On the Standardization Theorem for $\lambda\beta\eta$-Calculus

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Abstract

We present a new proof of the standardization theorem for $\lambda\beta\eta$-calculus, which is performed by inductions based on an inductive definition of $\beta\eta$-reducibility with a standard sequence.

1 Introduction

The standardization theorem is a fundamental theorem in reduction theory of λ-calculus, which states that if a λ-term M β-(or $\beta\eta$-)reduces to a λ-term N, then there is a "standard" β-(or $\beta\eta$-)reduction sequence from M to N. In [3], the author gave a simple proof of the theorem for β-reduction. This paper extends the result to $\beta\eta$-reduction: we give a simple proof of the (weak) standardization theorem for $\beta\eta$-reduction.

There have been some proofs of the standardization theorem in literature (e.g., [1, 2, 4, 5, 6, 7]). Compared with these, a feature of the presented proof is that we use an inductive definition (formal theory) of $\beta\eta$-reducibility with a standard sequence. In virtue of this definition, all the proof can be performed by easy inductions.

Applications of this method to proofs of other theorems (e.g., the leftmost reduction theorem for $\beta\eta$-reduction) and other calculi (e.g., term rewriting systems) are future studies.

2 Preliminaries

λ-terms are constructed by "application" and "abstraction" from variables (λ-terms of the form (MN) and $(\lambda x.M)$ are called an *application* and an *abstraction* respectively). Capital letters $A, B, \ldots$ denote arbitrary λ-terms, and small letters $x, y, \ldots$ denote arbitrary variables. $M[x := N]$ denotes the result of substituting N for all the free occurrences of x in M with adequate renaming of bound variables. The set of free variables in M is written by $\mathrm{FV}(M)$. We identify λ-terms that are mutually obtained by renaming of bound variables. In successive abstractions or applications, parentheses are omitted as follows: for example, $\lambda xyz.A = \lambda x.(\lambda y.(\lambda z.A))$, and $ABCD = ((AB)C)D$. (See, e.g., [1] for the basic notion and terminology.)

The binary relation $\to_{\beta\eta}$ (one-step $\beta\eta$-reducibility) on the set of λ-terms are defined as usual:

$$(\cdots((\lambda x.A)B)\cdots) \ \to_{\beta\eta}\ (\cdots(A[x:=B])\cdots) \text{ and}$$
$$(\cdots(\lambda x.(Cx))\cdots) \ \to_{\beta\eta}\ (\cdots C\cdots)$$

where $x \notin \mathrm{FV}(C)$. The subterms $(\lambda x.A)B$ and $\lambda x.(Cx)$ are called *$\beta\eta$-redexes* where the former and the latter are also called a *β-redex* and an *η-redex* respectively. Let R be an occurrence of a $\beta\eta$-redex in a λ-term M. We write $M \xrightarrow{R}_{\beta\eta} N$ if and only if N is obtained from M by contracting the redex R. The relation $\to^{\ell}_{\beta\eta}$ (one-step leftmost $\beta\eta$-reducibility) is defined as usual: $M \to^{\ell}_{\beta\eta} N$ if and only if $M \xrightarrow{R}_{\beta\eta} N$ for the leftmost $\beta\eta$-redex R in M, where a redex occurrence R_1 in M is said to be *to the left of* another redex occurence R_2 in M if they occur as follows.

$$M = (\cdots R_1 \cdots R_2 \cdots) \quad \text{or} \quad M = (\cdots \underbrace{(\cdots R_2 \cdots)}_{R_1} \cdots)$$

The binary relations $\twoheadrightarrow_{\beta\eta}$ and $\twoheadrightarrow^{\ell}_{\beta\eta}$ are the reflexive transitive closures of $\to_{\beta\eta}$ and $\to^{\ell}_{\beta\eta}$ respectively.

Suppose that $M \xrightarrow{R}_{\beta\eta} N$ and that P and Q are subterm occurrences of M and N respectively. We write $P \rightsquigarrow Q$ if Q is a *one step residual* of P. For example, if

$$M = \Big(\cdots P_1 \cdots \overbrace{\Big(\cdots(\lambda x.\underbrace{(\cdots P_3 \cdots)}_{A})\underbrace{(\cdots P_4 \cdots)}_{B}\cdots\Big)}^{P_2}\cdots P_5 \cdots\Big)$$

$$N = \Big(\cdots P_1 \cdots \overbrace{\Big(\cdots\underbrace{(\cdots P_3 \cdots)}_{A}[x := \underbrace{(\cdots P_4 \cdots)}_{B}]\cdots\Big)}^{Q_2}\cdots P_5 \cdots\Big)$$

