mathcal{T}))$*. If* ρ *is (*$\forall$ *right) or (*$\exists$ *left), then we have* **LJ2CD** $\vdash (\forall y(\#(\mathcal{S})[p{:=}y]) \Rightarrow \#(\mathcal{T}))$ *where* p *is the eigenvariable and* y *is a fresh propositional variable.*

Proof The proof is same as Lemma 22 except the case that ρ is ($\forall$ right). In that case, suppose $\mathcal{S} = (\Gamma \overset{\mathcal{T},f^+}{\Rightarrow} \Delta,\ \langle\rangle : A[x{:=}p])$, $\mathcal{T} = (\Gamma \overset{\mathcal{T},f}{\Rightarrow} \Delta,\ \langle\rangle : \forall x A)$, $\#(\mathcal{S}) = \big(C{\to}(D{\vee}A[x{:=}p])\big)$, and $\#(\mathcal{T}) = \big(C{\to}(D{\vee}\forall xA)\big)$, where $p \notin \mathrm{PP}(C{\to}(D{\vee}\forall xA))$ and $C{\to}(D{\vee}\forall xA)$ is a closed formula. Then a proof sketch for **LJ2CD** $\vdash (\forall y(\#(\mathcal{S})[p{:=}y]) \Rightarrow \#(\mathcal{T}))$ is as follows (y is a fresh propositional variable).

$$\dfrac{\dfrac{\dfrac{\dfrac{A[x{:=}y][y{:=}p] \Rightarrow A[x{:=}p]}{D{\vee}\forall yA[x{:=}y] \Rightarrow D{\vee}\forall xA}}{\forall y(D{\vee}A[x{:=}y]) \Rightarrow D{\vee}\forall xA}\ (\forall\text{-}\vee \text{ left) in Lemma 1}}{C{\to}\forall y(D{\vee}A[x{:=}y]) \Rightarrow C{\to}(D{\vee}\forall xA)}}{\forall y(C{\to}(D{\vee}A[x{:=}y])) \Rightarrow C{\to}(D{\vee}\forall xA)}\ (\forall\text{-}{\to} \text{ left) in Lemma 1}$$

QED

Lemma 27 *Let*

$$\frac{\mathcal{S}}{\mathcal{T}}\ (\rho) \quad or \quad \frac{\mathcal{S}_1 \quad \mathcal{S}_2}{\mathcal{T}}\ (\rho)$$

be an instance of an inference rule of **TLJ2CD***. The same claims as Lemma 26 hold even if the root node is not the operated node.*

Proof The proof is similar to Lemma 23, using Lemma 1. **QED**

Theorem 28 *Let* $\mathcal{T}$ *be a constant domain sequent tree. If* **TLJ2CD** $\vdash \mathcal{T}$*, then* **LJ2CD** $\vdash$ $\Rightarrow \#(\mathcal{T})$.

Proof Similar to Theorem 24, using Lemmas 26 and 27. **QED**

Corollary 29 *Let* A *be a closed formula. If* **TLJ2CD** $\vdash \overset{\{\langle\rangle\},f}{\Rightarrow} \langle\rangle : A$ *(where* $f(\langle\rangle) \supseteq \mathrm{PP}(A)$*), then* **LJ2CD** $\vdash \Rightarrow A$.

10 Disjunction free case

We define a sequent calculus $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}}$ to be the systems obtained from $\mathbf{LJ2}^{\mathrm{df}}$ by deleting (cut) rule. In this section we show the following:

$$(\mathbf{TLJ2CD}^{\mathrm{df}} \vdash \overset{\{\langle\rangle\},f}{\Rightarrow} \langle\rangle : A) \implies (\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Rightarrow A).$$

Compared with the previous Corollary 29, this is somewhat surprising because "$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash$" means not only needlessness of cut rule but also needlessness of constant domain axiom.

For a sequent tree $\Gamma \overset{T,f}{\Rightarrow} \Delta$ and a label $\langle\vec{n}\rangle \in T$, we define a set $\Gamma_{\langle\vec{n}\rangle\downarrow}$ of formulas by

$$\Gamma_{\langle\vec{n}\rangle\downarrow} \;=\; \{A \mid \langle m\rangle : A \in \Gamma \text{ for some label } \langle\vec{m}\rangle \in T \text{ such that } \langle\vec{m}\rangle \preceq \langle\vec{n}\rangle\}.$$

For example, if $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is the sequent tree of Figure 2, then $\Gamma_{\langle 2,1\rangle\downarrow} = \{\neg\neg q, r\to p, p, q\}$ and $\Gamma_{\langle 2\rangle\downarrow} = \{r\to p, p, q\}$.

