Cut-Elimination Theorem for the Intermediate Logic CD

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Abstract. The logic CD is an intermediate logic (stronger than intuitionistic logic and weaker than classical logic) which exactly corresponds to the Kripke models with constant domains. It is known that the logic CD has a Gentzen-type formulation called LD (which is same as LK except that $(\rightarrow\supset)$ and $(\rightarrow\neg)$ rules are replaced by the corresponding intuitionistic rules) and that the cut-elimination theorem does not hold for LD. In this paper, we present a modification of LD and prove that the cut-elimination holds for it. From this result we obtain a "weak" version of cut-elimination theorem for LD, saying that all "cuts" except some special forms can be eliminated from a proof in LD. Some properties of CD are obtained as its corollaries.

In this paper, we present a Gentzen-type cut-eliminable system for a natural variant of intuitionistic logic, called CD following [2]. CD is characterized by Kripke models with constant domains, from which the name CD comes. Syntactically, CD is obtained from intuitionistic logic by adding

$$\forall x(A(x)\vee B)\supset\forall xA(x)\vee B$$

where x does not occur freely in B. The importance of CD is well-established (e.g. its close relation to the notion of forcing in set theory ([1],[4])), and so this intermediate logic (stronger than intuitionistic logic and weaker than classical logic) is extensively studied in [1] $\sim$ [8] and [11].

CD also has a quite natural Gentzen-type formulation LD, that is, LK with ($\rightarrow\supset$) and ($\rightarrow\neg$) rules replaced by the corresponding intuitionistic (LJ) rules.[1] But unfortunately, the cut-elimination theorem does not hold for LD ([5], [6]). Moreover, it is known that even if we add any finite number of inference rules to LD, we can not get a cut-free system for CD ([5]). In this paper, we present a certain non-standard extension of LD, and prove the cut-elimination theorem for it. Moreover, we prove a weak version of cut-elimination theorem for LD. From these cut-elimination theorems, we also obtain some useful corollaries.

In section 1, we give the definition of LD and show some preliminary results. In section 2, we give a technical extension of LD called c-LD. Section 3 is devoted to prove the cut-elimination theorem for c-LD. In section 4, we prove the equivalence between LD and c-LD. We then show a version of "cut-elimination theorem" for LD which says that all "cuts" except some special forms can be eliminated from a proof in LD. In section 5, we apply our cut-elimination theorems to show the following properties of CD:

[1]Our LD is a little different from one with the same name in [11].

- If A is a theorem of CD and A has no negative occurrence of disjunction, then A is a theorem of intuitionistic logic.
- If $A \vee B$ is a theorem of CD, then either A or B is a theorem of CD. (Disjunction property.)
- If $\exists x A(x)$ is a theorem of CD, then $A(t)$ is a theorem of CD for some term t. (Existence property.)

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1 Sequent calculus LD

Our language consists of the following symbols:

1) *constant symbols*: $c_0, c_1, ...$;

2) *free variables*: $a_0, a_1, ...$;

3) *bound variables*: $x_0, x_1, ...$;

4) for each $n (n = 0, 1, ...)$, n-ary *predicate symbols*: $p_0^n, p_1^n, ...$;

5) *logical connectives*: $\bot$(falsefood), $\wedge$(conjunction), $\vee$(disjunction), $\supset$(implication);

6) *quantifiers*: $\forall$(universal quantifier), $\exists$(existential quantifier);

7) *auxiliary symbols*: (,), comma, $\rightarrow$.

Let A be a sequence of symbols, and let v_1, v_2 be symbols. Then by $A[v_1/v_2]$ we mean the expression obtained from A by replacing all the occurrences of v_1 by v_2.

Constant symbols and free variables are called *terms*.

Formulas are defined inductively as follows:

1) If p is an n-ary predicate symbol, and $t_1, ..., t_n$ are terms, then $pt_1...t_n$ is a formula;

2) $\bot$ is a formula;

3) If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \supset B)$ are formulas;

4) If A is a formula, a is a free variable, and x is a bound variable not occurring in A, then $\forall x A[a/x]$ and $\exists x A[a/x]$ are formulas.

When there is no fear of confusion, we will omit parentheses in formulas. The negation symbol "$\neg$" is not used formally, but it may be introduced by

$$\neg A = A \supset \bot.$$

If Γ and Δ are finite (possibly empty) sequences of formulas separated by commas, then $\Gamma \rightarrow \Delta$ is called a *sequent*, and Γ and Δ are called the *antecedent* and *succedent* of this sequent, respectively.

We will use letters $s, t, \ldots$ to represent terms, $a, b, c, \ldots$ to represent free variables, $x, y, z, \ldots$ to represent bound variables, $A, B, C, \ldots$ to represent formulas, $\Gamma, \Delta, \ldots$ to represent sequences of formulas, and $\mathcal{S}, \mathcal{T}, \ldots$ to represent sequents.

Sometimes, we write $A(a)$ to denote formula A with some specified occurrences of free variable a in A. Then, $A(v)$ means the expression obtained from A by replacing the specified occurrences of a by v.

Now, we define sequent calculus *LD.* The *axioms* of LD are

$$A \rightarrow A$$

and

$$\bot \rightarrow \quad .$$

The *inference rules* of LD are as follows:

$$(w\rightarrow)\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} \qquad (\rightarrow w)\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}$$

$$(c\rightarrow)\frac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} \qquad (\rightarrow c)\frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}$$

$$(e\rightarrow)\frac{\Gamma, A, B, \Pi \rightarrow \Delta}{\Gamma, B, A, \Pi \rightarrow \Delta} \qquad (\rightarrow e)\frac{\Gamma \rightarrow \Delta, A, B, \Sigma}{\Gamma \rightarrow \Delta, B, A, \Sigma}$$

$$(\wedge\rightarrow 1)\frac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} \qquad (\wedge\rightarrow 2)\frac{B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}$$

$$(\rightarrow\wedge)\frac{\Gamma \rightarrow \Delta, A \qquad \Pi \rightarrow \Sigma, B}{\Gamma, \Pi \rightarrow \Delta, \Sigma, A \wedge B}$$

$$(\vee\rightarrow)\frac{A,\Gamma\rightarrow\Delta \qquad B,\Pi\rightarrow\Sigma}{A\vee B,\Gamma,\Pi\rightarrow\Delta,\Sigma}$$

$$(\rightarrow\vee 1)\frac{\Gamma\rightarrow\Delta,A}{\Gamma\rightarrow\Delta,A\vee B} \qquad (\rightarrow\vee 2)\frac{\Gamma\rightarrow\Delta,B}{\Gamma\rightarrow\Delta,A\vee B}$$

$$(\supset\rightarrow)\frac{\Gamma\rightarrow\Delta,A \qquad B,\Pi\rightarrow\Sigma}{A\supset B,\Gamma,\Pi\rightarrow\Delta,\Sigma}$$

$$(\rightarrow\supset)\frac{A,\Gamma\rightarrow B}{\Gamma\rightarrow A\supset B}$$

$$(\forall\rightarrow)\frac{A(t),\Gamma\rightarrow\Delta}{\forall xA(x),\Gamma\rightarrow\Delta} \qquad (\rightarrow\forall)\frac{\Gamma\rightarrow\Delta,A(a)}{\Gamma\rightarrow\Delta,\forall xA(x)}$$

where a does not occur in the lower sequent.

$$(\exists\rightarrow)\frac{A(a),\Gamma\rightarrow\Delta}{\exists xA(x),\Gamma\rightarrow\Delta} \qquad (\rightarrow\exists)\frac{\Gamma\rightarrow\Delta,A(t)}{\Gamma\rightarrow\Delta,\exists xA(x)}$$

where a does not occur in the lower sequent.

$$\text{CUT}\frac{\Gamma\rightarrow\Delta,A \qquad A,\Pi\rightarrow\Sigma}{\Gamma,\Pi\rightarrow\Delta,\Sigma}$$

The *Proofs* in LD are defined inductively as follows:

1) if $\mathcal{S}$ is an axiom of LD, then $\mathcal{S}$ is a proof of $\mathcal{S}$ in LD;

2) if $\mathcal{P}_1$ is a proof of $\mathcal{S}_1$ in LD, and

$$\mathrm{J}\frac{\mathcal{S}_1}{\mathcal{S}}$$

is an inference rule of LD, then

$$\mathrm{J}\frac{\mathcal{P}_1}{\mathcal{S}}$$

is a proof of $\mathcal{S}$ in LD;

3) if $\mathcal{P}_1$ and $\mathcal{P}_2$ are proofs of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, in LD, and

$$\mathrm{J}\frac{\mathcal{S}_1 \qquad \mathcal{S}_2}{\mathcal{S}}$$

is an inference rule of LD, then

$$\mathrm{J}\frac{\mathcal{P}_1 \qquad \mathcal{P}_2}{\mathcal{S}}$$

is a proof of $\mathcal{S}$ in LD.

A *cut-free* proof means a proof which does not contain CUT.

If there is a proof $\mathcal{P}$ of $\mathcal{S}$ in LD, we say $\mathcal{S}$ is ***provable in*** LD.

Examples of proofs in LD:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A(a) \to A(a) \qquad\qquad B \to B}{A(a) \vee B \to A(a), B}\;(\vee\to)}{\forall x(A(x)\vee B) \to A(a), B}\;(\forall\to)}{\forall x(A(x)\vee B) \to B, A(a)}\;(\to e)}{\forall x(A(x)\vee B) \to B, \forall x A(x)}\;(\to\forall)}{\forall x(A(x)\vee B) \to B, \forall x A(x)\vee B}\;(\to\vee 1)}{\forall x(A(x)\vee B) \to \forall x A(x)\vee B, B}\;(\to e)}{\forall x(A(x)\vee B) \to \forall x A(x)\vee B, \forall x A(x)\vee B}\;(\to\vee 2)}{\forall x(A(x)\vee B) \to \forall x A(x)\vee B}\;(\to c)}{\to \forall x(A(x)\vee B)\supset(\forall x A(x)\vee B)}\;(\to\supset)
$$

where a does not occur in $\forall x A(x), B$.

$$
\cfrac{\begin{matrix}\vdots\ \text{(as above)}\\ \forall x(A(x)\vee B) \to B, \forall x A(x)\end{matrix} \qquad \cfrac{\cfrac{\forall x A(x) \to \forall x A(x)}{C, \forall x A(x) \to \forall x A(x)}\;(w\to)}{\forall x A(x) \to C\supset\forall x A(x)}\;(\to\supset)}{\forall x(A(x)\vee B) \to B, C\supset\forall x A(x)}\;\text{CUT}
$$

where a does not occur in $\forall x A(x), B$.

If we extend LD by adding the inference rule:

$$
(\to\supset_{LK})\;\frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A\supset B}\quad ,
$$

then we get sequent calculus LK for classical logic. If we replace $(\to\forall)$ in LD by

$$
(\to\forall_{LJ})\;\frac{\Gamma \to A(a)}{\Gamma \to \forall x A(x)}\quad ,
$$

where a does not occur in the lower sequent, then we get sequent calculus LJ' for intuitionistic logic ([10]).

The following results are well-known.

Proposition 1.1 **([2], [3], [6], [8], [11])** *The following four conditions are equivalent:*

1. *$\to A$ is provable in LD;*

2. *A is provable in intuitionistic predicate logic with the additional axiom schema:*

$$\forall x(B(x)\vee C)\supset(\forall xB(x)\vee C);$$

3. *A is valid in any Kripke-model having a constant domain;*

4. *A is valid in any complete Heyting algebra satisfying the following $\vee,\bigwedge$-distributive law:*

$$\bigwedge_{i\in I}(p_i \vee q) = \bigwedge_{i\in I} p_i \vee q.$$

When one of these conditions holds, we say A is a *theorem of CD* (CD means "Constant Domains").

Proposition 1.2 **([5], [6])** *The cut-elimination theorem fails for LD, i.e. there is a sequent which is provable in LD but not provable without a CUT.*

Actually, a sequent

$$\forall x(A(x)\vee B) \to B, C\supset\forall xA(x)$$

is a counter example of the cut-elimination theorem for LD. Moreover it is shown in [5] that: even if we add any finite number of inference rules to LD, we can not get a cut-free system for CD.

2 Introducing c-LD

In this section, we define a system *c-LD* which is a non-standard extension of LD. For this definition, we introduce a notion of "connections" which express "the dependency between the formulas in the antecedent and the formulas in the succedent of a sequent". With the help of this notion, we can extend $(\to\supset)$ rule in LD to a rule like $(\to \supset_{LK})$.

From now on, we use superscripts to distinguish occurrences of formulas in a sequent. For example, a sequent

$$A^0, A^1 \to A^2$$

has three occurrences of the same formula A.

We introduce a binary relation $\sim$, called *connection*, between the occurrences of formulas in the antecedent and those in succedent of a sequent. To denote, for instance, the connected pairs $A^1 \sim C^5$, $B^3 \sim C^4$ and $B^3 \sim C^5$ in a sequent

$$A^1, A^2, B^3 \to C^4, C^5 \quad ,$$

we write

$$A^1, A^2, B^3 \to C^4, C^5 \quad .$$

A sequent with connections is also called a sequent.

Now, we define the system c-LD which derives sequents with connections ("c" means "Connections").

The axioms of c-LD are

$$A \to A$$

and

$$\bot \to A.$$

c-LD has nineteen inference rules, each of which corresponds to a rule in LD.

A rule essentially different from the corresponding one in LD is

$$(\to\supset+)\frac{A^1, \Gamma \to \Delta, B^2}{\Gamma \to \Delta, (A \supset B)^3}$$

where for any formula γ in Γ and δ in Δ,

1) $A^1 \not\sim \delta$ at the upper sequent,

2) $\gamma \sim \delta$ at the lower sequent $\Leftrightarrow \gamma \sim \delta$ at the upper sequent,

3) $\gamma \sim (A \supset B)^3$ at the lower sequent $\Leftrightarrow \gamma \sim B^2$ at the upper sequent.

$(A \supset B)^3$ is called the *principal occurrence* of this inference. Recall that we do not admit the inference

$$\frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}$$

in LD unless Δ is empty. But by $(\to\supset+)$, we can infer ($\Gamma \to \Delta, A \supset B$) from ($A^1, \Gamma \to \Delta, B$) if $A^1 \not\sim \delta$ for any formula δ in Δ.

Other inference rules in c-LD have the same names and similar forms to the corresponding ones in LD, as follows:

- $(w\to)$:

$$(w\to)\frac{\Gamma \to \Delta}{A^1, \Gamma \to \Delta}$$

where the connections between Γ and Δ are inherited (i.e.

$\gamma \sim \delta$ at the lower sequent $\Leftrightarrow \gamma \sim \delta$ at the upper sequent

for any formulas γ in Γ and δ in Δ), and A^1 may or may not be connected to any formula δ in Δ at the lower sequent. A^1 is called the principal occurrence of this inference.

- $(\to w)$: symmetric to $(w\to)$.

- $(c\to)$:

$$(c\to)\frac{A^1, A^2, \Gamma \to \Delta}{A^3, \Gamma \to \Delta}$$

where the connections between Γ and Δ are inherited, and for any formula δ in Δ,

$A^3 \sim \delta$ at the lower sequent $\Leftrightarrow$ ($A^1 \sim \delta$ or $A^2 \sim \delta$) at the upper sequent.

A^3 is called the principal occurrence, and A^1 and A^2 are called the *contraction occurrences* of this inference.

- $(\to c)$: symmetric to $(c\to)$.

- $(e\to)$:

$$(e\to)\frac{\Gamma, A^1, B^2, \Pi \to \Delta}{\Gamma, B^2, A^1, \Pi \to \Delta}$$

where the connections between (Γ, Π) and Δ are inherited, and for any formula δ in Δ,

1) $A^1 \sim \delta$ at the lower sequent $\Leftrightarrow$ $A^1 \sim \delta$ at the upper sequent,

2) $B^2 \sim \delta$ at the lower sequent $\Leftrightarrow$ $B^2 \sim \delta$ at the upper sequent.

We will use a superscript in common for some different occurrences of a formula which uniquely correspond each other, as above.

- $(\to e)$: symmetric to $(e\to)$.

- $(\wedge\to 1)$:

$$(\wedge\to 1)\frac{A^1, \Gamma \to \Delta}{(A\wedge B)^2, \Gamma \to \Delta}$$

where the connections between Γ and Δ are inherited, and for any formula δ in Δ,

$(A\wedge B)^2 \sim \delta$ at the lower sequent $\Leftrightarrow$ $A^1 \sim \delta$ at the upper sequent.

$(A\wedge B)^2$ is called the principal occurrence of this inference.

- $(\wedge\to 2)$: similar to $(\wedge\to 1)$.