where $R = (\lambda x.A)B \neq P_2$ and $P_3 \neq x$, then $P_1 \rightsquigarrow P_1$, $P_2 \rightsquigarrow Q_2$, $P_3 \rightsquigarrow P_3[x := B]$, $P_4 \rightsquigarrow$ "each copy of P_4 in $A[x := B]$", and $P_5 \rightsquigarrow P_5$. (Note that the redex $(\lambda x.A)B$ has no residual.) Suppose that

$$(\cdots X \cdots) \to_{\beta\eta} \cdots \to_{\beta\eta} (\cdots Y \cdots).$$

We say that Y *is a residual of* X if

$$X \rightsquigarrow Z_1 \rightsquigarrow \cdots \rightsquigarrow Z_n \rightsquigarrow Y$$

for some $Z_1, \ldots, Z_n$ $(n \geq 0)$. A $\beta\eta$-reduction sequence

$$M_1 \xrightarrow{R_1}_{\beta\eta} M_2 \xrightarrow{R_2}_{\beta\eta} \cdots \xrightarrow{R_{n-1}}_{\beta\eta} M_n$$

is called *weakly standard* if the following condition is satisfied.

$\forall i$ [R_{i+1} is not a residual of a $\beta\eta$-redex occurrence P in M_i such that P is to the left of R_i]

Moreover, the sequence is called *strongly standard* if the following condition is satisfied.

$\forall i, \forall j > i$ [R_j is not a residual of a $\beta\eta$-redex occurrence P in M_i such that P is to the left of R_i]

(The distinction between "weak" and "strong" appeared in [2]. Note that these two conditions are equivalent if η-reduction does not exist.) For example, the sequence

$$\lambda x.(\mathbf{I}y(\mathbf{I}x)) \xrightarrow{\mathbf{I}y}_{\beta\eta} \lambda x.(y(\mathbf{I}x)) \xrightarrow{\mathbf{I}x}_{\beta\eta} \lambda x.(yx) \xrightarrow{\lambda x.(yx)}_{\beta\eta} y$$

is weakly and strongly standard; the sequence

$$\lambda x.(\mathbf{I}y(\mathbf{I}x)) \xrightarrow{\mathbf{I}x}_{\beta\eta} \lambda x.(\mathbf{I}yx) \xrightarrow{\lambda x.(\mathbf{I}yx)}_{\beta\eta} \mathbf{I}y \xrightarrow{\mathbf{I}y}_{\beta\eta} y$$

is not strongly standard but weakly standard; and the sequence

$$\lambda x.(\mathbf{I}yx) \xrightarrow{\mathbf{I}y}_{\beta\eta} \lambda x.(yx) \xrightarrow{\lambda x.(yx)}_{\beta\eta} y$$

is neither strongly nor weakly standard, where $\mathbf{I} = \lambda z.z$.

In this paper, we present a simple proof of the theorem:

Theorem 2.1 (Weak Standardization Theorem) *If $M \twoheadrightarrow_{\beta\eta} N$, then there is a weakly standard $\beta\eta$-reduction sequence from M to N.*

Other important theorems, which have been proved in the literature (e.g., [4, 6]), are there:

(Strong Standardization Theorem) *If $M \twoheadrightarrow_{\beta\eta} N$, then there is a strongly standard $\beta\eta$-reduction sequence from M to N.*

(Leftmost Reduction Theorem) *If $M \twoheadrightarrow_{\beta\eta} N$ and N is a $\beta\eta$-normal form, then $M \twoheadrightarrow^{\ell}_{\beta\eta} N$.*

We will discuss them in Section 4.

3 Proof

We define a "formal theory" which proves "formulas" of the forms $A \Rightarrow_{\text{hap}} B$ and $A \Rightarrow_{\text{st}} B$.

Axioms

(Id) $A \Rightarrow_{\text{hap}} A$.

(β) $(\lambda x.A)BC_1 \cdots C_n \Rightarrow_{\text{hap}} A[x:=B]C_1 \cdots C_n$, where $n \geq 0$.