Lemma 30 *If a (constant domain) disjunction free sequent tree $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is provable in* $\mathbf{TLJ2CD}^{\mathrm{df}}$*, then there exists a labelled formula $\langle\vec{n}\rangle : A \in \Delta$ such that the sequent $\Gamma_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$ is provable in* $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}}$.

Proof By induction on the $\mathbf{TLJ2CD}^{\mathrm{df}}$-proof of $\Gamma \overset{T,f}{\Rightarrow} \Delta$.

(Case 1) $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is an axiom $\langle\vec{n}\rangle : A, \Pi \overset{T,f}{\Rightarrow} \Sigma, \langle\vec{n}\rangle : A$ or $\langle\vec{n}\rangle : \bot, \Pi \overset{T,f}{\Rightarrow} \Sigma, \langle\vec{n}\rangle : A$. In this case, $\langle\vec{n}\rangle : A \in \Delta$ is the required labelled formula because both $A, \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$ and $\bot, \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$ are provable in $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}}$ by axiom and the weakening rule.

(Case 2) $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is inferred by (heredity) as

$$\frac{\langle\vec{m}, a\rangle : B,\ \Pi \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{m}\rangle : B,\ \Pi \overset{T,f}{\Rightarrow} \Delta}\ \text{(heredity)}$$

By the induction hypothesis, there is a labelled formula $\langle\vec{n}\rangle : A \in \Delta$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash (\langle\vec{m}, a\rangle : B, \Pi)_{\langle\vec{n}\rangle\downarrow} \Rightarrow A. \tag{13}$$

If $\langle\vec{m}, a\rangle \preceq \langle\vec{n}\rangle$, then condition (13) is equal to

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash (\langle\vec{m}\rangle : B, \Pi)_{\langle\vec{n}\rangle\downarrow} \Rightarrow A, \tag{14}$$

which is the required condition. If $\langle\vec{m}, a\rangle \not\preceq \langle\vec{n}\rangle$, then condition (13) is equal to

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A. \tag{15}$$

This (15) implies (14) using the weakening rule (if $\langle\vec{m}\rangle \preceq \langle\vec{n}\rangle$), or (15) = (14) (if $\langle\vec{m}\rangle \not\preceq \langle\vec{n}\rangle$).

(Case 3) $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is inferred by ($\to$ left) as

$$\frac{\Pi \overset{T,f}{\Rightarrow} \Delta,\ \langle\vec{m}\rangle : B \qquad \langle\vec{m}\rangle : C,\ \Pi \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{m}\rangle : B\to C,\ \Pi \overset{T,f}{\Rightarrow} \Delta}\ (\to \text{left})$$

By the induction hypotheses, there is a labelled formula $\langle\vec{n_1}\rangle : A_1 \in (\Delta, \langle\vec{m}\rangle : B)$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Pi_{\langle\vec{n_1}\rangle\downarrow} \Rightarrow A_1, \tag{16}$$

and there is a labelled formula $\langle\vec{n_2}\rangle : A_2 \in \Delta$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash (\langle\vec{m}\rangle : C, \Pi)_{\langle\vec{n_2}\rangle\downarrow} \Rightarrow A_2. \tag{17}$$

(Subcase 3-1) $(\langle\vec{n_1}\rangle : A_1) \neq (\langle\vec{m}\rangle : B)$ or $\langle\vec{m}\rangle \not\preceq \langle\vec{n_2}\rangle$. In this case, (16) and (17) imply

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A \tag{18}$$

for some $\langle\vec{n}\rangle : A \in \Delta$ (where $(\langle\vec{n}\rangle : A) = (\langle\vec{n_1}\rangle : A_1)$ or $(\langle\vec{n_2}\rangle : A_2)$), and (18) implies the required condition $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \langle\vec{m}\rangle : B \to C, \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$.

(Subcase 3-2) $(\langle\vec{n_1}\rangle : A_1) = (\langle\vec{m}\rangle : B)$ and $\langle\vec{m}\rangle \preceq \langle\vec{n_2}\rangle$. In this case, (16) and (17) are equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Pi_{\langle\vec{m}\rangle\downarrow} \Rightarrow B$ and $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash C, \Pi_{\langle\vec{n_2}\rangle\downarrow} \Rightarrow A_2$. Then, by ($\to$ left), we have $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash B \to C, \Pi_{\langle\vec{n_2}\rangle\downarrow} \Rightarrow A_2$, which is the required condition (note that $\Pi_{\langle\vec{m}\rangle\downarrow} \subseteq \Pi_{\langle\vec{n_2}\rangle\downarrow}$).