- $(\to\wedge)$:

$$(\to\wedge)\frac{\Gamma \to \Delta, A^1 \qquad\qquad \Pi \to \Sigma, B^2}{\Gamma, \Pi \to \Delta, \Sigma, (A\wedge B)^3}$$

where

1) the connections between Γ and Δ are inherited,

2) the connections between Π and Σ are inherited,

3) there is no connection between Γ and Σ at the lower sequent,

4) there is no connection between Π and Δ at the lower sequent,

5) for any formulas γ in Γ and π in Π,

   5-1) $\gamma \sim (A\wedge B)^3$ at the lower sequent $\Leftrightarrow$ $\gamma \sim A^1$ at the left-side upper sequent,

   5-2) $\pi \sim (A\wedge B)^3$ at the lower sequent $\Leftrightarrow$ $\pi \sim B^2$ at the right-side upper sequent.

$(A\wedge B)^3$ is called the principal occurrence of this inference.

- $(\vee\to)$: symmetric to $(\to\wedge)$.

- $(\to\vee 1)$: symmetric to $(\wedge\to 1)$.

• $(\to\vee 2)$: symmetric to $(\wedge\to 2)$.

• $(\supset\to)$:

$$(\supset\to)\ \frac{\Gamma \to \Delta, A^1 \qquad\qquad B^2, \Pi \to \Sigma}{(A \supset B)^3, \Gamma, \Pi \to \Delta, \Sigma}$$

where

1) the connections between Γ and Δ are inherited,

2) the connections between Π and Σ are inherited,

3) there is no connection between Π and Δ at the lower sequent,

4) there is no connection between $(A \supset B)^3$ and Δ at the lower sequent,

5) for any formulas γ in Γ and σ in Σ,

5-2) $(A \supset B)^3 \sim \sigma$ at the lower sequent $\Leftrightarrow$ $B^2 \sim \sigma$ at the right-side upper sequent,

5-3) $\gamma \sim \sigma$ at the lower sequent $\Leftrightarrow$ ($\gamma \sim A^1$ at the left-side upper sequent and $B^2 \sim \sigma$ at the right-side upper sequent).

$(A \supset B)^3$ is called the principal occurrence of this inference.

• $(\forall\to)$:

$$(\forall\to)\ \frac{A(t)^1, \Gamma \to \Delta}{\forall x A(x)^2, \Gamma \to \Delta}$$

where the connections between Γ and Δ are inherited, and for any formula δ in Δ,

$\forall x A(x)^2 \sim \delta$ at the lower sequent $\Leftrightarrow$ $A(t)^1 \sim \delta$ at the upper sequent.

$\forall x A(x)^2$ is called the principal occurrence of this inference.

• $(\to\forall)$:

$$(\rightarrow\forall)\frac{\Gamma \rightarrow \Delta, A(a)^1}{\Gamma \rightarrow \Delta, \forall x A(x)^2}$$

where a is a free variable not occuring in the lower sequent, the connections between Γ and Δ are inherited, and for any formula γ in Γ,

$\gamma \sim \forall x A(x)^2$ at the lower sequent $\Leftrightarrow$ $\gamma \sim A(a)^1$ at the upper sequent.

$\forall x A(x)^2$ is called the principal occurrence of this inference.

- $(\exists\rightarrow)$: symmetric to $(\rightarrow\forall)$.

- $(\rightarrow\exists)$: symmetric to $(\forall\rightarrow)$.

- CUT:

$$\text{CUT}\frac{\Gamma \rightarrow \Delta, A^1 \qquad\qquad A^2, \Pi \rightarrow \Sigma}{\Gamma, \Pi \rightarrow \Delta, \Sigma}$$

where

1) the connections between Γ and Δ are inherited,

2) the connections between Π and Σ are inherited,

3) there is no connection between Π and Δ at the lower sequent,

4) for any formulas γ in Γ and σ in Σ,

$\gamma \sim \sigma$ at the lower sequent $\Leftrightarrow$ ($\gamma \sim A^1$ at the left-side upper sequent and $A^2 \sim \sigma$ at the right-side upper sequent).

A^1 and A^2 are called the *cut occurrences* of this inference.

Notice

1. In c-LD, we don't adopt the axiom

$$\bot \rightarrow$$

for some technical reasons. Then it is obvious that if $\Gamma \rightarrow \Delta$ is provable in c-LD then Δ is not empty.

2. In ($\supset\rightarrow$) and CUT, it may happen that: for some formulas γ and σ, $\gamma \not\sim \sigma$ at the upper sequents but $\gamma \sim \sigma$ at the lower sequent. Such situation does not happen in other inference rules.

Examples of proofs in c-LD (these sequents are not cut-free provable in LD ([5],[6])):

$$\dfrac{A(a) \rightarrow A(a) \qquad\qquad B \rightarrow B}{A(a) \vee B \rightarrow A(a), B}\ (\vee\rightarrow)$$

$$\dfrac{A(a) \vee B \rightarrow A(a), B}{\forall x(A(x) \vee B) \rightarrow A(a), B}\ (\forall\rightarrow)$$

$$\dfrac{\forall x(A(x) \vee B) \rightarrow A(a), B}{\forall x(A(x) \vee B) \rightarrow B, A(a)}\ (\rightarrow e)$$

$$\dfrac{\forall x(A(x) \vee B) \rightarrow B, A(a)}{\forall x(A(x) \vee B) \rightarrow B, \forall x A(x)}\ (\rightarrow\forall)$$

$$\dfrac{\forall x(A(x) \vee B) \rightarrow B, \forall x A(x)}{C, \forall x(A(x) \vee B) \rightarrow B, \forall x A(x)}\ (w\rightarrow)$$

$$\dfrac{C, \forall x(A(x) \vee B) \rightarrow B, \forall x A(x)}{\forall x(A(x) \vee B) \rightarrow B, C \supset \forall x A(x)}\ (\rightarrow\supset+)$$

where a does not occur in $\forall x A(x), B$.

$$
(\rightarrow\exists)\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow \exists x A(x)} \qquad (\supset\rightarrow)\frac{A(a) \rightarrow A(a) \qquad \bot \rightarrow \bot}{A(a)\supset\bot, A(a) \rightarrow \bot}
$$

$$
(\vee\rightarrow)\frac{A(a) \rightarrow \exists x A(x) \qquad A(a)\supset\bot, A(a) \rightarrow \bot}{A(a)\vee(A(a)\supset\bot), A(a) \rightarrow \exists x A(x), \bot}
$$

$$
(\forall\rightarrow)\frac{A(a)\vee(A(a)\supset\bot), A(a) \rightarrow \exists x A(x), \bot}{\forall x(A(x)\vee(A(x)\supset\bot)), A(a) \rightarrow \exists x A(x), \bot}
$$

$$
(e\rightarrow)\frac{\forall x(A(x)\vee(A(x)\supset\bot)), A(a) \rightarrow \exists x A(x), \bot}{A(a), \forall x(A(x)\vee(A(x)\supset\bot)) \rightarrow \exists x A(x), \bot}
$$

$$
(\exists\rightarrow)\frac{A(a), \forall x(A(x)\vee(A(x)\supset\bot)) \rightarrow \exists x A(x), \bot}{\exists x A(x), \forall x(A(x)\vee(A(x)\supset\bot)) \rightarrow \exists x A(x), \bot}
$$

$$
(\rightarrow\supset+)\frac{\exists x A(x), \forall x(A(x)\vee(A(x)\supset\bot)) \rightarrow \exists x A(x), \bot}{\forall x(A(x)\vee(A(x)\supset\bot)) \rightarrow \exists x A(x), \exists x A(x)\supset\bot}
$$

where a does not occur in $\forall x A(x)$.

Theorem 2.1 *Suppose $\mathcal{P}$ is a proof of $\Gamma \rightarrow \Delta$ in LD. Then from $\mathcal{P}$, we can construct a proof $\mathcal{P}'$ of $\Gamma \rightarrow \Delta'$ with some connections in c-LD where:*

$\Delta' = \Delta$ *if Δ is not empty,*

$\Delta' = \bot$ *if Δ is empty.*

Proof We prove this theorem by induction on the number of inference rules in $\mathcal{P}$.

If $\mathcal{P}$ is of the form

$$
(\supset\rightarrow)\frac{\begin{matrix}\vdots\ \mathcal{P}_1 \\ \Pi \rightarrow C, A\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{P}_2 \\ B, \Sigma \rightarrow\end{matrix}}{A\supset B, \Pi, \Sigma \rightarrow C} ,
$$

then by induction hypothesis, we can get the following two proofs in c-LD:

$$
\begin{matrix}\vdots\ \mathcal{P}'_1 \\ \Pi \rightarrow C, A\end{matrix} \quad \text{and} \quad \begin{matrix}\vdots\ \mathcal{P}'_2 \\ B, \Sigma \rightarrow \bot.\end{matrix}
$$

So we can get $\mathcal{P}'$ as follows:

$$
\dfrac{
\dfrac{
\dfrac{\vdots\, \mathcal{P}_1' \atop \Pi \to C, A \qquad \vdots\, \mathcal{P}_2' \atop B, \Sigma \to \bot}{A \supset B, \Pi, \Sigma \to C, \bot}\,(\supset\to)
\qquad \bot \to C
}{A \supset B, \Pi, \Sigma \to C, C}\,\mathrm{CUT}
}{A \supset B, \Pi, \Sigma \to C}\,(\to c) \quad .
$$

If $\mathcal{P}$ is of other forms, we can construct $\mathcal{P}'$ similarly or more easily. □

The converse of this theorem will be proved in section 4.

3 Cut-elimination theorem for c-LD

In this section, we prove the cut-elimination theorem for c-LD. Our proof is based on the ordinary proof of the cut-elimination theorem for LK ([10]), but it requires a little more sophisticated argument to treat the connections.

Let $\mathcal{S}$ and $\mathcal{S}'$ be sequents with connections. If $\mathcal{S}'$ can be obtained from $\mathcal{S}$ by some applications of inference rules ($e\!\rightarrow$) and ($\rightarrow\! e$) in c-LD, then we say $\mathcal{S}'$ is a *permutation* of $\mathcal{S}$.

Let $\mathcal{S}$ be a sequent with connections and $\mathcal{S}'$ be a permutation of $\mathcal{S}$ where:

• $\mathcal{S} = \Gamma \rightarrow \Delta$;

• $\mathcal{S}' = \Gamma_0, \Gamma_1, ..., \Gamma_m \rightarrow \Delta_0, \Delta_1, ..., \Delta_n$ ($m, n \geq 0$, Γ_i and Δ_i may be empty);

• each Γ_i ($0 \leq i \leq m$) is a subsequence of Γ, i.e. the order of formulas in Γ_i is the same as in Γ;

• each Δ_i ($0 \leq i \leq n$) is a subsequence of Δ.

Then we say $\mathcal{S}'$ is an *(m,n)-permutation* of $\mathcal{S}$. To denote the boundaries between $\Gamma_0, \Gamma_1, ..., \Gamma_m$ and between $\Delta_0, \Delta_1, ..., \Delta_n$, we often write $\mathcal{S}'$ as:

$$\Gamma_0; \Gamma_1; ...; \Gamma_m \rightarrow \Delta_0; \Delta_1; ...; \Delta_n.$$

Obviously, the (0,0)-permutation of $\mathcal{S}$ is $\mathcal{S}$.

Lemma 3.1 *Suppose $\mathcal{P}$ is a cut-free proof of a sequent $\mathcal{S}$ in c-LD, and $\mathcal{T}$ is a (1,0)-permutation of $\mathcal{S}$ where $\mathcal{S} = (\Gamma \rightarrow \Delta)$, $\mathcal{T} = (\Gamma_0; \Gamma_1 \rightarrow \Delta)$, and there is no connection between Γ_1 and Δ at $\mathcal{T}$. Then from $\mathcal{P}$, we can construct a cut-free proof $\mathcal{P}'$ in c-LD of $(\Gamma_0 \rightarrow \Delta)$ with the same connections between Γ_0 and Δ as $\mathcal{T}$.*

Proof This lemma is proved by induction on the number of inference rules in $\mathcal{P}$. We only consider the following case: $\mathcal{P}$ is of the form

$$(\supset\rightarrow)\frac{\begin{array}{c}\vdots\mathcal{P}_1\\ \Theta\rightarrow\Lambda,A^1\end{array}\qquad\begin{array}{c}\vdots\mathcal{P}_2\\ B^2,\Pi\rightarrow\Sigma\end{array}}{(A\supset B)^3,\Theta,\Pi\rightarrow\Lambda,\Sigma}\ .$$

We call the above left-side upper sequent $\mathcal{S}_1$ and right-side upper sequent $\mathcal{S}_2$.

Subcase 1: $(A\supset B)^3$ is in Γ_0 at $\mathcal{T}$, i.e. $\mathcal{T}$ is of the form

$$(A\supset B)^3,\Theta_0,\Pi_0;\Theta_1,\Pi_1\rightarrow\Lambda,\Sigma.$$

Sub-subcase 1-1: $B^2\sim\sigma$ at $\mathcal{S}_2$ for some formula σ in Σ. Then consider two (1,0)-permutations $\mathcal{T}_1$ of $\mathcal{S}_1$ and $\mathcal{T}_2$ of $\mathcal{S}_2$ where

$\mathcal{T}_1=\Theta_0;\Theta_1\rightarrow\Lambda,A,$

$\mathcal{T}_2=B,\Pi_0;\Pi_1\rightarrow\Sigma.$

There is no connection between Θ_1 and (Λ,A) at $\mathcal{T}_1$ and between Π_1 and Σ at $\mathcal{T}_2$. So by induction hypotheses, we can get the following two proofs:

$$\begin{array}{c}\vdots\mathcal{P}_1'\\ \Theta_0\rightarrow\Lambda,A\ ,\end{array}\qquad\begin{array}{c}\vdots\mathcal{P}_2'\\ B,\Pi_0\rightarrow\Sigma\ ,\end{array}$$

and we can get $\mathcal{P}'$ as:

$$(\supset\rightarrow)\frac{\vdots\mathcal{P}_1'\qquad\qquad\vdots\mathcal{P}_2'}{A\supset B,\Theta_0,\Pi_0\rightarrow\Lambda,\Sigma}\ .$$

Sub-subcase 1-2: $B^2\not\sim\sigma$ at $\mathcal{S}_2$ for any formula σ in Σ. Then consider a (1,0)-permutation $\mathcal{T}_2$ of $\mathcal{S}_2$ where

$\mathcal{T}_2=\Pi_0;B,\Pi_1\rightarrow\Sigma.$

Then, there is no connection between (B,Π_1) and Σ at $\mathcal{T}_2$. So by induction hypothesis, we can get the following proof:

$$\begin{array}{c} \vdots\, \mathcal{P}_2' \\ \Pi_0 \rightarrow \Sigma\ , \end{array}$$

and we can get $\mathcal{P}'$ as:

$$\dfrac{\dfrac{\vdots\, \mathcal{P}_2'}{\text{some } (w\rightarrow),(\rightarrow w),(\rightarrow e)}}{A\supset B, \Theta_0, \Pi_0 \rightarrow \Lambda, \Sigma\ .}$$

Subcase 2: $(A\supset B)^3$ is in Γ_1 at $\mathcal{T}$, i.e. $\mathcal{T}$ is of the form

$$\Theta_0, \Pi_0; (A\supset B)^3, \Theta_1, \Pi_1 \rightarrow \Lambda, \Sigma.$$

Then consider a (1,0)-permutation $\mathcal{T}_2$ of $\mathcal{S}_2$ where

$\mathcal{T}_2 = \Pi_0; B, \Pi_1 \rightarrow \Sigma.$

There is no connection between (B, Π_1) and Σ at $\mathcal{T}_2$. So by induction hypothesis, we can get the following proof:

$$\begin{array}{c} \vdots\, \mathcal{P}_2' \\ \Pi_0 \rightarrow \Sigma\ , \end{array}$$

and we can get $\mathcal{P}'$ as:

$$\dfrac{\dfrac{\vdots\, \mathcal{P}_2'}{\text{some } (w\rightarrow),(\rightarrow w),(\rightarrow e)}}{\Theta_0, \Pi_0 \rightarrow \Lambda, \Sigma\ .}$$

If $\mathcal{P}$ is of other forms, we can construct $\mathcal{P}'$ similary. □

Lemma 3.2 *Suppose $\mathcal{P}$ is a proof of $\Gamma \rightarrow \Delta$ in c-LD, a is a free variable, and t is a term. Then from $\mathcal{P}$, we can construct a proof $\mathcal{P}'$ of $\Gamma[a/t] \rightarrow \Delta[a/t]$ (sequent obtained by* *replacing all the occurrences of a by t) in c-LD where $\mathcal{P}'$ has the same structure as $\mathcal{P}$, i.e. differences between $\mathcal{P}$ and $\mathcal{P}'$ are only the differences of some free variables.*

Lemma 3.3 *Suppose $\mathcal{P}$ is a cut-free proof of a sequent $\mathcal{S}$ in c-LD, $\mathcal{S}$ is of the form*

$$\Gamma \to \Delta_0, \bot^1, \Delta_1, ..., \bot^n, \Delta_n (n \geq 0)$$

and A is a formula. Then from $\mathcal{P}$, we can construct a cut-free proof $\mathcal{P}'$ of a sequent $\mathcal{S}'$ in c-LD where

$\mathcal{S}' = \Gamma \to \Delta_0, A^1, \Delta_1, ..., A^n, \Delta_n$

and for any formula γ in Γ and δ in $\Delta_0, \Delta_1, ..., \Delta_n$,

1) $\gamma \sim \delta$ at $\mathcal{S}' \Leftrightarrow \gamma \sim \delta$ at $\mathcal{S}$,

2) $\gamma \sim A^i$ at $\mathcal{S}' \Leftrightarrow \gamma \sim \bot^i$ at $\mathcal{S}$ (i = 1,...,n).