Rules

$$\frac{A \Rightarrow_{\mathrm{hap}} B \quad B \Rightarrow_{\mathrm{hap}} C}{A \Rightarrow_{\mathrm{hap}} C} \text{ (Tr)}$$

$$\frac{L \Rightarrow_{\mathrm{hap}} x}{L \Rightarrow_{\mathrm{st}} x} \text{ (Var)}$$

$$\frac{L \Rightarrow_{\mathrm{hap}} AB \quad A \Rightarrow_{\mathrm{st}} C \quad B \Rightarrow_{\mathrm{st}} D}{L \Rightarrow_{\mathrm{st}} CD} \text{ (App)}$$

$$\frac{L \Rightarrow_{\mathrm{hap}} \lambda x.A \quad A \Rightarrow_{\mathrm{st}} B}{L \Rightarrow_{\mathrm{st}} \lambda x.B} \text{ (Abs)}$$

$$\frac{L \Rightarrow_{\mathrm{hap}} \lambda x.A \quad A \Rightarrow_{\mathrm{st}} Bx}{L \Rightarrow_{\mathrm{st}} B} \text{ (η) with the proviso:}$$

$x \notin \mathrm{FV}(B)$ and B is not an abstraction.

We write $A \twoheadrightarrow_{\mathrm{hap}} B$ (or $A \twoheadrightarrow_{\mathrm{st}} B$) if and only if the formula $A \Rightarrow_{\mathrm{hap}} B$ (or $A \Rightarrow_{\mathrm{st}} B$, respectively) is provable in this system. ("hap" and "st" stand for "head reduction in application" and "standard".)

Theorem 3.1 *If $M \twoheadrightarrow_{\mathrm{hap}} N$, then $M \twoheadrightarrow^{\ell}_{\beta\eta} N$. If $M \twoheadrightarrow_{\mathrm{st}} N$, then there is a weakly standard $\beta\eta$-reduction sequence from M to N.*

Proof By induction on the proof of $M \Rightarrow_{\mathrm{hap}} N$ or $M \Rightarrow_{\mathrm{st}} N$. Here we show the only nontrivial case: The proof is

$$\frac{\begin{matrix}\vdots \\ M \Rightarrow_{\mathrm{hap}} \lambda x.A\end{matrix} \quad \begin{matrix}\vdots \\ A \Rightarrow_{\mathrm{st}} Nx\end{matrix}}{M \Rightarrow_{\mathrm{st}} N} \text{ (η)}$$

where $x \notin \mathrm{FV}(N)$ and N is not an abstraction. By the induction hypotheses, there are a leftmost reduction sequence $\mathcal{L}$ from M to $\lambda x.A$, and a weakly standard reduction sequence $\mathcal{S}$ from A to Nx. We will show that there is a weakly standard reduction sequence $\mathcal{S}^+$ from $\lambda x.A$ to N; then the concatenation of $\mathcal{L}$ and $\mathcal{S}^+$ is the required standard sequence from M to N. We will explain, based on an example, the definition of the sequence $\mathcal{S}^+$. If a λ-term P is of the form $P'x$ and $x \notin \mathrm{FV}(P')$, then we say that P is an *η-body for x*. Suppose that the weakly standard reduction sequence $\mathcal{S}$ is

$$A \xrightarrow{R_1}_{\beta\eta} A' \xrightarrow{R_2}_{\beta\eta} Bx \xrightarrow{R_3}_{\beta\eta} (\lambda y.(Cy))x \xrightarrow{(\lambda y.(Cy))x}_{\beta\eta} Cx \xrightarrow{R_5}_{\beta\eta} Nx$$

where

- Bx, $(\lambda y.(Cy))x$, Cx, and Nx are η-bodies for x;

- A' is not an η-body for x;
- $B \xrightarrow{R_3}_{\beta\eta} \lambda y.(Cy)$; and
- $C \xrightarrow{R_5}_{\beta\eta} N$.

Note that $R_3 = B$; otherwise, reduction of R_3 cannot make $\lambda y.(Cy)$ if B is an application, and S is not weakly standard if B is an abstraction. Then, the weakly standard sequence $\mathcal{S}^+$ is defined as follows.

$$\lambda x.A \xrightarrow{R_1}_{\beta\eta} \lambda x.A' \xrightarrow{R_2}_{\beta\eta} \lambda x.(Bx) \xrightarrow{\lambda x.(Bx)}_{\beta\eta} B \xrightarrow{R_3}_{\beta\eta} \lambda y.(Cy) \xrightarrow{\lambda y.(Cy)}_{\beta\eta} C \xrightarrow{R_5}_{\beta\eta} N.$$

∎

Lemma 3.2 *(1) If $M \twoheadrightarrow_{\text{hap}} N$, then $MP \twoheadrightarrow_{\text{hap}} NP$.*

(2) If $L \twoheadrightarrow_{\text{hap}} M \twoheadrightarrow_{\text{st}} N$, then $L \twoheadrightarrow_{\text{st}} N$.