(Case 4) $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is inferred by ($\to$ right) as

$$\frac{\langle\vec{m},a\rangle : B,\ \Gamma \overset{T^+,f^+}{\Rightarrow} \Sigma,\ \langle\vec{m},a\rangle : C}{\Gamma \overset{T,f}{\Rightarrow} \Sigma,\ \langle\vec{m}\rangle : B \to C}\ (\to \text{right})$$

where $\langle\vec{m},a\rangle$ is a leaf of T^+ and no other formula than B, C has the label $\langle\vec{m},a\rangle$. By the induction hypothesis, there is a labelled formula $\langle\vec{n}\rangle : A \in (\Sigma, \langle\vec{m},a\rangle : C)$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash (\langle\vec{m},a\rangle : B, \Gamma)_{\langle\vec{n}\rangle\downarrow} \Rightarrow A. \tag{19}$$

(Subcase 4-1) $(\langle\vec{n}\rangle : A) \neq (\langle\vec{m},a\rangle : C)$. In this case, (19) is equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$, which is the required condition (note that $\langle\vec{m},a\rangle \not\preceq \langle\vec{n}\rangle$).

(Subcase 4-2) $(\langle\vec{n}\rangle : A) = (\langle\vec{m},a\rangle : C)$. In this case, (19) is equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash B, \Gamma_{\langle\vec{m}\rangle\downarrow} \Rightarrow C$ because no other formula than B, C has the label $\langle\vec{m},a\rangle$. Then, by ($\to$ right), we have $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{m}\rangle\downarrow} \Rightarrow B \to C$, which is the required condition.

(Case 5) $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is inferred by ($\wedge$ left) as

$$\frac{\langle\vec{m}\rangle : B,\ \langle\vec{m}\rangle : C,\ \Pi \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{m}\rangle : B \wedge C,\ \Pi \overset{T,f}{\Rightarrow} \Delta}\ (\wedge \text{ left})$$

By the induction hypothesis, there is a labelled formula $\langle\vec{n}\rangle : A \in \Delta$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash (\langle\vec{m}\rangle : B, \langle\vec{m}\rangle : C, \Pi)_{\langle\vec{n}\rangle\downarrow} \Rightarrow A. \tag{20}$$

(Subcase 5-1) $\langle\vec{m}\rangle \not\preceq \langle\vec{n}\rangle$. In this case, (20) is equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$, which is the required condition.

(Subcase 5-2) $\langle\vec{m}\rangle \preceq \langle\vec{n}\rangle$. In this case, (20) is equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash B, C, \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$. Then, by ($\wedge$ left), we have $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash B \wedge C, \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$, which is the required condition.

(Case 6) $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is inferred by ($\wedge$ right) as

$$\frac{\Gamma \overset{T,f}{\Rightarrow} \Sigma,\ \langle\vec{m}\rangle : B \qquad \Gamma \overset{T,f}{\Rightarrow} \Sigma,\ \langle\vec{m}\rangle : C}{\Gamma \overset{T,f}{\Rightarrow} \Sigma,\ \langle\vec{m}\rangle : B \wedge C}\ (\wedge \text{ right})$$

By the induction hypotheses, there are labelled formulas $\langle\vec{n_1}\rangle : A_1 \in (\Sigma, \langle\vec{m}\rangle : B)$ and $\langle\vec{n_2}\rangle : A_2 \in (\Sigma, \langle\vec{m}\rangle : C)$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{n_1}\rangle\downarrow} \Rightarrow A_1 \quad \text{and} \quad \mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{n_2}\rangle\downarrow} \Rightarrow A_2. \tag{21}$$

(Subcase 6-1) $(\langle\vec{n_1}\rangle : A_1) \neq (\langle\vec{m}\rangle : B)$ or $(\langle\vec{n_2}\rangle : A_2) \neq (\langle\vec{m}\rangle : C)$. In this case, (21) implies that $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$ for some $\langle n\rangle : A \in \Sigma$. This is the required condition.

(Subcase 6-2) $(\langle\vec{n_1}\rangle : A_1) = (\langle\vec{m}\rangle : B)$ and $(\langle\vec{n_2}\rangle : A_2) = (\langle\vec{m}\rangle : C)$. In this case, (21) is equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{m}\rangle\downarrow} \Rightarrow B$ and $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{m}\rangle\downarrow} \Rightarrow C$. Then, by ($\wedge$ left), we have $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{m}\rangle\downarrow} \Rightarrow B \wedge C$, which is the required condition.