Lemma 3.2 and 3.3 are proved by easy induction on the number of inference rules in $\mathcal{P}$. So we omit these proofs.

To prove the cut-elimination theorem for c-LD, we introduce a new inference rule called MIX.

Let $\Gamma \to \Delta_0; \Delta_1$ be a (0,1)-permutation of a sequent $\Gamma \to \Delta$, and $\Pi_0, \Pi_1 \to \Sigma$ be a (1,0)-permutation of a sequent $\Pi \to \Sigma$. Moreover, we assume that Δ_1 and Π_1 are the (possibly empty) sequences consisting only of a formula A. Then, the following inference rule is called MIX:

$$\text{MIX}\frac{\Gamma \to \Delta \qquad\qquad \Pi \to \Sigma}{\Gamma, \Pi_0 \to \Delta_0, \Sigma}$$

where for any formulas γ in Γ, δ_0 in Δ_0, π_0 in Π_0, and σ in Σ,

1) $\gamma \sim \delta_0$ at the lower sequent $\Leftrightarrow \gamma \sim \delta_0$ at the left-side upper sequent,

2) $\gamma \sim \sigma$ at the lower sequent $\Leftrightarrow$ ($\gamma \sim A^i$ for some A^i in Δ_1 at the left-side upper sequent and $A^j \sim \sigma$ for some A^j in Π_1 at the right-side upper sequent),

3) $\pi_0 \not\sim \delta_0$ at the lower sequent,

4) $\pi_0 \sim \sigma$ at the lower sequent $\Leftrightarrow$ $\pi_0 \sim \sigma$ at the right-side upper sequent.

A is called the *mix formula*, and the occurrences of A in Δ_1 at the left-side upper sequent and those in Π_1 at the right-side upper sequent are called the *mix occurrences* of this MIX.

In the ordinary proof of the cut-elimination theorem for LK, "mix" deletes all occurrences of its "mix formula" at once. But in our case, MIX can delete any number of mix formulas. This difference is essential for our proof.

CUT is a special form of MIX, and MIX can be replaced by some applications of $(w\to)$,$(\to w)$,$(c\to)$,$(\to c)$,$(e\to)$,$(\to e)$ and CUT. Hence, in the rest of this section, we consider only MIX instead of CUT.

In a proof

$$\text{MIX}\ \frac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \Gamma \to \Delta\end{array} \qquad \begin{array}{c}\vdots\,\mathcal{P}_2\\ \Pi \to \Sigma\end{array}}{\Gamma, \Pi_0 \to \Delta_0, \Sigma}\ ,$$

we define the *grade* of this MIX to be the number of logical connectives and quantifiers occurring in the mix formula of this MIX. When $\Delta_0 = \Delta$ and $\Pi_0 = \Pi$, we define grade=0.

In any inference rule, if A^1 is an occurrence of a formula in the lower sequent and it is not the principal occurrence, then there is a corresponding occurrence A^1 in the upper sequent, and we write

$$(A^1 \text{ in the lower sequent}) \lhd (A^1 \text{ in the upper sequent}).$$

Moreover, in $(c\to)$ and $(\to c)$, if A^1 is the principal occurrence in the lower sequent and

A^2 is one of the contraction occurrences in the upper sequent, then we write

$$(A^1 \text{ in the lower sequent}) \lhd (A^2 \text{ in the upper sequent})$$

too.

In the above proof, we define the *left-rank* of this MIX to be the maximal length of the following sequence of the occurrences of the mix formula:

$$\begin{pmatrix} \text{mix occurrence } A \\ \text{in the left-side upper} \\ \text{sequent of this MIX} \end{pmatrix} \lhd (A \text{ in } ...) \lhd ... \lhd (A \text{ in } ...)$$

in $\mathcal{P}_1$. If $\Delta_0 = \Delta$, we define left-rank=0. Similarly, we define the *right-rank* of this MIX to be the maximal length of the sequence:

$$\begin{pmatrix} \text{mix occurrence } A \\ \text{in the right-side upper} \\ \text{sequent of this MIX} \end{pmatrix} \lhd (A \text{ in } ...) \lhd ... \lhd (A \text{ in } ...)$$

in $\mathcal{P}_2$. The *rank* of this MIX is defined as

$$\text{rank} = \text{left-rank} + \text{right-rank}.$$

It is obvious that, if a MIX has left-rank=0 or right-rank=0, then the MIX can be replaced by some applications of $(w\rightarrow)$, $(\rightarrow w)$, $(c\rightarrow)$, $(\rightarrow c)$, $(e\rightarrow)$ and $(\rightarrow e)$.

The following lemma is the core of the cut-elimination theorem.

Lemma 3.4 *Suppose $\mathcal{P}$ is a proof of a sequent S in c-LD where $\mathcal{P}$ contains no CUT and only one MIX at the last inference. Then from $\mathcal{P}$, we can construct a proof $\mathcal{P}'$ of S in c-LD where $\mathcal{P}'$ contains no CUT and no MIX. (The latter S has not only the same occurrences of formulas but the same connections as the former S .)*

Proof We prove this lemma by double induction on the grade g and rank r of the MIX in $\mathcal{P}$.

We assume that $\mathcal{P}$ is of the form

$$
\text{MIX}\ \frac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \mathcal{S}_1\end{array}\qquad\qquad \begin{array}{c}\vdots\,\mathcal{P}_2\\ \mathcal{S}_2\end{array}}{\mathcal{S}}\ ,
$$

and the mix formula is A.

Case 1: $r < 2$, i.e. left-rank=0 or right-rank=0.

As mentioned above, we can get the proof $\mathcal{P}'$ without a MIX.

Case 2: $r = 2$.

If left-rank=0 or right-rank=0, we can get $\mathcal{P}'$. So we assume that left-rank=right-rank=1.

We distingush cases according to the form of $\mathcal{P}_1$ and $\mathcal{P}_2$.

Subcase 2-1: $\mathcal{S}_1$ is the axiom $A \to A$,i.e. $\mathcal{P}$ is of the form

$$
\text{MIX}\ \frac{A \to A \qquad\qquad \begin{array}{c}\vdots\,\mathcal{P}_2\\ A, \Pi \to \Sigma\end{array}}{A, \Pi \to \Sigma}
$$

(since the right-rank=1, $\mathcal{S}_2$ is of this form). In this case, we take $\mathcal{P}_2$ for $\mathcal{P}'$.

Subcase 2-2: $\mathcal{S}_1$ is the axiom $\bot \to A$,i.e. $\mathcal{P}$ is of the form

$$\dfrac{\bot \to A \qquad \begin{array}{c} \vdots\, \mathcal{P}_2 \\ A^1, \Pi \to \Sigma \end{array}}{\bot, \Pi \to \Sigma}\ \text{MIX} \quad .$$

Sub-subcase 2-2-1: $A^1 \sim B$ for some formula B in Σ at $\mathcal{S}_2$. Then we can get $\mathcal{P}'$ as

$$\dfrac{\bot \to B}{\bot, \Pi \to \Sigma', B, \Sigma''}\ \text{some } (w\to), (\to w), (e\to), (\to e) \quad .$$

When we apply $(w\to)$ and $(\to w)$, we can assign arbitrary connections to the principal occurrences. So we can assign the same connections as $\mathcal{S}$ to this last sequent.

Sub-subcase 2-2-2: $A^1 \not\sim B$ for any formula B in Σ at $\mathcal{S}_2$. Then by lemma 3.1, we can get the proof:

$$\begin{array}{c} \vdots\, \mathcal{P}_2' \\ \Pi \to \Sigma \end{array}$$

which contains no MIX. So by $(w\to)$, we can get the required proof $\mathcal{P}'$.

Subcase 2-3: the last inference of $\mathcal{P}_1$ is $(\to w)$, i.e. $\mathcal{P}$ is of the form

$$\dfrac{\dfrac{\begin{array}{c} \vdots\, \mathcal{P}_{1-1} \\ \Gamma \to \Delta \end{array}}{\Gamma \to \Delta, A} \qquad \begin{array}{c} \vdots\, \mathcal{P}_2 \\ A, \Pi \to \Sigma \end{array}}{\Gamma, \Pi \to \Delta, \Sigma}\ \text{MIX} \quad .$$

Then we can get $\mathcal{P}'$ as

$$\dfrac{\dfrac{\vdots\,\mathcal{P}_{1-1}}{\Gamma \to \Delta}}{\Gamma, \Pi \to \Delta, \Sigma}\ \text{some } (w\!\to), (\to\! w), (e\!\to), (\to\! e) \quad .$$

Subcase 2-4: $\mathcal{S}_2$ is the axiom $A \to A$. This case is similar to subcase 2-1.

Subcase 2-5: $\mathcal{S}_2$ is the axiom $\perp \to B$, i.e. $\mathcal{P}$ is of the form

$$\text{MIX}\dfrac{\dfrac{\vdots\,\mathcal{P}_1}{\Gamma \to \Delta, \perp} \qquad \perp \to B}{\Gamma \to \Delta, B\ .}$$

Then by applying lemma 3.3 to $\mathcal{P}_1$, we can get $\mathcal{P}'$ as:

$$\dfrac{\vdots\,\mathcal{P}'_1}{\Gamma \to \Delta, B\ .}$$

Subcase 2-6: the last inference of $\mathcal{P}_2$ is $(w\!\to)$. This case is similar to subcase 2-3.

Subcase 2-7: the last inferences of $\mathcal{P}_1$ and $\mathcal{P}_2$ are $(\to\!\wedge)$ and $(\wedge\!\to\!1)$, respectively, i.e. $\mathcal{P}$ is of the form

$$\text{MIX}\dfrac{(\to\!\wedge)\dfrac{\dfrac{\vdots\,\mathcal{P}_{1-1}}{\Gamma \to \Delta, B} \qquad \dfrac{\vdots\,\mathcal{P}_{1-2}}{\Theta \to \Lambda, C}}{\Gamma, \Theta \to \Delta, \Lambda, B\wedge C} \qquad (\wedge\!\to\!1)\dfrac{\dfrac{\vdots\,\mathcal{P}_{2-1}}{B, \Pi \to \Sigma}}{B\wedge C, \Pi \to \Sigma}}{\Gamma, \Theta, \Pi \to \Delta, \Lambda, \Sigma \quad .}$$

Then consider the following proof $\mathcal{Q}$:

$$
\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{P}_{1-1} \qquad\qquad \vdots\ \mathcal{P}_{2-1}}{\Gamma \rightarrow \Delta, B \qquad\qquad B, \Pi \rightarrow \Sigma}\ \text{MIX}}{\Gamma, \Pi \rightarrow \Delta, \Sigma}}{\dfrac{\text{some } (w\rightarrow), (\rightarrow w), (e\rightarrow), (\rightarrow e)}{\Gamma, \Theta, \Pi \rightarrow \Delta, \Lambda, \Sigma\ .}}
$$

To this last sequent, we can assgin the same connections as $\mathcal{S}$, and the grade of this MIX is less than g. So by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Subcase 2-8: the last inferences of $\mathcal{P}_1$ and $\mathcal{P}_2$ are $(\rightarrow\wedge)$ and $(\wedge\rightarrow 2)$.

Subcase 2-9: the last inferences of $\mathcal{P}_1$ and $\mathcal{P}_2$ are $(\rightarrow\vee 1)$ and $(\vee\rightarrow)$.

Subcase 2-10: the last inferences of $\mathcal{P}_1$ and $\mathcal{P}_2$ are $(\rightarrow\vee 2)$ and $(\vee\rightarrow)$.

Subcase 2-8 $\sim$ 2-10 are similar to subcase 2-7.

Subcase 2-11: the last inferences of $\mathcal{P}_1$ and $\mathcal{P}_2$ are $(\rightarrow\supset+)$ and $(\supset\rightarrow)$, i.e. $\mathcal{P}$ is of the form

$$
\dfrac{(\rightarrow\supset+)\dfrac{\dfrac{\vdots\ \mathcal{P}_{1-1}}{B^1, \Gamma \rightarrow \Delta, C^2}}{\Gamma \rightarrow \Delta, B \supset C} \qquad (\supset\rightarrow)\dfrac{\dfrac{\vdots\ \mathcal{P}_{2-1}}{\Theta \rightarrow \Lambda, B^3} \qquad \dfrac{\vdots\ \mathcal{P}_{2-2}}{C^4, \Pi \rightarrow \Sigma}}{B \supset C, \Theta, \Pi \rightarrow \Lambda, \Sigma}}{\Gamma, \Theta, \Pi \rightarrow \Delta, \Lambda, \Sigma\ .}\ \text{MIX}
$$

We call the above three sequents $(B^1, \Gamma \rightarrow \Delta, C^2)$, $(\Theta \rightarrow \Lambda, B^3)$ and $(C^4, \Pi \rightarrow \Sigma)$ by $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$, respectively.

We first assume that $B^1 \sim C^2$ at $\mathcal{T}_1$. Then consider the following proof $\mathcal{Q}$:

$$
\text{MIX}\;\dfrac{\text{MIX}\;\dfrac{\begin{matrix}\vdots\ \mathcal{P}_{2-1}\\ \Theta \to \Lambda, B^3\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{P}_{1-1}\\ B^1, \Gamma \to \Delta, C^2\end{matrix}}{\Theta, \Gamma \to \Lambda, \Delta, C} \qquad \begin{matrix}\vdots\ \mathcal{P}_{2-2}\\ C^4, \Pi \to \Sigma\end{matrix}}{\dfrac{\dfrac{\Theta, \Gamma, \Pi \to \Lambda, \Delta, \Sigma}{\text{some } (\to e), (\to e)}}{\Gamma, \Theta, \Pi \to \Delta, \Lambda, \Sigma\ .}}
$$

We can show that for any formulas γ in Γ, δ in Δ, θ in Θ, λ in Λ, π in Π and σ in Σ, the following 1) $\sim$ 9) hold in both $\mathcal{P}$ and $\mathcal{Q}$:

1) $\gamma \sim \delta$ at the last sequent $\Leftrightarrow \gamma \sim \delta$ at $\mathcal{T}_1$.

2) $\gamma \not\sim \lambda$ at the last sequent.

3) $\gamma \sim \sigma$ at the last sequent $\Leftrightarrow$ ($\gamma \sim C^2$ at $\mathcal{T}_1$ and $C^4 \sim \sigma$ at $\mathcal{T}_3$).

4) $\theta \not\sim \delta$ at the last sequent.

5) $\theta \sim \lambda$ at the last sequent $\Leftrightarrow \theta \sim \lambda$ at $\mathcal{T}_2$.

6) $\theta \sim \sigma$ at the last sequent $\Leftrightarrow$ ($\theta \sim B^3$ at $\mathcal{T}_2$ and $C^4 \sim \sigma$ at $\mathcal{T}_3$).

7) $\pi \not\sim \delta$ at the last sequent.

8) $\pi \not\sim \lambda$ at the last sequent.

9) $\pi \sim \sigma$ at the last sequent $\Leftrightarrow \pi \sim \sigma$ at $\mathcal{T}_3$.

This means the last sequent in $\mathcal{Q}$ has the same connections as $\mathcal{S}$. By the way, the grades of two MIX's occurring in $\mathcal{Q}$ are less than g. So by induction hypotheses, we can get the required proof $\mathcal{P}'$.

If $B^1 \not\sim C^2$ at $\mathcal{T}_1$, we shall take $\mathcal{Q}$ as

$$
\text{MIX}\dfrac{\text{MIX}\dfrac{\begin{matrix}\vdots\,\mathcal{P}_{2-1}\\ \Theta\to\Lambda,B^3\end{matrix}\qquad (c\to)\dfrac{(w\to)\dfrac{\begin{matrix}\vdots\,\mathcal{P}_{1-1}\\ B^1,\Gamma\to\Delta,C^2\end{matrix}}{B,B,\Gamma\to\Delta,C}}{B,\Gamma\to\Delta,C}}{\Theta,\Gamma\to\Lambda,\Delta,C}\qquad\begin{matrix}\vdots\,\mathcal{P}_{2-2}\\ C^4,\Pi\to\Sigma\end{matrix}}{\dfrac{\dfrac{\Theta,\Gamma,\Pi\to\Lambda,\Delta,\Sigma}{\text{some }(\to e),(\to e)}}{\Gamma,\Theta,\Pi\to\Delta,\Lambda,\Sigma}}\ ,
$$

and we can get $\mathcal{P}'$ similarly.