(3) If $M \twoheadrightarrow_{\text{hap}} N$, then $M[z := P] \twoheadrightarrow_{\text{hap}} N[z := P]$.

Proof (1) By induction on the proof of $M \Rightarrow_{\text{hap}} N$. (2) If $M \Rightarrow_{\text{st}} N$ is proved, then there must be a premise $M \Rightarrow_{\text{hap}} P$ for certain P. From this and the assumption $L \twoheadrightarrow_{\text{hap}} M$, we can infer $L \Rightarrow_{\text{hap}} P$ by the rule (Tr). Then $L \Rightarrow_{\text{st}} N$ is proved by the rule that infers $M \Rightarrow_{\text{st}} N$. (3) By induction on the proof of $M \Rightarrow_{\text{hap}} N$. ∎

Lemma 3.3 *If $M \twoheadrightarrow_{\text{st}} N$ and $P \twoheadrightarrow_{\text{st}} y$, then $M[z := P] \twoheadrightarrow_{\text{st}} N[z := y]$.*

Proof By induction on the proof of $M \Rightarrow_{\text{st}} N$. We divide cases according to the last inference of the proof. The substitution $[z := P]$ and $[z := y]$ will be represented by $[P]$ and $[y]$ for short.

(Case 1): The last inference is

$$\dfrac{\begin{array}{c}\vdots\ \pi \\ M \Rightarrow_{\text{hap}} x\end{array}}{M \Rightarrow_{\text{st}} x}\ (\text{Var})$$

where $N = x \neq z$. In this case, we have

$$\dfrac{\begin{array}{c}\vdots\ \pi \text{ and Lemma 3.2(3)} \\ M[P] \Rightarrow_{\text{hap}} x\end{array}}{M[P] \Rightarrow_{\text{st}} x.}\ (\text{Var})$$

(Case 2): The last inference is

$$\dfrac{\begin{array}{c}\vdots\ \pi \\ M \Rightarrow_{\text{hap}} z\end{array}}{M \Rightarrow_{\text{st}} z}\ (\text{Var})$$

where $N = z$. In this case, we have

$$\dfrac{\begin{array}{c}\vdots\ \pi \text{ and Lemma 3.2(3)}\\ M[P] \Rightarrow_{\text{hap}} P\end{array} \qquad \begin{array}{c}\vdots\ \text{assumption of the lemma}\\ P \Rightarrow_{\text{st}} y\end{array}}{M[P] \Rightarrow_{\text{st}} y.}\ \text{Lemma 3.2(2)}$$

(Case 3): The last inference is

$$\dfrac{\begin{array}{c}\vdots\ \pi\\ M \Rightarrow_{\text{hap}} AB\end{array} \quad \begin{array}{c}\vdots\ \sigma\\ A \Rightarrow_{\text{st}} C\end{array} \quad \begin{array}{c}\vdots\ \tau\\ B \Rightarrow_{\text{st}} D\end{array}}{M \Rightarrow_{\text{st}} CD}\ (\text{App})$$

where $N = CD$. In this case, we have

$$\dfrac{\begin{array}{c}\vdots\ \pi \text{ and Lem.3.2(3)}\\ M[P] \Rightarrow_{\text{hap}} (AB)[P]\end{array} \qquad \begin{array}{c}\vdots\ \text{i.h. for } \sigma\\ A[P] \Rightarrow_{\text{st}} C[y]\end{array} \qquad \begin{array}{c}\vdots\ \text{i.h. for } \tau\\ B[P] \Rightarrow_{\text{st}} D[y]\end{array}}{M[P] \Rightarrow_{\text{st}} (CD)[y].}\ (\text{App})$$

(Case 4): The last inference is

$$\dfrac{\begin{array}{c}\vdots\ \pi\\ M \Rightarrow_{\text{hap}} \lambda x.A\end{array} \quad \begin{array}{c}\vdots\ \sigma\\ A \Rightarrow_{\text{st}} B\end{array}}{M \Rightarrow_{\text{st}} \lambda x.B}\ (\text{Abs})$$

where $N = \lambda x.B$. If $x \neq z$, then we have

$$\dfrac{\begin{array}{c}\vdots\ \pi \text{ and Lem.3.2(3)}\\ M[P] \Rightarrow_{\text{hap}} (\lambda x.A)[P]\end{array} \qquad \begin{array}{c}\vdots\ \text{i.h. for } \sigma\\ A[P] \Rightarrow_{\text{st}} B[y]\end{array}}{M[P] \Rightarrow_{\text{st}} (\lambda x.B)[y]}\ (\text{Abs})$$

where we assume that $x \notin \text{FV}(P, y)$ (otherwise we rename x). If $x = z$, then $(\lambda x.X)[z := Z] = \lambda x.X$, and the induction hypothesis is not necessary.