(Case 7) $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is inferred by ($\forall$ left) as

$$\frac{\langle\vec{m}\rangle : B[x{:=}C],\ \Pi \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{m}\rangle : \forall x B,\ \Pi \overset{T,f}{\Rightarrow} \Delta}\ (\forall \text{ left})$$

By the induction hypothesis, there is a labelled formula $\langle\vec{n}\rangle : A \in \Delta$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash (\langle\vec{m}\rangle : B[x{:=}C],\ \Pi)_{\langle\vec{n}\rangle\downarrow} \Rightarrow A. \tag{22}$$

(Subcase 7-1) $\langle\vec{m}\rangle \not\preceq \langle\vec{n}\rangle$. In this case, (22) is equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$, which is the required condition.

(Subcase 7-2) $\langle\vec{m}\rangle \preceq \langle\vec{n}\rangle$. In this case, (22) is equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash B[x{:=}C], \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$. Then, by ($\forall$ left), we have $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \forall x B, \Pi_{\langle\vec{n}\rangle\downarrow} \Rightarrow A$, which is the required condition.

(Case 8) $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is inferred by ($\forall$ right$_{\mathrm{cd}}$) as

$$\frac{\Gamma \overset{T,f^+}{\Rightarrow} \Sigma,\ \langle\vec{m}\rangle : B[x{:=}p]}{\Gamma \overset{T,f}{\Rightarrow} \Sigma,\ \langle\vec{m}\rangle : \forall x B}\ (\forall \text{ right}_{\mathrm{cd}})$$

where p does not occur in the lower sequent tree. By the induction hypothesis, there is a labelled formula $\langle\vec{n}\rangle : A \in (\Sigma, \langle\vec{m}\rangle : B[x{:=}p])$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{n}\rangle\downarrow} \Rightarrow A. \tag{23}$$

(Subcase 8-1) $(\langle\vec{n}\rangle : A) \neq (\langle\vec{m}\rangle : B[x{:=}p])$. In this case, (23) is the required condition.

(Subcase 8-2) $(\langle\vec{n}\rangle : A) = (\langle\vec{m}\rangle : B[x{:=}p])$. In this case, (23) is equal to $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Gamma_{\langle\vec{m}\rangle\downarrow} \Rightarrow B[x{:=}p]$. Then, by ($\forall$ right), we have $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Pi_{\langle\vec{m}\rangle\downarrow} \Rightarrow \forall x B$, which is the required condition.

The other cases — ($\exists$ left) and ($\exists$ right) — are similar to Cases 7 and 8. **QED**

Corollary 31 *Let A be a disjunction free closed formula. If the sequent tree $\overset{\{\langle\rangle\},f}{\Rightarrow} \langle\rangle : A$ (where $f(\langle\rangle) \supseteq \mathrm{PP}(A)$) is provable in* $\mathbf{TLJ2CD}^{\mathrm{df}}$*, then the sequent $\Rightarrow A$ is provable in* $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}}$.

Proof By the previous Lemma 30. **QED**

(Remark) If we try to prove Lemma 30 for **TLJ2CD** instead of $\mathbf{TLJ2CD}^{\mathrm{df}}$, we get into difficulties in the following case: $\Gamma \overset{T,f}{\Rightarrow} \Delta$ is inferred by ($\vee$ left) as

$$\frac{\langle\vec{m}\rangle : B,\ \Pi \overset{T,f}{\Rightarrow} \Delta \qquad \langle\vec{m}\rangle : C,\ \Pi \overset{T,f}{\Rightarrow} \Delta}{\langle\vec{m}\rangle : B \vee C,\ \Pi \overset{T,f}{\Rightarrow} \Delta}\ (\vee \text{ left})$$

By the induction hypotheses, there are labelled formulas $\langle\vec{n_1}\rangle : A_1 \in \Delta$ and $\langle\vec{n_2}\rangle : A_2 \in \Delta$ such that

$$\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash (\langle\vec{m}\rangle : B, \Pi)_{\langle\vec{n_1}\rangle\downarrow} \Rightarrow A_1 \quad \text{and} \quad \mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash (\langle\vec{m}\rangle : C, \Pi)_{\langle\vec{n_2}\rangle\downarrow} \Rightarrow A_2$$

However, if $(\langle\vec{n_1}\rangle : A_1) \neq (\langle\vec{n_2}\rangle : A_2)$, $\langle\vec{m}\rangle \preceq \langle\vec{n_1}\rangle$ and $\langle\vec{m}\rangle \preceq \langle\vec{n_2}\rangle$, then neither $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash B \vee C, \Pi_{\langle\vec{n_1}\rangle\downarrow} \Rightarrow A_1$ nor $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash B \vee C, \Pi_{\langle\vec{n_2}\rangle\downarrow} \Rightarrow A_2$ can be shown.