Subcase 2-12: the last inferences of $\mathcal{P}_1$ and $\mathcal{P}_2$ are $(\to\forall)$ and $(\forall\to)$, i.e. $\mathcal{P}$ is of the form

$$
\text{MIX}\dfrac{(\to\forall)\dfrac{\begin{matrix}\vdots\,\mathcal{P}_{1-1}\\ \Gamma\to\Delta,B(a)\end{matrix}}{\Gamma\to\Delta,\forall xB(x)}\qquad(\forall\to)\dfrac{\begin{matrix}\vdots\,\mathcal{P}_{2-1}\\ B(t),\Pi\to\Sigma\end{matrix}}{\forall xB(x),\Pi\to\Sigma}}{\Gamma,\Pi\to\Delta,\Sigma}\ .
$$

Then by lemma 3.2, we can get a proof:

$$
\begin{matrix}\vdots\,\mathcal{P}'_{1-1}\\ \Gamma\to\Delta,B(t)\end{matrix}
$$

without a MIX . Then consider the following proof $\mathcal{Q}$:

$$
\text{MIX}\dfrac{\begin{matrix}\vdots\,\mathcal{P}'_{1-1}\\ \Gamma\to\Delta,B(t)\end{matrix}\qquad\begin{matrix}\vdots\,\mathcal{P}_{2-1}\\ B(t),\Pi\to\Sigma\end{matrix}}{\Gamma,\Pi\to\Delta,\Sigma}
$$

The grade of this MIX is less than g, so by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Subcase 2-13: the last inferences of $\mathcal{P}_1$ and $\mathcal{P}_2$ are $(\rightarrow\exists)$ and $(\exists\rightarrow)$. This case is similar to subcase 2-12.

Case 3: $r > 2$.

Subcase 3-1: left-rank>1 and any mix occurrence of that MIX is not the principal occurrence of the last inference of $\mathcal{P}_1$. We distinguish cases according to the form of $\mathcal{P}_1$.

Sub-subcase 3-1-1: the last inference of $\mathcal{P}_1$ is $(w\rightarrow)$, i.e. $\mathcal{P}$ is of the form

$$\text{MIX}\dfrac{(w\rightarrow)\dfrac{\dfrac{\vdots\ \mathcal{P}_{1-1}}{\Gamma \rightarrow \Delta}}{B, \Gamma \rightarrow \Delta} \qquad \dfrac{\vdots\ \mathcal{P}_2}{\Pi \rightarrow \Sigma}}{B, \Gamma, \Pi_0 \rightarrow \Delta_0, \Sigma}\ .$$

Then consider the following proof $\mathcal{Q}$:

$$(w\rightarrow)\dfrac{\text{MIX}\dfrac{\dfrac{\vdots\ \mathcal{P}_{1-1}}{\Gamma \rightarrow \Delta} \qquad \dfrac{\vdots\ \mathcal{P}_2}{\Pi \rightarrow \Sigma}}{\Gamma, \Pi_0 \rightarrow \Delta_0, \Sigma}}{B, \Gamma, \Pi_0 \rightarrow \Delta_0, \Sigma}\ .$$

We can assign the same connections as $\mathcal{S}$ to the last sequent in $\mathcal{Q}$, and the rank of this MIX is less than r (since the left-rank decreases). So by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Sub-subcase 3-1-2: the last inference of $\mathcal{P}_1$ is $(\rightarrow w)$.

Sub-subcase 3-1-3: the last inference of $\mathcal{P}_1$ is $(c\!\to)$.

Sub-subcase 3-1-4: the last inference of $\mathcal{P}_1$ is $(\to\! c)$.

Sub-subcase 3-1-5: the last inference of $\mathcal{P}_1$ is $(e\!\to)$.

Sub-subcase 3-1-6: the last inference of $\mathcal{P}_1$ is $(\to\! e)$.

Sub-subcase 3-1-7: the last inference of $\mathcal{P}_1$ is $(\wedge\!\to\!1)$.

Sub-subcase 3-1-8: the last inference of $\mathcal{P}_1$ is $(\wedge\!\to\!2)$.

Sub-subcase 3-1-2 $\sim$ 3-1-8 are similar to sub-subcase 3-1-1.

Sub-subcase 3-1-9: the last inference of $\mathcal{P}_1$ is $(\to\!\wedge)$, i.e. $\mathcal{P}$ is of the form

$$\text{MIX}\ \dfrac{(\to\wedge)\ \dfrac{\begin{array}{c}\vdots\ \mathcal{P}_{1-1}\\ \Gamma\to\Delta,B\end{array}\qquad \begin{array}{c}\vdots\ \mathcal{P}_{1-2}\\ \Theta\to\Lambda,C\end{array}}{\Gamma,\Theta\to\Delta,\Lambda,B\wedge C}\qquad \begin{array}{c}\vdots\ \mathcal{P}_2\\ \Pi\to\Sigma\end{array}}{\Gamma,\Theta,\Pi_0\to\Delta_0,\Lambda_0,B\wedge C,\Sigma}\ .$$

Then consider the following proof $\mathcal{Q}$:

$$\dfrac{\dfrac{(\to\wedge)\ \dfrac{\dfrac{\text{MIX}\ \dfrac{\begin{array}{c}\vdots\ \mathcal{P}_{1-1}\\ \Gamma\to\Delta,B\end{array}\qquad \begin{array}{c}\vdots\ \mathcal{P}_{2}\\ \Pi\to\Sigma\end{array}}{\Gamma,\Pi_0\to\Delta_0,B,\Sigma}}{\text{some }(\to e)}}{\Gamma,\Pi_0\to\Delta_0,\Sigma,B}\qquad \dfrac{\dfrac{\text{MIX}\ \dfrac{\begin{array}{c}\vdots\ \mathcal{P}_{1-2}\\ \Theta\to\Lambda,C\end{array}\qquad \begin{array}{c}\vdots\ \mathcal{P}_{2}\\ \Pi\to\Sigma\end{array}}{\Theta,\Pi_0\to\Lambda_0,C,\Sigma}}{\text{some }(\to e)}}{\Theta,\Pi_0\to\Lambda_0,\Sigma,C}}{\Gamma,\Pi_0,\Theta,\Pi_0\to\Delta_0,\Sigma,\Lambda_0,\Sigma,B\wedge C}}{\dfrac{\text{some }(c\to),(\to c),(e\to),(\to e)}{\Gamma,\Theta,\Pi_0\to\Delta_0,\Lambda_0,B\wedge C,\Sigma\ .}}$$

The ranks of these two MIX's in $\mathcal{Q}$ are less than r, and the last sequent in $\mathcal{Q}$ has the same connections as $\mathcal{S}$. So by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Sub-subcase 3-1-10: the last inference of $\mathcal{P}_1$ is $(\vee\rightarrow)$. This case is similar to sub-subcase 3-1-9.

Sub-subcase 3-1-11: the last inference of $\mathcal{P}_1$ is $(\rightarrow\vee 1)$.

Sub-subcase 3-1-12: the last inference of $\mathcal{P}_1$ is $(\rightarrow\vee 2)$.

Sub-subcase 3-1-11 and 3-1-12 are similar to sub-subcase 3-1-1.

Sub-subcase 3-1-13: the last inference of $\mathcal{P}_1$ is $(\supset\rightarrow)$, i.e. $\mathcal{P}$ is of the form

$$\text{MIX}\ \dfrac{(\supset\rightarrow)\ \dfrac{\begin{matrix}\vdots\ \mathcal{P}_{1-1}\\ \Gamma\rightarrow\Delta,B^1\end{matrix}\qquad\begin{matrix}\vdots\ \mathcal{P}_{1-2}\\ C^2,\Theta\rightarrow\Lambda\end{matrix}}{B\supset C,\Gamma,\Theta\rightarrow\Delta,\Lambda}\qquad\begin{matrix}\vdots\ \mathcal{P}_2\\ \Pi\rightarrow\Sigma\end{matrix}}{(B\supset C)^3,\Gamma,\Theta,\Pi_0\rightarrow\Delta_0,\Lambda_0,\Sigma}\ .$$

Then consider the following proof $\mathcal{Q}$:

$$\dfrac{\dfrac{(\supset\rightarrow)\ \dfrac{\dfrac{\dfrac{\text{MIX}\ \dfrac{\begin{matrix}\vdots\ \mathcal{P}_{1-1}\\ \Gamma\rightarrow\Delta,B^1\end{matrix}\qquad\begin{matrix}\vdots\ \mathcal{P}_2\\ \Pi\rightarrow\Sigma\end{matrix}}{\Gamma,\Pi_0\rightarrow\Delta_0,B,\Sigma}}{\text{some }(\rightarrow e)}}{\Gamma,\Pi_0\rightarrow\Delta_0,\Sigma,B}\qquad \text{MIX}\ \dfrac{\begin{matrix}\vdots\ \mathcal{P}_{1-2}\\ C^2,\Theta\rightarrow\Lambda\end{matrix}\qquad\begin{matrix}\vdots\ \mathcal{P}_2\\ \Pi\rightarrow\Sigma\end{matrix}}{C,\Theta,\Pi_0\rightarrow\Lambda_0,\Sigma}}{B\supset C,\Gamma,\Pi_0,\Theta,\Pi_0\rightarrow\Delta_0,\Sigma,\Lambda_0,\Sigma}}{\text{some }(c\rightarrow),(\rightarrow c),(e\rightarrow),(\rightarrow e)}}{(B\supset C)^3,\Gamma,\Theta,\Pi_0\rightarrow\Delta_0,\Lambda_0,\Sigma}\ .$$

The ranks of these two MIX's are less than r. Now, we check up the connections at the last sequent in $\mathcal{Q}$. The above three sequents $(\Gamma\rightarrow\Delta,B^1)$, $(C^2,\Theta\rightarrow\Lambda)$ and $(\Pi\rightarrow\Sigma)$ are called $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$, respectively. For any formulas γ in Γ, δ_0 in Δ_0, θ in Θ, λ_0 in Λ_0, π_0 in Π_0 and σ in Σ, the following 1) $\sim$ 12) hold in both $\mathcal{P}$ and $\mathcal{Q}$:

1) $(B\supset C)^3 \not\sim \delta_0$ at the last sequent.

2) $(B\supset C)^3 \sim \lambda_0$ at the last sequent $\Leftrightarrow C^2 \sim \lambda_0$ at $\mathcal{T}_2$.

3) $(B\supset C)^3 \sim \sigma$ at the last sequent $\Leftrightarrow$ ($C^2 \sim A^i$ at $\mathcal{T}_2$ and $A^j \sim \sigma$ at $\mathcal{T}_3$) for some mix occurrences A^i, A^j.

4) $\gamma \sim \delta_0$ at the last sequent $\Leftrightarrow \gamma \sim \delta_0$ at $\mathcal{T}_1$.

5) $\gamma \sim \lambda_0$ at the last sequent $\Leftrightarrow$ ($\gamma \sim B^1$ at $\mathcal{T}_1$ and $C^2 \sim \lambda_0$ at $\mathcal{T}_2$).

6) $\gamma \sim \sigma$ at the last sequent $\Leftrightarrow$ (($\gamma \sim A^i$ at $\mathcal{T}_1$ or ($\gamma \sim B^1$ at $\mathcal{T}_1$ and $C^2 \sim A^j$ at $\mathcal{T}_2$)) and $A^k \sim \sigma$ at $\mathcal{T}_3$) for some mix occurrences A^i, A^j, A^k.

7) $\theta \not\sim \delta_0$ at the last sequent.

8) $\theta \sim \lambda_0$ at the last sequent $\Leftrightarrow \theta \sim \lambda_0$ at $\mathcal{T}_2$.

9) $\theta \sim \sigma$ at the last sequent $\Leftrightarrow$ ($\theta \sim A^i$ at $\mathcal{T}_2$ and $A^j \sim \sigma$ at $\mathcal{T}_3$) for some mix occurrence A^i, A^j.

10) $\pi_0 \not\sim \delta_0$ at the last sequent.

11) $\pi_0 \not\sim \lambda_0$ at the last sequent.

12) $\pi_0 \sim \sigma$ at the last sequent $\Leftrightarrow \pi_0 \sim \sigma$ at $\mathcal{T}_3$.

This means the last sequent in $\mathcal{Q}$ has the same connections as $\mathcal{S}$. So by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Sub-subcase 3-1-14: the last inference of $\mathcal{P}_1$ is $(\rightarrow\supset+)$, i.e. $\mathcal{P}$ is of the form

$$\dfrac{(\rightarrow\supset+)\dfrac{\dfrac{\vdots\ \mathcal{P}_{1-1}}{B, \Gamma \rightarrow \Delta, C}}{\Gamma \rightarrow \Delta, B\supset C} \qquad \dfrac{\vdots\ \mathcal{P}_2}{\Pi \rightarrow \Sigma}}{\Gamma, \Pi_0 \rightarrow \Delta_0, B\supset C, \Sigma}\,\text{MIX} \; .$$

Then consider the following proof $\mathcal{Q}$:

$$\dfrac{\dfrac{\vdots\ \mathcal{P}_{1-1}}{B, \Gamma \to \Delta, C} \qquad \dfrac{\vdots\ \mathcal{P}_2}{\Pi \to \Sigma}}{\dfrac{B, \Gamma, \Pi_0 \to \Delta_0, C, \Sigma}{\dfrac{\text{some } (\to e)}{B^1, \Gamma, \Pi_0 \to \Delta_0, \Sigma, C\ .}}}\ \text{MIX}$$

At this last sequent, $B^1 \not\sim \phi$ for any formula ϕ in (Δ_0, Σ). Then we can get the following proof:

$$\dfrac{\vdots\ \mathcal{Q}}{\dfrac{\Gamma, \Pi_0 \to \Delta_0, \Sigma, B \supset C}{\dfrac{\text{some } (\to e)}{\Gamma, \Pi_0 \to \Delta_0, B \supset C, \Sigma\ .}}}\ (\to\supset+)$$

The rank of this MIX in $\mathcal{Q}$ is less than r, and this last sequent has the same connections as $\mathcal{S}$. So we can get the required proof $\mathcal{P}'$ by induction hypothesis.

Sub-subcase 3-1-15: the last inference of $\mathcal{P}_1$ is $(\forall\to)$. This case is similar to sub-subcase 3-1-1.

Sub-subcase 3-1-16: the last inference of $\mathcal{P}_1$ is $(\to\forall)$, i.e. $\mathcal{P}$ is of the form

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{P}_{1-1}}{\Gamma \to \Delta, B(a)}}{\Gamma \to \Delta, \forall x B(x)}\ (\to\forall) \qquad \dfrac{\vdots\ \mathcal{P}_2}{\Pi \to \Sigma}}{\Gamma, \Pi_0 \to \Delta_0, \forall x B(x), \Sigma\ .}\ \text{MIX}$$

Then using lemma 3.2, we can get the following proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{P}'_{1-1} \\ \Gamma \to \Delta, B(b)\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{P}_2 \\ \Pi \to \Sigma\end{matrix}}{\Gamma, \Pi_0 \to \Delta_0, B(b), \Sigma}\,\text{MIX}}{\text{some } (\to e)}}{\Gamma, \Pi_0 \to \Delta_0, \Sigma, B(b)}}{\Gamma, \Pi_0 \to \Delta_0, \Sigma, \forall x B(x)}\,(\to\forall)}{\dfrac{\text{some } (\to e)}{\Gamma, \Pi_0 \to \Delta_0, \forall x B(x), \Sigma}}
$$

where b is a new free variable. The rank of this MIX is less than r, so by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Sub-subcase 3-1-17: the last inference of $\mathcal{P}_1$ is $(\exists\to)$. This case is similar to sub-subcase 3-1-16.

Sub-subcase 3-1-18: the last inference of $\mathcal{P}_1$ is $(\to\exists)$. This case is similar to sub-subcase 3-1-1.

Subcase 3-2: left-rank>1 and one of the mix occurrences of that MIX is the principal occurrence of the last inference of $\mathcal{P}_1$. We distinguish cases according to the form of $\mathcal{P}_1$.

Sub-subcase 3-2-1: the last inference of $\mathcal{P}_1$ is $(\to w)$, i.e. $\mathcal{P}$ is of the form

$$
\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{P}_{1-1} \\ \Gamma \to \Delta\end{matrix}}{\Gamma \to \Delta, A}\,(w\to) \qquad \begin{matrix}\vdots\ \mathcal{P}_2 \\ \Pi \to \Sigma\end{matrix}}{\Gamma, \Pi_0 \to \Delta_0, \Sigma}\,\text{MIX} \ .
$$

Then consider the following proof $\mathcal{Q}$:

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_{1-1}\\ \Gamma \to \Delta\end{array} \qquad \begin{array}{c}\vdots\,\mathcal{P}_2\\ \Pi \to \Sigma\end{array}}{\Gamma, \Pi_0 \to \Delta_0, \Sigma\ .}\ \text{MIX}$$

We call this last sequent $\mathcal{S}'$. Now, rank of this MIX is less than r, so by induction hypothesis, we can get a proof of $\mathcal{S}'$ without a MIX. But $\mathcal{S}'$ may not have the same connections as $\mathcal{S}$. That is, for any formulas ϕ in (Γ, Π_0) and ψ in (Δ_0, Σ),

$\phi \sim \psi$ at $\mathcal{S}' \Rightarrow \phi \sim \psi$ at $\mathcal{S}$,

but the converse may not be true. Then we apply some ($w\!\to$), ($\to\! w$), ($c\!\to$), ($\to\! c$), ($e\!\to$) and ($\to\! e$) to $\mathcal{Q}$ in order to add necessary connections, and we get the required proof $\mathcal{P}'$.