(Case 5): The last inference is

$$\dfrac{\begin{array}{c}\vdots\ \pi\\ M \Rightarrow_{\text{hap}} \lambda x.A\end{array} \quad \begin{array}{c}\vdots\ \sigma\\ A \Rightarrow_{\text{st}} Nx\end{array}}{M \Rightarrow_{\text{st}} N}\ (\eta)$$

where $x \notin \text{FV}(N)$ and N is not an abstraction. If $x \neq z$, then we have

$$\dfrac{\begin{array}{c}\vdots\ \pi \text{ and Lem.3.2(3)}\\ M[P] \Rightarrow_{\text{hap}} (\lambda x.A)[P]\end{array} \qquad \begin{array}{c}\vdots\ \text{i.h. for } \sigma\\ A[P] \Rightarrow_{\text{st}} (Nx)[y]\end{array}}{M[P] \Rightarrow_{\text{st}} N[y]}\ (\eta)$$

where we assume that $x \notin \text{FV}(P, y)$ (otherwise we rename x). Note that $(Nx)[y] = (N[y])x$ and the proviso for $N[y]$ is satisfied. If $x = z$, then the induction hypothesis is not necessary. ∎

Lemma 3.4 *If $L \twoheadrightarrow_{\mathrm{st}} (\lambda x_1 \cdots x_m.M)y_1 \cdots y_m F_1 \cdots F_n$, then $L \twoheadrightarrow_{\mathrm{st}} M[x_1 := y_1, \ldots, x_m := y_m]F_1 \cdots F_n$, where $m \geq 1$, $n \geq 0$ and $x_1, \ldots, x_m$ are distinct.* (Note that $[x_1 := y_1, \ldots, x_m := y_m]$ is a simultaneous substitution, and this will be represented by $[\vec{y}]$ for short.)

Proof By induction on the proof of $L \Rightarrow_{\mathrm{st}} (\lambda x_1 \cdots x_m.M)y_1 \cdots y_m F_1 \cdots F_n$. We divide cases according to the last inference of the proof.

(Case 1): $n = 0$, and the proof is

$$\frac{\begin{array}{c}\vdots\,\pi\\ L \Rightarrow_{\mathrm{hap}} AB\end{array}\quad \begin{array}{c}\vdots\,\sigma\\ A \Rightarrow_{\mathrm{st}} (\lambda x_1 \cdots x_m.M)y_1 \cdots y_{m-1}\end{array}\quad \begin{array}{c}\vdots\,\tau\\ B \Rightarrow_{\mathrm{st}} y_m\end{array}}{L \Rightarrow_{\mathrm{st}} (\lambda x_1 \cdots x_m.M)y_1 \cdots y_m.}\ (\mathrm{App})$$

If $m \geq 2$, then by the induction hypothesis for σ, there is a proof

$$\begin{array}{c}\vdots\,\sigma'\\ A \Rightarrow_{\mathrm{st}} (\lambda x_m.M)[\vec{y'}]\end{array}$$

where $[\vec{y'}] = [x_1 := y_1, \ldots, x_{m-1} := y_{m-1}]$. If $m = 1$, then we define $\sigma' = \sigma$. In any case, the proof σ' must be of the form

$$\frac{\begin{array}{c}\vdots\,\phi\\ A \Rightarrow_{\mathrm{hap}} \lambda x_m.C\end{array}\quad \begin{array}{c}\vdots\,\psi\\ C \Rightarrow_{\mathrm{st}} M[\vec{y'}]\end{array}}{A \Rightarrow_{\mathrm{st}} \lambda x_m.(M[\vec{y'}]).}\ (\mathrm{Abs})$$

(Note that $x_m \notin \{y_1, \ldots, y_{m-1}\}$; otherwise we rename x_m.) Then we have

$$\frac{\begin{array}{c}\vdots\,(\dagger)\\ L \Rightarrow_{\mathrm{hap}} C[x_m := B]\end{array}\quad \begin{array}{c}\vdots\,(\ddagger)\\ C[x_m := B] \Rightarrow_{\mathrm{st}} M[\vec{y}]\end{array}}{L \Rightarrow_{\mathrm{st}} M[\vec{y}]}\ \text{Lem. 3.2(2)}$$