11 Conclusions and future study

Combining all the results of Sections 2–10, we obtain main theorems of this paper.

Theorem 32 (Completeness of LJ2) *For any closed formula A, the following four conditions are equivalent.*

(1) $\mathbf{LJ2} \vdash \Rightarrow A$.

(2) $\mathbf{LJ2}^{\mathrm{df}} \vdash \Rightarrow A^{\mathrm{df}}$.

(3) $\mathbf{TLJ2} \vdash \overset{T,f}{\Rightarrow} \langle\rangle : A$ *where* $T = \{\langle\rangle\}$ *and* $f(\langle\rangle) = \mathrm{PP}(A)$.

(4) *A is valid in any full Kripke model.*

Proof By Theorems 6, 11, 19, and Corollary 25. **QED**

Theorem 33 (Completeness of LJ2CD) *For any closed formula A, the following three conditions are equivalent.*

(1) $\mathbf{LJ2CD} \vdash \Rightarrow A$.

(2) $\mathbf{TLJ2CD} \vdash \overset{T,f}{\Rightarrow} \langle\rangle : A$ *where* $T = \{\langle\rangle\}$ *and* $f(\langle\rangle) = \mathrm{PP}(A)$.

(3) *A is valid in any constant domain full Kripke model.*

Proof By Theorems 14, 20, and Corollary 29. **QED**

Theorem 34 (Completeness and cut-elimination for disjunction free formulas) *For any disjunction free closed formula A, the following seven conditions are equivalent.*

(1) $\mathbf{LJ2} \vdash \Rightarrow A$.

(2) $\mathbf{LJ2}^{\mathrm{df}} \vdash \Rightarrow A$.

(3) $\mathbf{LJ2}^{\mathrm{df}}_{\mathrm{cutfree}} \vdash \Rightarrow A$.

(4) $\mathbf{TLJ2CD}^{\mathrm{df}} \vdash \overset{T,f}{\Rightarrow} \langle\rangle : A$ *where* $T = \{\langle\rangle\}$ *and* $f(\langle\rangle) = \mathrm{PP}(A)$.

(5) *A is valid in any full Kripke model.*

(6) *A is valid in any df-full Kripke model.*

(7) *A is valid in any constant domain df-full Kripke model.*

Proof ($1 \Leftrightarrow 2$) is shown by Corollary 7. ($1 \Leftrightarrow 5$) is shown by Theorem 32. ($2 \Rightarrow 6$) is shown by Theorem 12. ($6 \Rightarrow 5$), ($6 \Rightarrow 7$), and ($3 \Rightarrow 2$) are trivial. ($7 \Rightarrow 4$) is shown by Theorem 21. ($4 \Rightarrow 3$) is shown by Corollary 31. **QED**

In the rest of this paper, we give some remarks and problems for future study.

An essentially equivalent result to ($2 \Rightarrow 7 \Rightarrow 4 \Rightarrow 3$) of Theorem 34 (in other words, semantical cut-elimination for $\mathbf{LJ2}^{\mathrm{df}}$) was already proved by Prawitz [4] by using Beth model instead of Kripke model. However I do not know whether this results can be extended to the language with disjunction. So we give:

Problem 1 *Does the cut-elimination theorem hold for* **LJ2**?

Let $C = \forall x\big(p\vee((p\to x)\vee\neg x)\big)\to\big(p\vee\forall x((p\to x)\vee\neg x)\big)$. In Section 5, we show an example of full Kripke model in which the formula C is not valid. Then, by Theorems 32 and 34, there is a constant domain df-full Kripke model in which C^{df} is not valid.

Problem 2 *Find a simple constant domain df-full Kripke model in which the formula C^{df} is not valid.*

Note that C is valid in any constant domain full Kripke model (because C is an instance of constant domain axiom). A solution to this problem should be a good example to show the distinction between $A\vee B$ and $\forall x((A\to x)\to(B\to x)\to x)$.

In the study of nonclassical logics, there have been a lot of results on propositional and predicate intermediate (between classical and intuitionistic) logics. However, I do not know substantial studies on second order propositional intermediate logics. (Theorem 33 — completeness of the constant domain intermediate logic — is the only result on second order propositional intermediate logics I know.) So we give:

Problem 3 *Develop studies on second order propositional intermediate logics.*

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