Sub-subcase 3-2-2: the last inference of $\mathcal{P}_1$ is ($\to\! c$). This case is similar to, or easier than sub-subcase 3-2-1.

Sub-subcase 3-2-3: the last inference of $\mathcal{P}_1$ is ($\to\!\wedge$), i.e. $\mathcal{P}$ is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_{1-1}\\ \Gamma \to \Delta, B\end{array} \qquad \begin{array}{c}\vdots\,\mathcal{P}_{1-2}\\ \Theta \to \Lambda, C\end{array}}{\Gamma, \Theta \to \Delta, \Lambda, B\wedge C}\,(\to\wedge) \qquad \begin{array}{c}\vdots\,\mathcal{P}_2\\ \Pi \to \Sigma\end{array}}{\Gamma, \Theta, \Pi_0 \to \Delta_0, \Lambda_0, \Sigma}\ \text{MIX}$$

where $A = B\wedge C$. Then, by the same way as sub-subcase 3-1-9, we can get the following proof without a MIX:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, \Pi_0 \to \Delta_0, \Sigma, B\end{array} \qquad \begin{array}{c}\vdots\\ \Theta, \Pi_0 \to \Lambda_0, \Sigma, C\end{array}}{\Gamma, \Pi_0, \Theta, \Pi_0 \to \Delta_0, \Sigma, \Lambda_0, \Sigma, B\wedge C\ .}\,(\to\wedge)$$

Then by using this proof, we can get the following proof:

$$
\dfrac{(\rightarrow\wedge)\dfrac{\begin{array}{c}\vdots \\ \Gamma, \Pi_0 \rightarrow \Delta_0, \Sigma, B\end{array} \qquad \begin{array}{c}\vdots \\ \Theta, \Pi_0 \rightarrow \Lambda_0, \Sigma, C\end{array}}{\Gamma, \Pi_0, \Theta, \Pi_0 \rightarrow \Delta_0, \Sigma, \Lambda_0, \Sigma, B\wedge C} \qquad \begin{array}{c}\vdots \mathcal{P}_2 \\ \Pi \rightarrow \Sigma\end{array}}{\dfrac{\dfrac{\Gamma, \Pi_0, \Theta, \Pi_0, \Pi_0 \rightarrow \Delta_0, \Sigma, \Lambda_0, \Sigma, \Sigma}{\text{some } (c\rightarrow), (\rightarrow c), (e\rightarrow), (\rightarrow e)}}{\Gamma, \Theta, \Pi_0 \rightarrow \Delta_0, \Lambda_0, \Sigma\ .}}\ \text{MIX}
$$

Since the left-rank of this MIX is 1, the rank of this MIX is less than r. And this last sequent has the same connections as $\mathcal{S}$. So by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Sub-subcase 3-2-4: the last inference of $\mathcal{P}_1$ is $(\rightarrow\vee 1)$.

Sub-subcase 3-2-5: the last inference of $\mathcal{P}_1$ is $(\rightarrow\vee 2)$.

Sub-subcase 3-2-4 and 3-2-5 are similar to, or easier than sub-subcase 3-2-1.

Sub-subcase 3-2-6: the last inference of $\mathcal{P}_1$ is $(\rightarrow\supset+)$, i.e. $\mathcal{P}$ is of the form

$$
\text{MIX}\ \dfrac{(\rightarrow\supset+)\dfrac{\begin{array}{c}\vdots \mathcal{P}_{1-1} \\ B, \Gamma \rightarrow \Delta, C\end{array}}{\Gamma \rightarrow \Delta, B\supset C} \qquad \begin{array}{c}\vdots \mathcal{P}_2 \\ \Pi \rightarrow \Sigma\end{array}}{\Gamma, \Pi_0 \rightarrow \Delta_0, \Sigma}
$$

where $A = B\supset C$. Then, by the same way as sub-subcase 3-1-14, we can get the following proof without a MIX:

$$
(\rightarrow\supset+)\dfrac{\begin{array}{c}\vdots \\ B, \Gamma, \Pi_0 \rightarrow \Delta_0, \Sigma, C\end{array}}{\Gamma, \Pi_0 \rightarrow \Delta_0, \Sigma, B\supset C\ .}
$$

Then by using this proof, we can get the following proof:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ B,\Gamma,\Pi_0 \to \Delta_0,\Sigma,C\end{array}}{\Gamma,\Pi_0 \to \Delta_0,\Sigma,B\supset C}(\to\supset+) \qquad \begin{array}{c}\vdots\ \mathcal{P}_2\\ \Pi \to \Sigma\end{array}}{\dfrac{\dfrac{\Gamma,\Pi_0,\Pi_0 \to \Delta_0,\Sigma,\Sigma}{\text{some } (c\to),(\to c),(e\to),(\to e)}}{\Gamma,\Pi_0 \to \Delta_0,\Sigma\ .}}\text{MIX}$$

Since the left-rank of this MIX is 1, the rank of this MIX is less than r. And this last sequent has the same connections as $\mathcal{S}$. So by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Sub-subcase 3-2-7: the last inference of $\mathcal{P}_1$ is $(\to\forall)$, i.e. $\mathcal{P}$ is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{P}_{1-1}\\ \Gamma \to \Delta, B(a)\end{array}}{\Gamma \to \Delta, \forall x B(x)}(\to\forall) \qquad \begin{array}{c}\vdots\ \mathcal{P}_2\\ \Pi \to \Sigma\end{array}}{\Gamma,\Pi_0 \to \Delta_0,\Sigma}\text{MIX}$$

where $A = \forall x B(x)$. By the same way as sub-subcase 3-1-16, we can get the following proof without a MIX:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma,\Pi_0 \to \Delta_0,\Sigma,B(b)\end{array}}{\Gamma,\Pi_0 \to \Delta_0,\Sigma,\forall x B(x)\ .}(\to\forall)$$

Then by using this proof, we can get the following proof:

$$
\cfrac{(\to\forall)\cfrac{\begin{array}{c}\vdots\\ \Gamma,\Pi_0\to\Delta_0,\Sigma,B(b)\end{array}}{\Gamma,\Pi_0\to\Delta_0,\Sigma,\forall x B(x)} \qquad \begin{array}{c}\vdots\,\mathcal{P}_2\\ \Pi\to\Sigma\end{array}}{\cfrac{\Gamma,\Pi_0,\Pi_0\to\Delta_0,\Sigma,\Sigma}{\cfrac{\text{some } (c\to),(\to c),(e\to),(\to e)}{\Gamma,\Pi_0\to\Delta_0,\Sigma\ .}}}\ \text{MIX}
$$

Since the left-rank of this MIX is 1, the rank of this MIX is less than r. And this last sequent has the same connections as $\mathcal{S}$. So by induction hypothesis, we can get the required proof $\mathcal{P}'$.

Sub-subcase 3-2-8: the last inference of $\mathcal{P}_1$ is $(\to\exists)$. This case is similar to, or easier than sub-subcase 3-2-1.

Subcase 3-3: right-rank>1 and any mix occurrence of that MIX is not the principal occurrence of the last inference of $\mathcal{P}_2$.

We only show the case: $\mathcal{P}$ is of the form

$$
\cfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \Gamma\to\Delta\end{array} \qquad (\to\supset+)\cfrac{\begin{array}{c}\vdots\,\mathcal{P}_{2-1}\\ B,\Pi\to\Sigma,C\end{array}}{\Pi\to\Sigma,B\supset C}}{\Gamma,\Pi_0\to\Delta_0,\Sigma,B\supset C\ \ .}\ \text{MIX}
$$

We can get the following proof $\mathcal{Q}$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \Gamma\rightarrow\Delta\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{P}_{2-1}\\ B,\Pi\rightarrow\Sigma,C\end{array}}{\Gamma,B,\Pi_0\rightarrow\Delta_0,\Sigma,C}\,\text{MIX}}{\dfrac{\text{some }(e\rightarrow)}{B^1,\Gamma,\Pi_0\rightarrow\Delta_0,\Sigma,C\ .}}$$

At this last sequent, $B^1 \not\sim \phi$ for any ϕ in (Δ_0,Σ). So we can get the following proof:

$$(\rightarrow\supset+)\dfrac{\vdots\,\mathcal{Q}}{\Gamma,\Pi_0\rightarrow\Delta_0,\Sigma,B\supset C\ .}$$

The rank of this MIX in $\mathcal{Q}$ is less than r, and this last sequent has the same connections as $\mathcal{S}$. So we can get the required proof $\mathcal{P}'$ by induction hypothesis.

Other cases in subcase 3-3 are left to the reader.

Subcase 3-4: right-rank>1 and one of the mix occurrences of that MIX is the principal occurrence of the last inference of $\mathcal{P}_2$.

We only show the case: $\mathcal{P}$ is of the form

$$\text{MIX}\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \Gamma\rightarrow\Delta\end{array}\qquad (\supset\rightarrow)\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_{2-1}\\ \Theta\rightarrow\Lambda,B^1\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{P}_{2-2}\\ C^2,\Pi\rightarrow\Sigma\end{array}}{B\supset C,\Theta,\Pi\rightarrow\Lambda,\Sigma}}{\Gamma,\Theta_0,\Pi_0\rightarrow\Delta_0,\Lambda,\Sigma}$$

where $A = B\supset C$. Then consider the following proof $\mathcal{Q}$:

$$
\text{MIX}\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \Gamma\to\Delta\end{array}\qquad (\supset\to)\dfrac{\text{MIX}\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \Gamma\to\Delta\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{P}_{2-1}\\ \Theta\to\Lambda,B^1\end{array}}{\Gamma,\Theta_0\to\Delta_0,\Lambda,B}\qquad \dfrac{\text{MIX}\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \Gamma\to\Delta\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{P}_{2-2}\\ C^2,\Pi\to\Sigma\end{array}}{\Gamma,C,\Pi_0\to\Delta_0,\Sigma}}{\dfrac{\text{some }(e\to)}{C,\Gamma,\Pi_0\to\Delta_0,\Sigma}}}{B\supset C,\Gamma,\Theta_0,\Gamma,\Pi_0\to\Delta_0,\Lambda,\Delta_0,\Sigma}}{\dfrac{\Gamma,\Gamma,\Theta_0,\Gamma,\Pi_0\to\Delta_0,\Delta_0,\Lambda,\Delta_0,\Sigma}{\dfrac{\text{some }(c\to),(\to c),(e\to),(\to e)}{\Gamma,\Theta_0,\Pi_0\to\Delta_0,\Lambda,\Sigma\ .}}}
$$

The ranks of these three MIX's are less than r. Now, we check up the connections at this last sequent. The above three sequents $(\Gamma\to\Delta)$, $(\Theta\to\Lambda,B^1)$ and $(C^2,\Pi\to\Sigma)$ are called $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$, respectively. For any formulas γ in Γ, δ_0 in Δ_0, θ_0 in Θ_0, λ in Λ, π_0 in Π_0 and σ in Σ, the following 1) $\sim$ 9) hold in both $\mathcal{P}$ and $\mathcal{Q}$:

1) $\gamma\sim\delta_0$ at the last sequent $\Leftrightarrow$ $\gamma\sim\delta_0$ at $\mathcal{T}_1$.

2) $\gamma\sim\lambda$ at the last sequent $\Leftrightarrow$ ($\gamma\sim A^i$ at $\mathcal{T}_1$ and $A^j\sim\lambda$ at $\mathcal{T}_2$) for some mix occurrence A^i,A^j.

3) $\gamma\sim\sigma$ at the last sequent $\Leftrightarrow$ ($\gamma\sim A^i$ at $\mathcal{T}_1$ and ($C^2\sim\sigma$ at $\mathcal{T}_3$ or $A^j\sim\sigma$ at $\mathcal{T}_3$)) for some mix occrrences A^i,A^j.

4) $\theta_0\not\sim\delta_0$ at the last sequent.

5) $\theta_0\sim\lambda$ at the last sequent $\Leftrightarrow$ $\theta_0\sim\lambda$ at $\mathcal{T}_2$.

6) $\theta_0\sim\sigma$ at the last sequent $\Leftrightarrow$ ($\theta_0\sim B^1$ at $\mathcal{T}_2$ and $C^2\sim\sigma$ at $\mathcal{T}_3$).

7) $\pi_0\not\sim\delta_0$ at the last sequent.

8) $\pi_0\not\sim\lambda$ at the last sequent.

9) $\pi_0\sim\sigma$ at the last sequent $\Leftrightarrow$ $\pi_0\sim\sigma$ at $\mathcal{T}_3$.

This means the last sequent of $\mathcal{Q}$ has the same connections as $\mathcal{S}$. So by induction hypotheses, we can get the required proof $\mathcal{P}'$.

Other cases in subcase 3-4 are left to the reader.

This completes the proof of lemma 3.4. □

Theorem 3.5 (cut-elimination theorem for c-LD) *Suppose $\mathcal{P}$ is a proof of a sequent $\mathcal{S}$ in c-LD. Then from $\mathcal{P}$, we can construct a proof $\mathcal{P}'$ of $\mathcal{S}$ in c-LD which does not contain CUT and MIX.*

Proof CUT is a special form of MIX. So by lemma 3.4, we can eliminate all CUT's in $\mathcal{P}$ one after another. □

4 Cut-elimination theorem for LD

In this section, we prove the converse of theorem 2.1 and a weak cut-elimination theorem for LD.

Let $A_1, ..., A_n$ be formulas and $\Gamma \rightarrow \Delta$ be a sequent. Then by $\bigvee(A_1, ..., A_n)$ and $\bigwedge(A_1, ..., A_n)$, we mean the formulas $A_1 \vee ... \vee A_n$ and $A_1 \wedge ... \wedge A_n$, respectively. If $n = 0$, we define $\bigvee() = \bot$ and $\bigwedge() = \bot \supset \bot$. Moreover by $\{\Gamma; \Delta\}$, we mean the formula $\bigwedge(\Gamma) \supset \bigvee(\Delta)$ which represents the sequent $\Gamma \rightarrow \Delta$.

Special sequents are defined inductively as follows:

1) Sequents

$$\{\Gamma; \Delta\} \rightarrow \{A, \Gamma; \Delta\}$$

$$\{\Gamma; \Delta\} \rightarrow \{\Gamma; \Delta, A\}$$

$$\{A, A, \Gamma; \Delta\} \rightarrow \{A, \Gamma; \Delta\}$$

$$\{\Gamma; \Delta, A, A\} \rightarrow \{\Gamma; \Delta, A\}$$

$$\{\Gamma, A, B, \Pi; \Delta\} \rightarrow \{\Gamma, B, A, \Pi; \Delta\}$$

$$\{\Gamma; \Delta, A, B, \Sigma\} \rightarrow \{\Gamma; \Delta, B, A, \Sigma\}$$

$$\{A, \Gamma; \Delta\} \rightarrow \{A \wedge B, \Gamma; \Delta\}$$

$$\{B, \Gamma; \Delta\} \rightarrow \{A \wedge B, \Gamma; \Delta\}$$

$$\{\Gamma; \Delta, A\} \wedge \{\Pi; \Sigma, B\} \rightarrow \{\Gamma, \Pi; \Delta, \Sigma, A \wedge B\}$$

$$\{A, \Gamma; \Delta\} \wedge \{B, \Pi; \Sigma\} \rightarrow \{A \vee B, \Gamma, \Pi; \Delta, \Sigma\}$$

$$\{\Gamma; \Delta, A\} \rightarrow \{\Gamma; \Delta, A \vee B\}$$

$$\{\Gamma; \Delta, B\} \rightarrow \{\Gamma; \Delta, A \vee B\}$$

$$\{\Gamma;\Delta,A\}\wedge\{B,\Pi;\Sigma\}\rightarrow\{A\supset B,\Gamma,\Pi;\Delta,\Sigma\}$$

$$\{A,\Gamma;B\}\rightarrow\{\Gamma;A\supset B\}$$

$$\{A(t),\Gamma;\Delta\}\rightarrow\{\forall xA(x),\Gamma;\Delta\}$$

$$\forall x\{\Gamma;\Delta,A(x)\}\rightarrow\{\Gamma;\Delta,\forall xA(x)\}$$

$$\forall x\{A(x),\Gamma;\Delta\}\rightarrow\{\exists xA(x),\Gamma;\Delta\}$$

$$\{\Gamma;\Delta,A(t)\}\rightarrow\{\Gamma;\Delta,\exists xA(x)\}$$

$$\{\Gamma;\Delta,\{A,\Pi;\Sigma\}\}\rightarrow\{\Gamma,A;\Delta,\{\Pi;\Sigma\}\}$$

are all special sequents; [2]

2) If $A\rightarrow B$ is a special sequent, then

$$\{\Gamma;\Delta,A\}\rightarrow\{\Gamma;\Delta,B\}$$

is a special sequent.