where $(\dagger)$ is

$$\frac{\begin{array}{c}\vdots\,\pi\\ L \Rightarrow_{\mathrm{hap}} AB\end{array}\quad \dfrac{\begin{array}{c}\vdots\,\phi\\ A \Rightarrow_{\mathrm{hap}} \lambda x_m.C\end{array}}{AB \Rightarrow_{\mathrm{hap}} (\lambda x_m.C)B}\ \text{Lem.3.2(1)}\quad \begin{array}{c}(\text{axiom } \beta)\\ (\lambda x_m.C)B \Rightarrow_{\mathrm{hap}} C[x_m := B]\end{array}}{L \Rightarrow_{\mathrm{hap}} C[x_m := B]}\ (\mathrm{Tr})\times 2$$

and $(\ddagger)$ is obtained by ψ, τ and Lemma 3.3.

(Case 2): $n \geq 1$, and the proof is

$$\frac{\begin{array}{c}\vdots\,\pi\\ L \Rightarrow_{\mathrm{hap}} AB\end{array}\quad \begin{array}{c}\vdots\,\sigma\\ A \Rightarrow_{\mathrm{st}} (\lambda x_1 \cdots x_m.M)y_1 \cdots y_m F_1 \cdots F_{n-1}\end{array}\quad \begin{array}{c}\vdots\,\tau\\ B \Rightarrow_{\mathrm{st}} F_n\end{array}}{L \Rightarrow_{\mathrm{st}} (\lambda x_1 \cdots x_m.M)y_1 \cdots y_m F_1 \cdots F_n.}\ (\mathrm{App})$$

In this case, we have

$$\dfrac{\overset{\vdots\ \pi}{L \Rightarrow_{\text{hap}} AB} \quad \overset{\vdots\ \text{i.h. for } \sigma}{A \Rightarrow_{\text{st}} M[\vec{y}]F_1 \cdots F_{n-1}} \quad \overset{\vdots\ \tau}{B \Rightarrow_{\text{st}} F_n}}{L \Rightarrow_{\text{st}} M[\vec{y}]F_1 \cdots F_n.}\ (\text{App})$$

(Case 3): The proof is

$$\dfrac{\overset{\vdots\ \pi}{L \Rightarrow_{\text{hap}} \lambda z.A} \quad \overset{\vdots\ \sigma}{A \Rightarrow_{\text{st}} (\lambda x_1 \cdots x_m.M)y_1 \cdots y_m F_1 \cdots F_n z}}{L \Rightarrow_{\text{st}} (\lambda x_1 \cdots x_m.M)y_1 \cdots y_m F_1 \cdots F_n}\ (\eta)$$

where $z \notin \text{FV}((\lambda x_1 \cdots x_m.M)y_1 \cdots y_m F_1 \cdots F_n)$. If $M[\vec{y}]F_1 \cdots F_n$ is not an abstraction, then we have

$$\dfrac{\overset{\vdots\ \pi}{L \Rightarrow_{\text{hap}} \lambda z.A} \quad \overset{\vdots\ \text{i.h. for } \sigma}{A \Rightarrow_{\text{st}} M[\vec{y}]F_1 \cdots F_n z}}{L \Rightarrow_{\text{st}} M[\vec{y}]F_1 \cdots F_n.}\ (\eta)$$

Note that $z \notin \text{FV}(M[\vec{y}]F_1 \cdots F_n) \subseteq \text{FV}((\lambda x_1 \cdots x_m.M)y_1 \cdots y_m F_1 \cdots F_n)$. If $M[\vec{y}]F_1 \cdots F_n$ is an abstraction, then $n = 0$, $M = \lambda v.B$, and σ is

$$\overset{\vdots\ \sigma}{A \Rightarrow_{\text{st}} (\lambda x_1 \cdots x_m v.B)y_1 \cdots y_m z.}$$

where $v \notin \{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ (otherwise, we rename v). Then we have

$$\dfrac{\overset{\vdots\ \pi}{L \Rightarrow_{\text{hap}} \lambda z.A} \quad \overset{\vdots\ \text{i.h. for } \sigma}{A \Rightarrow_{\text{st}} B[\vec{y}, z]}}{L \Rightarrow_{\text{st}} \lambda z.(B[\vec{y}, z]) = (\lambda v.B)[\vec{y}]}\ (\text{Abs})$$

where $[\vec{y}, z] = [x_1 := y_1, \ldots, x_m := y_m, v := z]$. ∎

We consider an inference rule:

$$\dfrac{L \Rightarrow_{\text{hap}} \lambda x.A \quad A \Rightarrow_{\text{st}} Bx}{L \Rightarrow_{\text{st}} B}\ (\eta^+) \text{ with the proviso: } x \notin \text{FV}(B).$$

This is an extension of (η) rule (B may be an abstraction).