Sequent calculus *LDS* is defined as follows ("S" means "Special sequent"). The axioms of LDS are the same as LD. The inference rules of LDS are the same as LD except CUT. We adopt the following rules SP.CUT and SP.CUT2 in place of CUT:

$$\text{SP.CUT}\ \frac{\Gamma\rightarrow\Delta,A \qquad A\rightarrow B}{\Gamma\rightarrow\Delta,B}$$

where $A\rightarrow B$ is a special sequent.

$$\text{SP.CUT2}\ \frac{\Gamma\rightarrow\Delta,\{A,\Pi;\Sigma\} \qquad \{A,\Pi;\Sigma\},A\rightarrow\{\Pi;\Sigma\}}{\Gamma,A\rightarrow\Delta,\{\Pi;\Sigma\}}\ .$$

[2] In our notation, x does not occur in $(\Gamma,\Delta,A())$ at $\forall x\{\Gamma;\Delta,A(x)\}\rightarrow\{\Gamma;\Delta,\forall xA(x)\}$ and at $\forall x\{A(x),\Gamma;\Delta\}\rightarrow\{\exists xA(x),\Gamma;\Delta\}$, by the definition of formulas in section 1.

LDS is a subsystem of LD.

Sequent calculus *LDS′* is obtained from LDS by replacing SP.CUT and SP.CUT2 by the following rules:

$$\text{SP.CUT}'\frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, B}$$

where $A \to B$ is a special sequent.

$$\text{SP.CUT2}'\frac{\Gamma \to \Delta, \{A, \Pi; \Sigma\}}{\Gamma, A \to \Delta, \{\Pi; \Sigma\}}\ .$$

Lemma 4.1 *Each special sequent is provable in LD without a CUT. A sequent*

$$\{A, \Gamma; \Delta\}, A \to \{\Gamma; \Delta\}$$

is provable in LD without a CUT. (We can actually construct proofs of these.)

The proof of this lemma is not difficult, but needs a large space. So we omit this.

Theorem 4.2 *LDS and LDS′ are equivalent, i.e. if $\mathcal{P}$ is a proof of $\mathcal{S}$ in LDS (LDS′), then we can construct a proof of $\mathcal{S}$ in LDS′ (LDS) from $\mathcal{P}$.*

Proof This is obvious by lemma 4.1. □

The following theorem and corollaries are the main results in this section.

Theorem 4.3 (converse of theorem 2.1) *Suppose $\mathcal{P}$ is a proof of a sequent $\Gamma \to \Delta$ with some connections in c-LD. Then from $\mathcal{P}$, we can construct a proof of $\Gamma \to \Delta$ (without connections) in LDS (hence, in LD). If $\Delta = \bot$, we can also construct a proof of $\Gamma \to$ in LDS.*

Corollary 4.4 (cut-elimination theorem for LD) *Suppose $\mathcal{P}$ is a proof of $\Gamma \rightarrow \Delta$ in LD. Then from $\mathcal{P}$, we can construct a proof of $\Gamma \rightarrow \Delta$ in LDS.*

Corollary 4.5 *The following five conditions are equivalent:*

1. *$\rightarrow A$ is provable in LD;*

2. *$\rightarrow A$ is provable in c-LD;*

3. *$\rightarrow A$ is provable in c-LD without a CUT;*

4. *$\rightarrow A$ is provable in LDS;*

5. *$\rightarrow A$ is provable in LDS/.*[3]

The proofs of these corollaries are straightforward by proposition 1.1, theorem 2.1, 3.5, 4.2 and 4.3.

To prove theorem 4.3, we introduce a system *c-LDS/* which derives sequents with connections. The axioms of c-LDS/ are the same as c-LD. The inference rules of c-LDS/ are the same as c-LD except that we adopt the following rules ($\rightarrow\supset$), SP.CUT′ and SP.CUT2′ in place of ($\rightarrow\supset+$) and CUT:

$$(\rightarrow\supset)\frac{A, \Gamma \rightarrow B^1}{\Gamma \rightarrow (A\supset B)^2}$$

where for any formula γ in Γ,

$\gamma \sim (A\supset B)^2$ at the lower sequent $\Leftrightarrow$ $\gamma \sim B^1$ at the upper sequent.

[3]In [7], Ono independently proposes a cut-free formulation of CD which is similar to our LDS/.

$$\text{SP.CUT}'\frac{\Gamma \rightarrow \Delta, A^1}{\Gamma \rightarrow \Delta, B^2}$$

where $A \rightarrow B$ is a special sequent, connections between Γ and Δ are inherited, and for any formula γ in Γ,

$\gamma \sim B^2$ at the lower sequent $\Leftrightarrow \gamma \sim A^1$ at the upper sequent.

$$\text{SP.CUT2}'\frac{\Gamma \rightarrow \Delta, \{A, \Pi; \Sigma\}^1}{\Gamma, A^2 \rightarrow \Delta, \{\Pi; \Sigma\}^3}$$

where connections between Γ and Δ are inherited, there is no connection between A^2 and Δ at the lower sequent, $A^2 \sim \{\Pi; \Sigma\}^3$ at the lower sequent, and for any formulas γ in Γ,

$\gamma \sim \{\Pi; \Sigma\}^3$ at the lower sequent $\Leftrightarrow \gamma \sim \{A, \Pi; \Sigma\}^1$ at the upper sequent.

Except for the difference of axioms, c-LDS/ is LDS/ simply with connections.

Now, if the following lemmas are proved, we can prove theorem 4.3 easily.

Lemma 4.6 *The rule ($\rightarrow\supset+$) is derivable in c-LDS/, i.e. if $\mathcal{P}$ is a proof of a sequent S = ($A^1, \Gamma \rightarrow \Delta, B^2$) in c-LDS/ and there is no connections between A^1 and Δ at S , then from $\mathcal{P}$, we can construct a proof of a sequent S' = ($\Gamma \rightarrow \Delta, (A \supset B)^3$) in c-LDS/ where the connections between Γ and Δ are inherited and for any formula γ in Γ,*

$\gamma \sim (A \supset B)^3$ at $S' \Leftrightarrow \gamma \sim B^2$ at S .

Lemma 4.7 *If $\mathcal{P}$ is a proof of $\Gamma \rightarrow \bot$ in LDS, then from $\mathcal{P}$, we can construct a proof of $\Gamma \rightarrow$ in LDS.*

Proof of theorem 4.3 Let $\mathcal{P}$ be a proof of $\Gamma \rightarrow \Delta$ in c-LD. Then by theorem 3.5, we can get a cut-free proof $\mathcal{P}'$ of $\Gamma \rightarrow \Delta$ in c-LD.

Now, let $\mathcal{P}'$ be of the form:

$$\begin{array}{c} \vdots\, \mathcal{Q} \\ (\rightarrow\supset+)\dfrac{A, \Pi \rightarrow \Sigma, B}{\Pi \rightarrow \Sigma, A\supset B} \\ \vdots \\ \Gamma \rightarrow \Delta \end{array}$$

where $\mathcal{Q}$ does not contain $(\rightarrow\supset+)$. Then $\mathcal{Q}$ is a proof in c-LDS/, since $\mathcal{Q}$ does not contain CUT and $(\rightarrow\supset+)$. So by applying lemma 4.6 to $\mathcal{Q}$, we can get the following proof:

$$\begin{array}{c} \vdots\, \mathcal{Q}' \\ \Pi \rightarrow \Sigma, A\supset B \\ \vdots \\ \Gamma \rightarrow \Delta \end{array}$$

where $\mathcal{Q}'$ is a proof in c-LDS/. And by iteration of this "$(\rightarrow\supset+)$ elimination process", we can get a proof $\mathcal{P}''$ of $\Gamma \rightarrow \Delta$ in c-LDS/ in the end. Then, if we ignore the connections in $\mathcal{P}''$, this is a proof of $\Gamma \rightarrow \Delta$ in LDS/, and by theorem 4.2, we can get a proof of $\Gamma \rightarrow \Delta$ in LDS.

When $\Delta = \bot$, we can also get a proof $\mathcal{P}'''$ of $\Gamma \rightarrow$ in LDS by lemma 4.7. □

Lemma 4.7 is easily shown by the following lemma.

Lemma 4.8 *Suppose $\mathcal{P}$ is a proof of*

$$\Gamma \rightarrow \Delta_0, \bot, \Delta_1, ..., \bot, \Delta_n \quad (n \geq 0)$$

in LDS. Then from $\mathcal{P}$, we can construct a proof of

$$\Gamma \rightarrow \Delta_0, \Delta_1, ..., \Delta_n$$

in LDS.

Proof This lemma is proved by easy induction on the number of inference rules in $\mathcal{P}$.

We only notice that: at the inference

$$\text{SP.CUT}\frac{\Gamma \to \Delta, A \qquad\qquad A \to B}{\Gamma \to \Delta, B} \ ,$$

$B \neq \bot$, and at the inference

$$\text{SP.CUT2}\frac{\Gamma \to \Delta, A \qquad\qquad A, B \to C}{\Gamma, B \to \Delta, C} \ ,$$

$C \neq \bot$. □

Lemma 4.6 can not be proved directly. We will prove a stronger statement in lemma 4.10 by induction.

For it we need the following lemma.

Lemma 4.9 *Suppose $\mathcal{P}$ is a proof of a sequent $\mathcal{S}$ in c-LDS! and $\mathcal{T}$ is a (1,0)-permutation of $\mathcal{S}$, where $\mathcal{S} = (\ \Gamma \to \Delta\ )$, $\mathcal{T} = (\ \Gamma_0; \Gamma_1 \to \Delta\ )$, and there is no connection between Γ_1 and Δ at $\mathcal{T}$. Then from $\mathcal{P}$, we can construct a proof $\mathcal{P}'$ in c-LDS! of $(\ \Gamma_0 \to \Delta\ )$ with the same connections between Γ_0 and Δ as $\mathcal{T}$.*

Proof This lemma is proved similarly to lemma 3.1. □

Lemma 4.10 *Suppose $\mathcal{P}$ is a proof of a sequent $\mathcal{S}$ in c-LDS! and $\mathcal{T}$ is an (n,n)-permutation ($n \geq 0$) of $\mathcal{S}$ where:*

- $\mathcal{S} = \Gamma \to \Delta$,
- $\mathcal{T} = \Gamma_0; \Gamma_1; ...; \Gamma_n \to \Delta_0; \Delta_1; ...; \Delta_n$,
- *for any i ($1 \leq i \leq n$) and j ($0 \leq j \leq n$), if $i \neq j$ then there is no connection between Γ_i and Δ_j at $\mathcal{T}$ (we will call this condition as* the connection condition*).*

Then from $\mathcal{P}$, we can construct a proof $\mathcal{P}'$ of a sequent $\mathcal{S}'$ in c-LDS! where :

- $\mathcal{S}' = \Gamma_0 \to \Delta_0, \{\Gamma_1; \Delta_1\}^1, \ldots, \{\Gamma_n; \Delta_n\}^n$,
- *the connections between* Γ_0 *and* Δ_0 *at* $\mathcal{S}'$ *are the same as those at* $\mathcal{T}$,
- *for any* i $(1 \le i \le n)$ *and any formula* γ_0 *in* Γ_0,

  $\gamma_0 \sim \{\Gamma_i; \Delta_i\}^i$ *at* $\mathcal{S}' \Leftrightarrow \gamma_0 \sim \delta_i$ *for some formula* δ_i *in* Δ_i *at* $\mathcal{T}$.

Proof We prove this lemma by induction on the number of inference rules in $\mathcal{P}$. We distinguish cases according to the form of $\mathcal{P}$.

Case 1: $\mathcal{P}$ is of the form

$A^1 \to A^2$.

Subcase 1-1: A^2 is in Δ_0 at $\mathcal{T}$. In this case, A^1 is in Γ_0 (since $A^1 \sim A^2$ and the connection condition). So we can get $\mathcal{P}'$ as follows:

$$\cfrac{\cfrac{A \to A}{\text{some } (\to w)}}{A \to A, \{;\}, \ldots, \{;\}}\ .$$

Subcase 1-2: A^2 is in $\Delta_k (k > 0)$ and A^1 is in Γ_0 at $\mathcal{T}$. Then we can get $\mathcal{P}'$ as follows:

$$\cfrac{\cfrac{(\to\supset)\cfrac{(w\to)\cfrac{A \to A}{\bot\supset\bot, A \to A}}{A \to (\bot\supset\bot)\supset A}}{\text{some } (\to w), (\to e)}}{A \to \{;\}, \ldots, \{;A\}, \ldots, \{;\}} \qquad (\ \{;A\} = (\bot\supset\bot)\supset A\ ).$$

Subcase 1-3: A^2 is in $\Delta_k (k > 0)$ and A^1 is not in Γ_0 at $\mathcal{T}$. In this case, A^1 is in Γ_k (since $A^1 \sim A^2$ and the connection condition). So we can get $\mathcal{P}'$ as follows:

$$
\dfrac{\dfrac{\dfrac{A \to A}{\to A \supset A}\,(\to\supset)}{\text{some } (\to w),\ (\to e)}}{\to \{;\}, \ldots, \{A;A\}, \ldots, \{;\}} \qquad (\ \{A;A\} = A \supset A\ ).
$$

Case 2: $\mathcal{P}$ is of the form

$$\bot \to A\ .$$

This case is similar to case 1.

In the following cases, we assume that $\mathcal{P}$ is of the form

$$
\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \mathcal{S}_1\end{array}}{\mathcal{S}}\,\mathrm{J}
$$

or

$$
\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \mathcal{S}_1\end{array} \qquad \begin{array}{c}\vdots\,\mathcal{P}_2\\ \mathcal{S}_2\end{array}}{\mathcal{S}}\,\mathrm{J}\ .
$$

Case 3: $\mathrm{J} = (w\to)$, i.e. $\mathcal{P}$ is of the form

$$
\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ \Pi \to \Delta\end{array}}{A^1, \Pi \to \Delta}\,(w\to)\ .
$$

Subcase 3-1: A^1 is in Γ_0 at $\mathcal{T}$. Let

$\mathcal{T} = A^1, \Pi_0; \Pi_1; \ldots; \Pi_n \to \Delta_0; \Delta_1; \ldots; \Delta_n$.

Then consider an (n, n)-permutation $\mathcal{T}_1$ of $\mathcal{S}_1$ as:

$$\mathcal{T}_1 = \Pi_0; \Pi_1; ...; \Pi_n \rightarrow \Delta_0; \Delta_1; ...; \Delta_n.$$

The connection condition holds for $\mathcal{T}_1$. So by induction hypothesis, we can get the following proof:

$$\begin{array}{c} \vdots\, \mathcal{P}_1' \\ \Pi_0 \rightarrow \Delta_0, \{\Pi_1; \Delta_1\}, ..., \{\Pi_n; \Delta_n\} \end{array} \quad ,$$

and we can get $\mathcal{P}'$ as:

$$(w\rightarrow)\dfrac{\vdots\, \mathcal{P}_1'}{A, \Pi_0 \rightarrow \Delta_0, \{\Pi_1; \Delta_1\}, ..., \{\Pi_n; \Delta_n\}} \ .$$

At the inference $(w\rightarrow)$, we can assign arbitrary connections to the principal occurrence. So we can assign the required connections to this last sequent.

Subcase 3-2: A^1 is in $\Gamma_k (k > 0)$ at $\mathcal{T}$. By the same way as above, we can get a proof:

$$\begin{array}{c} \vdots\, \mathcal{P}_1' \\ \Pi_0 \rightarrow \Delta_0, \{\Pi_1; \Delta_1\}, ..., \{\Pi_n; \Delta_n\} \end{array} \ .$$

Then, we can get $\mathcal{P}'$ as follows:

$$\begin{array}{c} \vdots\, \mathcal{P}_1' \\ \hline \text{some } (\rightarrow e) \\ \hline \Pi_0 \rightarrow \Delta_0, \{\Pi_1; \Delta_1\}, ..., \{\Pi_n; \Delta_n\}, \{\Pi_k; \Delta_k\} \\ \hline \text{SP.CUT}' \qquad \Pi_0 \rightarrow \Delta_0, \{\Pi_1; \Delta_1\}, ..., \{\Pi_n; \Delta_n\}, \{A, \Pi_k; \Delta_k\} \\ \hline \text{some } (\rightarrow e) \\ \hline \Pi_0 \rightarrow \Delta_0, \{\Pi_1; \Delta_1\}, ..., \{A, \Pi_k; \Delta_k\}, ..., \{\Pi_n; \Delta_n\} \ . \end{array}$$

It is obvious that this last sequent has the required connections.

Case 4: J = $(\rightarrow w)$. This case is similar to case 3.