Lemma 3.5 *The rule (η^+) is admissible; that is, if $L \twoheadrightarrow_{\text{hap}} \lambda x.A$, $A \twoheadrightarrow_{\text{st}} Bx$, and $x \notin \text{FV}(B)$, then $L \twoheadrightarrow_{\text{st}} B$.*

Proof If B is not an abstraction, then (η^+) is just the rule (η). If $B = \lambda y.C$, then we have

$$\dfrac{L \Rightarrow_{\text{hap}} \lambda x.A \quad \dfrac{A \Rightarrow_{\text{st}} (\lambda y.C)x}{A \Rightarrow_{\text{st}} C[y := x]}\ \text{Lem.3.4}}{L \Rightarrow_{\text{st}} \lambda x.(C[y := x]) = \lambda y.C.}\ (\text{Abs})$$

Note that $x \notin \text{FV}(C)$. ∎

Lemma 3.6 *If $M \twoheadrightarrow_{\text{st}} N$ and $P \twoheadrightarrow_{\text{st}} Q$, then $M[z := P] \twoheadrightarrow_{\text{st}} N[z := Q]$.*

Proof Similar to the proof of Lemma 3.3 ($y = Q$). The only difference is that $N[z{:=}Q]$ may be an abstraction, in Case 5, that is in contravention of the proviso of (η) rule. In such a case, we use the rule (η^+) (Lemma 3.5) to infer $M[P] \twoheadrightarrow_{\text{st}} N[Q]$. $\blacksquare$

Lemma 3.7 *If $L \twoheadrightarrow_{\text{st}} (\lambda x.M)NF_1 \cdots F_n$, then $L \twoheadrightarrow_{\text{st}} M[x := N]F_1 \cdots F_n$, where $n \geq 0$.*

Proof Similar to the proof of Lemma 3.4. ($m = 1$ and $y_m = N$. In Case 1, we use Lemma 3.6 instead of 3.3. In Case 3, we use (η^+) rule (Lemma 3.5).) $\blacksquare$

Lemma 3.8 *If $L \twoheadrightarrow_{\text{st}} M \rightarrow_{\beta\eta} N$, then $L \twoheadrightarrow_{\text{st}} N$.*

Proof By induction on the proof of $L \Rightarrow_{\text{st}} M$. We divide cases according to the last inference of the proof. Let R be the occurrence of the $\beta\eta$-redex such that $M \xrightarrow{R}_{\beta\eta} N$.

(Case 1): The proof is

$$\dfrac{\overset{\vdots\ \pi}{L \Rightarrow_{\text{hap}} AB} \quad \overset{\vdots\ \sigma}{A \Rightarrow_{\text{st}} C} \quad \overset{\vdots\ \tau}{B \Rightarrow_{\text{st}} D}}{L \Rightarrow_{\text{st}} CD}\ (\text{App})$$

where $M = CD$.

(Subcase 1-1): $R = CD$; that is, $C = \lambda x.E$, $M = (\lambda x.E)D$ and $N = E[x := D]$. This is a special case of Lemma 3.7.

(Subcase 1-2): R is in C; that is, $C \xrightarrow{R}_{\beta\eta} C'$ and $N = C'D$. In this case, we have

$$\dfrac{\overset{\vdots\ \pi}{L \Rightarrow_{\text{hap}} AB} \quad \overset{\vdots\ \text{i.h. for } \sigma}{A \Rightarrow_{\text{st}} C'} \quad \overset{\vdots\ \tau}{B \Rightarrow_{\text{st}} D}}{L \Rightarrow_{\text{st}} C'D.}\ (\text{App})$$

(Subcase 1-3): R is in D. This is similar to Subcase 1-2.

(Case 2): The proof is

$$\dfrac{\overset{\vdots\ \pi}{L \Rightarrow_{\text{hap}} \lambda x.A} \quad \overset{\vdots\ \sigma}{A \Rightarrow_{\text{st}} B}}{L \Rightarrow_{\text{st}} \lambda x.B}\ (\text{Abs})$$

where $M = \lambda x.B$.

(Subcase 2-1): $R = \lambda x.B$; that is, $B = Nx$, $M = \lambda x.(Nx)$, and $x \notin \text{FV}(N)$. This is just the rule (η^+) (Lemma 3.5).