Case 5: J = $(e\rightarrow)$, i.e. $\mathcal{P}$ is of the form

$$(e\rightarrow)\dfrac{\begin{array}{c}\vdots\ \mathcal{P}_1\\ \Pi, A^1, B^2, \Theta \rightarrow \Delta\end{array}}{\Pi, B^2, A^1, \Theta \rightarrow \Delta} \quad .$$

Subcase 5-1: B^2 and A^1 are in Γ_0 at $\mathcal{T}$. This case is similar to subcase 3-1.

Subcase 5-2: B^2 and A^1 are in $\Gamma_k (k > 0)$ at $\mathcal{T}$. This case is similar to subcase 3-2.

Subcase 5-3: B^2 is in Γ_i, A^1 is in Γ_j, and $i \neq j$ at $\mathcal{T}$. Then $\mathcal{T}$ is also an (n,n)-permutation of $\mathcal{S}_1$ satisfying the connection condition. So we can get the required proof $\mathcal{P}'$ by induction hypothesis.

Case 6: J = $(\rightarrow e)$. This case is similar to case 5.

Case 7: J = $(c\rightarrow)$.

Case 8: J = $(\rightarrow c)$.

Case 9: J = $(\wedge\rightarrow 1)$.

Case 10: J = $(\wedge\rightarrow 2)$.

Case 7 $\sim$ 10 are similar to case 3.

Case 11: J = $(\rightarrow\wedge)$, i.e. $\mathcal{P}$ is of the form

$$
\begin{array}{c}
\vdots\ \mathcal{P}_1 \qquad\qquad\qquad \vdots\ \mathcal{P}_2 \\
(\rightarrow\wedge)\dfrac{\Theta \rightarrow \Lambda, A^1 \qquad\qquad \Pi \rightarrow \Sigma, B^2}{\Theta, \Pi \rightarrow \Lambda, \Sigma, (A\wedge B)^3} \ .
\end{array}
$$

Subcase 11-1: $(A\wedge B)^3$ is in Δ_0 at $\mathcal{T}$. Let

$\mathcal{T} = \Theta_0, \Pi_0; \Theta_1, \Pi_1; ...; \Theta_n, \Pi_n \rightarrow \Lambda_0, \Sigma_0, (A\wedge B)^3; \Lambda_1, \Sigma_1; ...; \Lambda_n, \Sigma_n$.

Then consider two (n,n)-permutations $\mathcal{T}_1$ of $\mathcal{S}_1$ and $\mathcal{T}_2$ of $\mathcal{S}_2$ as:

$\mathcal{T}_1 = \Theta_0; \Theta_1; ...; \Theta_n \rightarrow \Lambda_0, A^1; \Lambda_1; ...; \Lambda_n$,

$\mathcal{T}_2 = \Pi_0; \Pi_1; ...; \Pi_n \rightarrow \Sigma_0, B^2; \Sigma_1; ...; \Sigma_n$.

The connection condition holds for $\mathcal{T}_1$ and for $\mathcal{T}_2$. So by induction hypotheses, we can get the following two proofs:

$$
\begin{array}{c}
\vdots\ \mathcal{P}_1' \\
\Theta_0 \rightarrow \Lambda_0, A, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_n; \Lambda_n\} \ ,
\end{array}
$$

and

$$
\begin{array}{c}
\vdots\ \mathcal{P}_2' \\
\Pi_0 \rightarrow \Sigma_0, B, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\} \ .
\end{array}
$$

Then we can get $\mathcal{P}'$ as:

$$
\dfrac{(\rightarrow\wedge)\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{P}_1'}{\text{some } (\rightarrow e)}}{\Theta_0 \rightarrow \Lambda_0, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_n; \Lambda_n\}, A} \qquad \dfrac{\dfrac{\vdots\ \mathcal{P}_2'}{\text{some } (\rightarrow e)}}{\Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, B}}{\Theta_0, \Pi_0 \rightarrow \Lambda_0, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_n; \Lambda_n\}, \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, A\wedge B}}{\dfrac{\text{some } (\rightarrow e), \text{SP.CUT}', (\rightarrow c)}{\Theta_0, \Pi_0 \rightarrow \Lambda_0, \Sigma_0, A\wedge B, \{\Theta_1, \Pi_1; \Lambda_1, \Sigma_1\}, ..., \{\Theta_n, Pi_n; \Lambda_n, \Sigma_n\}}}
$$

It is easy to show that this last sequent has the required connections.

Subcase 11-2: $(A \wedge B)^3$ is in $\Delta_k (k > 0)$ at $\mathcal{T}$. Let

$$\mathcal{T} = \Theta_0, \Pi_0; \Theta_1, \Pi_1; ...; \Theta_n, \Pi_n \rightarrow \Lambda_0, \Sigma_0; \Lambda_1, \Sigma_1; ...; \Lambda_k, \Sigma_k, (A \wedge B)^3; ...; \Lambda_n, \Sigma_n.$$

Then consider two (n, n)-permutations $\mathcal{T}_1$ of $\mathcal{S}_1$ and $\mathcal{T}_2$ of $\mathcal{S}_2$ as:

$$\mathcal{T}_1 = \Theta_0; \Theta_1; ...; \Theta_n \rightarrow \Lambda_0; \Lambda_1; ...; \Lambda_k, A^1; ...; \Lambda_n,$$

$$\mathcal{T}_2 = \Pi_0; \Pi_1; ...; \Pi_n \rightarrow \Sigma_0; \Sigma_1; ...; \Sigma_k, B^2; ...; \Sigma_n.$$

The connection condition holds for $\mathcal{T}_1$ and for $\mathcal{T}_2$. So by induction hypotheses, we can get the following two proofs:

$$\begin{array}{c} \vdots\ \mathcal{P}_1' \\ \Theta_0 \rightarrow \Lambda_0, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_k; \Lambda_k, A\}, ..., \{\Theta_n; \Lambda_n\} \end{array},$$

and

$$\begin{array}{c} \vdots\ \mathcal{P}_2' \\ \Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_k; \Sigma_k, B\}, ..., \{\Pi_n; \Sigma_n\} \end{array}.$$

Then we can get $\mathcal{P}'$ as:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{P}_1'}{\text{some } (\rightarrow e)}}{\Theta_0 \rightarrow \Lambda_0, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_n; \Lambda_n\}, \{\Theta_k; \Lambda_k, A\}} \quad \dfrac{\dfrac{\vdots\ \mathcal{P}_2'}{\text{some } (\rightarrow e)}}{\Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Pi_k; \Sigma_k, B\}}}{\Theta_0, \Pi_0 \rightarrow \Lambda_0, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_n; \Lambda_n\}, \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Theta_k; \Lambda_k, A\} \wedge \{\Pi_k; \Sigma_k, B\}}(\rightarrow \wedge)}{\dfrac{\text{some } (\rightarrow e), \text{SP.CUT}', (\rightarrow c)}{\Theta_0, \Pi_0 \rightarrow \Lambda_0, \Sigma_0, \{\Theta_1, \Pi_1; \Lambda_1, \Sigma_1\}, ..., \{\Theta_k, \Pi_k; \Lambda_k, \Sigma_k, A \wedge B\}, ..., \{\Theta_n, \Pi_n; \Lambda_n, \Sigma_n\}}}.$$

It is easy to show that this last sequent has the required connections.

Case 12: J = $(\vee\rightarrow)$. This case is similar to case 11.

Case 13: J = $(\rightarrow \vee 1)$.

Case 14: J = $(\rightarrow \vee 2)$.

Case 13 and 14 are similar to case 3.

Case 15: J = ($\supset\rightarrow$), i.e. $\mathcal{P}$ is of the form

$$\dfrac{\begin{matrix} \vdots \mathcal{P}_1 & & \vdots \mathcal{P}_2 \\ \Theta \rightarrow \Lambda, A^1 & & B^2, \Pi \rightarrow \Sigma \end{matrix}}{(A \supset B)^3, \Theta, \Pi \rightarrow \Lambda, \Sigma} (\supset\rightarrow) \ .$$

Subcase 15-1: $(A \supset B)^3$ is in Γ_0 at $\mathcal{T}$. Let

$\mathcal{T} = (A \supset B)^3, \Theta_0, \Pi_0; \Theta_1, \Pi_1; ...; \Theta_n, \Pi_n \rightarrow \Lambda_0, \Sigma_0; \Lambda_1, \Sigma_1; ...; \Lambda_n, \Sigma_n.$

Sub-subcase 15-1-1: $\theta_+ \not\sim A^1$ for any formula θ_+ in $\Theta_1, ..., \Theta_n$ at $\mathcal{S}_1$. Then consider two (n,n)-permutations $\mathcal{T}_1$ of $\mathcal{S}_1$ and $\mathcal{T}_2$ of $\mathcal{S}_2$ as:

$\mathcal{T}_1 = \Theta_0; \Theta_1; ...; \Theta_n \rightarrow \Lambda_0, A^1; \Lambda_1; ...; \Lambda_n,$

$\mathcal{T}_2 = B^2, \Pi_0; \Pi_1; ...; \Pi_n \rightarrow \Sigma_0; \Sigma_1; ...; \Sigma_n.$

The connection condition holds for $\mathcal{T}_1$ and for $\mathcal{T}_2$. So by induction hypotheses, we can get the following two proofs:

$$\begin{matrix} \vdots \mathcal{P}'_1 \\ \Theta_0 \rightarrow \Lambda_0, A, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_n; \Lambda_n\} \end{matrix}$$

where for any formulas θ_0 in Θ_0, λ_0 in Λ_0 and for any i $(1 \leq i \leq n)$,

1) $\theta_0 \sim \lambda_0$ at this last sequent $\Leftrightarrow$ $\theta_0 \sim \lambda_0$ at $\mathcal{S}_1$,

2) $\theta_0 \sim A$ at this last sequent $\Leftrightarrow$ $\theta_0 \sim A^1$ at $\mathcal{S}_1$,

3) $\theta_0 \sim \{\Theta_i; \Lambda_i\}$ at this last sequent $\Leftrightarrow$ $\theta_0 \sim \lambda_i$ for some formula λ_i in Λ_i at $\mathcal{S}_1$,

and

$$\begin{matrix} \vdots \mathcal{P}'_2 \\ B, \Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\} \end{matrix}$$

where for any formulas π_0 in Π_0, σ_0 in Σ_0 and for any i $(1 \leq i \leq n)$,

1) $B \sim \sigma_0$ at this last sequent $\Leftrightarrow$ $B^2 \sim \sigma_0$ at $\mathcal{S}_2$,

2) $B \sim \{\Pi_i; \Sigma_i\}$ at this last sequent $\Leftrightarrow$ $B^2 \sim \sigma_i$ for some formula σ_i in Σ_i at $\mathcal{S}_2$,

3) $\pi_0 \sim \sigma_0$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_0$ at $\mathcal{S}_2$,

4) $\pi_0 \sim \{\Pi_i; \Sigma_i\}$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_i$ for some formula σ_i in Σ_i at $\mathcal{S}_2$.

So by using these proofs, we can get a proof:

$$
(\supset\to)\frac{\dfrac{\dfrac{\vdots\ \mathcal{P}_1'}{\text{some }(\to e)}}{\Theta_0 \to \Lambda_0, \{\Theta_1; \Lambda_1\}, \ldots, \{\Theta_n; \Lambda_n\}, A} \qquad \begin{array}{c}\vdots\ \mathcal{P}_2' \\ B, \Pi_0 \to \Sigma_0, \{\Pi_1; \Sigma_1\}, \ldots, \{\Pi_n; \Sigma_n\}\end{array}}{\dfrac{A\supset B, \Theta_0, \Pi_0 \to \Lambda_0, \{\Theta_1; \Lambda_1\}, \ldots, \{\Theta_n; \Lambda_n\}, \Sigma_0, \{\Pi_1; \Sigma_1\}, \ldots, \{\Pi_n; \Sigma_n\}}{\dfrac{\text{some }(\to e), \text{SP.CUT}', (\to c)}{A\supset B, \Theta_0, \Pi_0 \to \Lambda_0, \Sigma_0, \{\Theta_1, \Pi_1; \Lambda_1, \Sigma_1\}, \ldots, \{\Theta_n, \Pi_n; \Lambda_n, \Sigma_n\}}}}
$$

where for any formulas θ_0 in Θ_0, π_0 in Π_0, λ_0 in Λ_0, σ_0 in Σ_0 and for any i $(1 \leq i \leq n)$,

1) $A\supset B \not\sim \lambda_0$ at this last sequent,

2) $A\supset B \sim \sigma_0$ at this last sequent $\Leftrightarrow$ $B^2 \sim \sigma_0$ at $\mathcal{S}_2$,

3) $A\supset B \sim \{\Theta_i, \Pi_i; \Lambda_i, \Sigma_i\}$ at this last sequent $\Leftrightarrow$ $B^2 \sim \sigma_i$ for some formula σ_i in Σ_i at $\mathcal{S}_2$,

4) $\theta_0 \sim \lambda_0$ at this last sequent $\Leftrightarrow$ $\theta_0 \sim \lambda_0$ at $\mathcal{S}_1$,

5) $\theta_0 \sim \sigma_0$ at this last sequent $\Leftrightarrow$ ($\theta_0 \sim A^1$ at $\mathcal{S}_1$ and $B^2 \sim \sigma_0$ at $\mathcal{S}_2$),

6) $\theta_0 \sim \{\Theta_i, \Pi_i; \Lambda_i, \Sigma_i\}$ at this last sequent $\Leftrightarrow$ ($\theta_0 \sim \lambda_i$ for some formula λ_i in Λ_i at $\mathcal{S}_1$ or ($\theta_0 \sim A^1$ at $\mathcal{S}_1$ and $B^2 \sim \sigma_i$ for some formula σ_i in Σ_i at $\mathcal{S}_2$)),

7) $\pi_0 \not\sim \lambda_0$ at this last sequent,

8) $\pi_0 \sim \sigma_0$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_0$ at $\mathcal{S}_2$,

9) $\pi_0 \sim \{\Theta_i, \Pi_i; \Lambda_i, \Sigma_i\}$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_i$ for some formula σ_i in Σ_i at $\mathcal{S}_2$.

This is the required proof $\mathcal{P}'$.

Sub-subcase 15-1-2: $\theta_k \sim A^1$ for some formula θ_k in $\Theta_k (k > 0)$ at $\mathcal{S}_1$ and $B^2 \not\sim \sigma$ for any formula σ in Σ at $\mathcal{S}_2$. Then consider an (n, n)-permutation $\mathcal{T}_2$ of $\mathcal{S}_2$ as:

$\mathcal{T}_2 = B^2, \Pi_0; \Pi_1; ...; \Pi_n \rightarrow \Sigma_0; \Sigma_1; ...; \Sigma_n.$

The connection condition holds for $\mathcal{T}_2$. So by induction hypothesis, we can get the following proof:

$$\begin{array}{c} \vdots \mathcal{P}_2' \\ B, \Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\} \end{array}$$

where B has no connection at this last sequent. So, by lemma 4.9, we can get the following proof:

$$\begin{array}{c} \vdots \mathcal{P}_2'' \\ \Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \end{array}$$

and by applying some $(w\rightarrow)$, $(\rightarrow w)$, $(e\rightarrow)$, $(\rightarrow e)$ and SP.CUT′ to this, we can get the required $\mathcal{P}'$ as:

$$\frac{\vdots}{A \supset B, \Theta_0, \Pi_0 \rightarrow \Lambda_0, \Sigma_0, \{\Theta_1, \Pi_1; \Lambda_1, \Sigma_1\}, ..., \{\Theta_n, \Pi_n; \Lambda_n, \Sigma_n\}}\ .$$

We can assign the required cannections to this last sequent.