(Subcase 2-2): R is in B; that is, $B \xrightarrow{R}_{\beta\eta} B'$, and $N = \lambda x.B'$. Then we have

$$\dfrac{\begin{matrix} \vdots\ \pi & \vdots\ \text{i.h. for } \sigma \\ L \Rightarrow_{\text{hap}} \lambda x.A & A \Rightarrow_{\text{st}} B' \end{matrix}}{L \Rightarrow_{\text{st}} \lambda x.B'.}\ (\text{Abs})$$

(Case 3): The proof is

$$\dfrac{\begin{matrix} \vdots\ \pi & \vdots\ \sigma \\ L \Rightarrow_{\text{hap}} \lambda x.A & A \Rightarrow_{\text{st}} Mx \end{matrix}}{L \Rightarrow_{\text{st}} M}\ (\eta)$$

where $x \notin \text{FV}(M)$ and M is not an abstraction. In this case, we have

$$\dfrac{\begin{matrix} \vdots\ \pi & \vdots\ \text{i.h. for } \sigma \text{ (because } Mx \rightarrow_{\beta\eta} Nx) \\ L \Rightarrow_{\text{hap}} \lambda x.A & A \Rightarrow_{\text{st}} Nx \end{matrix}}{L \Rightarrow_{\text{st}} N.}\ (\eta^+)\ (\text{Lem. 3.5})$$

Note that $x \notin \text{FV}(N) \subseteq \text{FV}(M)$. ∎

Lemma 3.9 $M \twoheadrightarrow_{\text{st}} M$.

Proof By induction on the structure of M. ∎

Theorem 3.10 *If $M \twoheadrightarrow_{\beta\eta} N$, then $M \twoheadrightarrow_{\text{st}} N$.*

Proof Suppose $M = M_1 \rightarrow_{\beta\eta} M_2 \rightarrow_{\beta\eta} \cdots \rightarrow_{\beta\eta} M_k = N$. We can show $M \twoheadrightarrow_{\text{st}} M_i$ for $i = 1, \ldots, k$, by Lemmas 3.9 and 3.8. ∎

Now the Weak Standardization Theorem 2.1 is obvious by Theorems 3.1 and 3.10.

4 Remarks

(1) It seems that we can prove, by the similar proof of Theorem 3.1, the strong version of it: *If $M \twoheadrightarrow_{\text{st}} N$, then there is a strongly standard $\beta\eta$-reduction sequence from M to N.* Then the Strong Standardization Theorem for $\beta\eta$-reduction is proved.

(2) In the case of β-reduction, the Leftmost Reduction Theorem is an easy corollary to the Standardization Theorem because we have the following: *If $M \xrightarrow{R_1}_{\beta} \cdots \xrightarrow{R_n}_{\beta} N$ is a standard β-reduction sequence and N is a β-normal form, then $R_1, \ldots, R_n$ are all the leftmost occurrences of β-redexes.* However, this does not hold for $\beta\eta$-reduction: there are counterexamples

$$\lambda x.\Big((\lambda y.(yy))(\mathbf{I}x)\Big) \xrightarrow{\mathbf{I}x}_{\beta\eta} \lambda x.\Big((\lambda y.(yy))x\Big) \xrightarrow{\lambda x.((\lambda y.(yy))x)}_{\beta\eta} \lambda y.(yy)$$

and

$$\lambda x.\Big(\mathbf{I}(\lambda y.(yy))x\Big) \xrightarrow{\mathbf{I}(\lambda y.(yy))}_{\beta\eta} \lambda x.\Big((\lambda y.(yy))x\Big) \xrightarrow{(\lambda y.(yy))x}_{\beta\eta} \lambda x.(xx)$$

which are strongly standard and end with $\beta\eta$-normal forms, but neither of them is a leftmost reduction sequence. Note that these can be regarded as the sequences

$$\lambda x.\Big((\lambda y.(yy))(\mathbf{I}x)\Big) \xrightarrow{\mathbf{I}x}_{\beta\eta} \lambda x.\Big((\lambda y.(yy))x\Big) \xrightarrow{(\lambda y.(yy))x}_{\beta\eta} \lambda x.(xx)$$

and

$$\lambda x.\Big(\mathbf{I}(\lambda y.(yy))x\Big) \xrightarrow{\mathbf{I}(\lambda y.(yy))}_{\beta\eta} \lambda x.\Big((\lambda y.(yy))x\Big) \xrightarrow{\lambda x.((\lambda y.(yy))x)}_{\beta\eta} \lambda y.(yy),$$

neither of which is weakly standard; then these are not counterexamples. The overlap of redexes $\lambda x.\Big((\lambda y.(yy))x\Big)$ causes a problem.

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