Sub-subcase 15-1-3: $\theta_k \sim A^1$ for some formula θ_k in $\Theta_k (k > 0)$ at $\mathcal{S}_1$ and $B^2 \sim \sigma$ for some formula σ in Σ at $\mathcal{S}_2$. Then consider two (n, n)-permutations $\mathcal{T}_1$ of $\mathcal{S}_1$ and $\mathcal{T}_2$ of $\mathcal{S}_2$ as:

$\mathcal{T}_1 = \Theta_0; \Theta_1; ...; \Theta_n \rightarrow \Lambda_0; \Lambda_1; ...; \Lambda_k, A^1; ...; \Lambda_n,$

$\mathcal{T}_2 = \Pi_0; \Pi_1; ...; B^2, \Pi_k; ...; \Pi_n \rightarrow \Sigma_0; \Sigma_1; ...; \Sigma_n.$

Then the connection condition holds for $\mathcal{T}_1$ and for $\mathcal{T}_2$. So by induction hypotheses, we can get the following two proofs:

$$\vdots\ \mathcal{P}_1'$$

$$\Theta_0 \to \Lambda_0, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_k; \Lambda_k, A\}, ..., \{\Theta_n; \Lambda_n\}$$

where for any formulas θ_0 in Θ_0, λ_0 in Λ_0 and for any i $(1 \le i \le n,\ i \ne k)$,

1) $\theta_0 \sim \lambda_0$ at this last sequent $\Leftrightarrow$ $\theta_0 \sim \lambda_0$ at $\mathcal{S}_1$,

2) $\theta_0 \sim \{\Theta_i; \Lambda_i\}$ at this last sequent $\Leftrightarrow$ $\theta_0 \sim \lambda_i$ for some formula λ_i in Λ_i at $\mathcal{S}_1$,

3) $\theta_0 \sim \{\Theta_k; \Lambda_k, A\}$ at this last sequent $\Leftrightarrow$ ($\theta_0 \sim \lambda_k$ for some formula λ_k in Λ_k at $\mathcal{S}_1$ or $\theta_0 \sim A^1$ at $\mathcal{S}_1$),

and

$$\vdots\ \mathcal{P}_2'$$

$$\Pi_0 \to \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{B, \Pi_k; \Sigma_k\}, ..., \{\Pi_n; \Sigma_n\}$$

where for any formulas π_0 in Π_0, σ_0 in Σ_0 and for any i $(1 \le i \le n, i \ne k)$,

1) $\pi_0 \sim \sigma_0$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_0$ at $\mathcal{S}_2$,

2) $\pi_0 \sim \{\Pi_i; \Sigma_i\}$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_i$ for some formula σ_i in Σ_i at $\mathcal{S}_2$,

3) $\pi_0 \sim \{B, \Pi_k; \Sigma_k\}$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_k$ for some formula σ_k in Σ_k at $\mathcal{S}_2$.

So by using these proofs, we can get the following proof:

$$\vdots\ \mathcal{P}_1' \qquad\qquad \vdots\ \mathcal{P}_2'$$

$$\text{some } (\to e) \qquad\qquad \text{some } (\to e)$$

$$\dfrac{\Theta_0 \to \Lambda_0, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_n; \Lambda_n\}, \{\Theta_k; \Lambda_k, A\} \qquad \Pi_0 \to \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{B, \Pi_k; \Sigma_k\}}{\Theta_0, \Pi_0 \to \Lambda_0, \{\Theta_1; \Lambda_1\}, ..., \{\Theta_n; \Lambda_n\}, \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Theta_k; \Lambda_k, A\} \wedge \{B, \Pi_k; \Sigma_k\}}\ (\to\wedge)$$

$$\text{some } (\to e), \text{SP.CUT}', (\to c)$$

$$\text{SP.CUT2}'\ \dfrac{\Theta_0, \Pi_0 \to \Lambda_0, \Sigma_0, \{\Theta_1, \Pi_1; \Lambda_1, \Sigma_1\}, ..., \{\Theta_n, \Pi_n; \Lambda_n, \Sigma_n\}, \{A \supset B, \Theta_k, \Pi_k; \Lambda_k, \Sigma_k\}}{\Theta_0, \Pi_0, A \supset B \to \Lambda_0, \Sigma_0, \{\Theta_1, \Pi_1; \Lambda_1, \Sigma_1\}, ..., \{\Theta_n, \Pi_n; \Lambda_n, \Sigma_n\}, \{\Theta_k, \Pi_k; \Lambda_k, \Sigma_k\}}$$

$$\text{some } (e \to), (\to e)$$

$$A \supset B, \Theta_0, \Pi_0 \to \Lambda_0, \Sigma_0, \{\Theta_1, \Pi_1; \Lambda_1, \Sigma_1\}, ..., \{\Theta_n, \Pi_n; \Lambda_n, \Sigma_n\}$$

where for any formulas θ_0 in Θ_0, π_0 in Π_0, λ_0 in Λ_0, σ_0 in Σ_0 and for any i $(1 \le i \le n,$

$i \neq k$),

1) $A \supset B \not\sim \lambda_0$ at this last sequent,

2) $A \supset B \not\sim \sigma_0$ at this last sequent,

3) $A \supset B \not\sim \{\Theta_i, \Pi_i; \Lambda_i, \Sigma_i\}$ at this last sequent,

4) $A \supset B \sim \{\Theta_k, \Pi_k; \Lambda_k, \Sigma_k\}$ at this last sequent,

5) $\theta_0 \sim \lambda_0$ at this last sequent $\Leftrightarrow$ $\theta_0 \sim \lambda_0$ at $\mathcal{S}_1$,

6) $\theta_0 \not\sim \sigma_0$ at this last sequent,

7) $\theta_0 \sim \{\Theta_i, \Pi_i; \Lambda_i, \Sigma_i\}$ at this last sequent $\Leftrightarrow$ $\theta_0 \sim \lambda_i$ for some formula λ_i in Λ_i at $\mathcal{S}_1$,

8) $\theta_0 \sim \{\Theta_k, \Pi_k; \Lambda_k, \Sigma_k\}$ at this last sequent $\Leftrightarrow$ ($\theta_0 \sim \lambda_k$ for some formula λ_k in Λ_k at $\mathcal{S}_1$ or $\theta_0 \sim A^1$ at $\mathcal{S}_1$),

9) $\pi_0 \not\sim \lambda_0$ at this last sequent,

10) $\pi_0 \sim \sigma_0$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_0$ at $\mathcal{S}_2$,

11) $\pi_0 \sim \{\Theta_i, \Pi_i; \Lambda_i, \Sigma_i\}$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_i$ for some formula σ_i in Σ_i at $\mathcal{S}_2$,

12) $\pi_0 \sim \{\Theta_k, \Pi_k; \Lambda_k, \Sigma_k\}$ at this last sequent $\Leftrightarrow$ $\pi_0 \sim \sigma_k$ for some formula σ_k in Σ_k at $\mathcal{S}_2$.

This is the required proof $\mathcal{P}'$.

Subcase 15-2: $(A \supset B)^3$ is in $\Gamma_k (k > 0)$ at $\mathcal{T}$.

Sub-subcase 15-2-1: $B^2 \not\sim \sigma$ for any formula σ in Σ at $\mathcal{S}_2$. This case is similar to 15-1-2.

Sub-subcase 15-2-2: $B^2 \sim \sigma$ for some formula σ in Σ at $\mathcal{S}_2$. This case is similar to 15-1-3.

Case 16: J = $(\to\supset)$, i.e. $\mathcal{P}$ is of the form

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{P}_1\\ A^1,\Gamma\to B^2\end{array}}{\Gamma\to(A\supset B)^3}\,(\to\supset)\quad .$$

Subcase 16-1: $(A\supset B)^3$ is in Δ_0 at $\mathcal{T}$. Let

$\mathcal{T}=\Gamma_0;\Gamma_1;...;\Gamma_n\to(A\supset B)^3;;...;$.

Then consider an $(n+1,n+1)$-permutation $\mathcal{T}_1$ of $\mathcal{S}_1$ as:

$\mathcal{T}_1=\Gamma_0;\Gamma_1;...;\Gamma_n;A^1\to;;...;;B^2$.

The connection condition holds for $\mathcal{T}_1$. So by induction hypothesis, we can get the following proof:

$$\begin{array}{c}\vdots\,\mathcal{P}_1'\\ \Gamma_0\to\{\Gamma_1;\},...,\{\Gamma_n;\},\{A;B\}\end{array}$$

where for any formula γ_0 in Γ_0 and any i $(1\le i\le n)$,

1) $\gamma_0\not\sim\{\Gamma_i;\}$ at this last sequent,

2) $\gamma_0\sim\{A;B\}$ at this last sequent $\Leftrightarrow\gamma_0\sim B^2$ at $\mathcal{S}_1$.

Then we can get the required proof $\mathcal{P}'$ as:

$$\dfrac{\dfrac{\vdots\,\mathcal{P}_1'}{\text{some }(\to e)}}{\Gamma_0\to A\supset B,\{\Gamma_1;\},...,\{\Gamma_n;\}}\qquad(\{A;B\}=A\supset B).$$

Subcase 16-2: $(A\supset B)^3$ is in $\Delta_k(k>0)$ at $\mathcal{T}$. Let

$\mathcal{T}=\Gamma_0;\Gamma_1;...;\Gamma_n\to;;...;(A\supset B)^3;...;$.

Then consider an (n,n)-permutation $\mathcal{T}_1$ of $\mathcal{S}_1$ as:

$\mathcal{T}_1=\Gamma_0;\Gamma_1;...;A^1,\Gamma_k;...;\Gamma_n\to;;...;B^2;...;$.

The connection condition holds for $\mathcal{T}_1$. So by induction hypothesis, we can get the following proof.

$$\begin{array}{c} \vdots \mathcal{P}_1' \\ \Gamma_0 \to \{\Gamma_1;\}, ..., \{A, \Gamma_k; B\}, ..., \{\Gamma_n;\} \end{array}$$

where for any formula γ_0 in Γ_0 and any i $(1 \le i \le n, \ \ i \ne k)$,

1) $\gamma_0 \not\sim \{\Gamma_i;\}$ at this last sequent,

2) $\gamma_0 \sim \{A, \Gamma_k; B\}$ at this last sequent $\Leftrightarrow \gamma_0 \sim B^2$ at $\mathcal{S}_1$.

So we can get the required proof $\mathcal{P}'$ as:

$$\begin{array}{c} \vdots \mathcal{P}_1' \\ \hline \text{some } (\to e), \text{SP.CUT}' \\ \hline \Gamma_0 \to \{\Gamma_1;\}, ..., \{\Gamma_k; A \supset B\}, ..., \{\Gamma_n;\} \ . \end{array}$$

Case 17: J = $(\forall\!\to)$.

Case 18: J = $(\to\!\forall)$.

Case 19: J = $(\exists\!\to)$.

Case 20: J = $(\to\!\exists)$.

Case 21: J = SP.CUT$'$.

Case 17 $\sim$ 21 are similar to case 3.

Case 22: J = SP.CUT2$'$, i.e. $\mathcal{P}$ is of the form

$$\begin{array}{c} \vdots \mathcal{P}_1 \\ \text{SP.CUT2}' \dfrac{\Pi \to \Sigma, \{A, \Theta; \Lambda\}^1}{\Pi, A^2 \to \Sigma, \{\Theta; \Lambda\}^3} \ . \end{array}$$

Subcase 22-1: $\{\Theta; \Lambda\}^3$ is in Δ_0 at $\mathcal{T}$. In this case, A^2 is in Γ_0 at $\mathcal{T}$ (since $A^2 \sim \{\Theta; \Lambda\}^3$

and the connection condition for $\mathcal{T}$). So

$$\mathcal{T} = \Pi_0, A^2; \Pi_1; ...; \Pi_n \to \Sigma_0, \{\Theta; \Lambda\}^3; \Sigma_1; ...; \Sigma_n.$$

Then consider an (n,n)-permutation $\mathcal{T}_1$ of $\mathcal{S}_1$ as:

$$\mathcal{T}_1 = \Pi_0; \Pi_1; ...; \Pi_n \to \Sigma_0, \{A, \Theta; \Lambda\}^1; \Sigma_1; ...; \Sigma_n.$$

The connection condition holds for $\mathcal{T}_1$. So by induction hypothesis, we can get the following proof:

$$\begin{array}{c} \vdots\ \mathcal{P}_1' \\ \Pi_0 \to \Sigma_0, \{A, \Theta; \Lambda\}, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\} \end{array},$$

and we can get the requirted proof $\mathcal{P}'$ as:

$$\begin{array}{rc} & \vdots\ \mathcal{P}_1' \\ \cline{2-2} & \text{some } (\to e) \\ \cline{2-2} & \Pi_0 \to \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{A, \Theta; \Lambda\} \\ \text{SP.CUT2}' \!\!\!& \hrulefill \\ & \Pi_0, A \to \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Theta; \Lambda\} \\ \cline{2-2} & \text{some } (\to e) \\ \cline{2-2} & \Pi_0, A \to \Sigma_0, \{\Theta; \Lambda\}, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}\ . \end{array}$$

It is easy to show that this last sequent has the required connections.

Subcase 22-2: $\{\Theta; \Lambda\}^3$ is in $\Delta_k (k > 0)$ and A^2 is in Γ_k at $\mathcal{T}$. Let

$$\mathcal{T} = \Pi_0; \Pi_1; ...; \Pi_k, A^2; ...; \Pi_n \to \Sigma_0; \Sigma_1; ...; \Sigma_k, \{\Theta; \Lambda\}^3; ...; \Sigma_n.$$

Then consider an (n,n)-permutation $\mathcal{T}_1$ of $\mathcal{S}_1$ as:

$$\mathcal{T}_1 = \Pi_0; \Pi_1; ...; \Pi_n \to \Sigma_0; \Sigma_1; ...; \Sigma_k, \{A, \Theta; \Lambda\}^1; ...; \Sigma_n.$$

The connection condition holds for $\mathcal{T}_1$. So by induction hypothesis, we can get the following proof:

$$\begin{array}{c} \vdots\ \mathcal{P}_1' \\ \Pi_0 \to \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_k; \Sigma_k, \{A, \Theta; \Lambda\}\}, ..., \{\Pi_n; \Sigma_n\} \end{array},$$

and we can get the requirted proof $\mathcal{P}'$ as:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\, \mathcal{P}_1'}{\text{some } (\rightarrow e)}}{\Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Pi_k; \Sigma_k, \{A, \Theta; \Lambda\}\}}}{\Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Pi_k, A; \Sigma_k, \{\Theta; \Lambda\}\}}\text{SP.CUT}'}{\dfrac{\text{some } (\rightarrow e)}{\Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_k, A; \Sigma_k, \{\Theta; \Lambda\}\}, ..., \{\Pi_n; \Sigma_n\}}}\ .$$

It is easy to show that this last sequent has the required connections.

Subcase 22-3: $\{\Theta; \Lambda\}^3$ is in $\Delta_k (k > 0)$ and A^2 is in $\Gamma_l (l \neq k)$ at $\mathcal{T}$. In this case $l = 0$ (since $A^2 \sim \{\Theta; \Lambda\}^3$ and the connection condition). Let

$\mathcal{T} = \Pi_0, A^2; \Pi_1; ...; \Pi_n \rightarrow \Sigma_0; \Sigma_1; ...; \Sigma_k, \{\Theta; \Lambda\}^3; ...; \Sigma_n.$

Then consider an (n, n)-permutation $\mathcal{T}_1$ of $\mathcal{S}_1$ as:

$\mathcal{T}_1 = \Pi_0; \Pi_1; ...; \Pi_n \rightarrow \Sigma_0; \Sigma_1; ...; \Sigma_k, \{A, \Theta; \Lambda\}^1; ...; \Sigma_n.$

The connection condition holds for $\mathcal{T}_1$. So by induction hypothesis, we can get the following proof:

$$\begin{array}{c} \vdots\, \mathcal{P}_1' \\ \Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_k; \Sigma_k, \{A, \Theta; \Lambda\}\}, ..., \{\Pi_n; \Sigma_n\}\ , \end{array}$$

and we can get the requirted proof $\mathcal{P}'$ as:

$$\vdots\, \mathcal{P}_1'$$

$$\frac{\frac{\text{SP.CUT2}'\;\frac{\frac{\text{SP.CUT}'\;\frac{\frac{\vdots\,\mathcal{P}_1'}{\text{some } (\rightarrow e)}}{\Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Pi_k; \Sigma_k, \{A, \Theta; \Lambda\}\}}}{\Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Pi_k, A; \Sigma_k, \{\Theta; \Lambda\}\}}}{\text{some SP.CUT}'}}{\Pi_0 \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{A, \Pi_k; \Sigma_k, \{\Theta; \Lambda\}\}}}{\Pi_0, A \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_n; \Sigma_n\}, \{\Pi_k; \Sigma_k, \{\Theta; \Lambda\}\}}$$

$$\frac{\text{some } (\rightarrow e)}{\Pi_0, A \rightarrow \Sigma_0, \{\Pi_1; \Sigma_1\}, ..., \{\Pi_k; \Sigma_k, \{\Theta; \Lambda\}\}, ..., \{\Pi_n; \Sigma_n\}\ .}$$

It is easy to show that this last sequent has the required connections.

This completes the proof of lemma 4.10. □

Proof of lemma 4.6 Lemma 4.6 is a special case of lemma 4.10. □

5 Applications of cut-elimination theorems

In this section, we show some applications of our cut-elimination theorems.

First, we prove the following theorem.

Theorem 5.1 *If a formula A is a theorem of CD and A has no occurrence of "$\vee$" in negative, then A is a theorem of intuitionistic logic.*

In [2], Gabbay shows that: if A is a theorem of CD and A has no occurrence of "$\vee$", then A is a theorem of intuitionistic logic. Our theorem refines this result.

To prove theorem 5.1, we show the following lemma.

Lemma 5.2 *Suppose $\mathcal{P}$ is a proof of a sequent $\mathcal{S} = \Gamma \to \Delta$ (with connections) in c-LD, and $\mathcal{P}$ does not contain $(\vee\!\to)$. Then from $\mathcal{P}$, we can construct a proof $\mathcal{P}'$ of a sequent $\mathcal{S}' = \Gamma_0 \to A$ (without connection) in LJ! (or LJ) where $(\Gamma_0; \Gamma_1 \to A^1; \Delta_1)$ is a (1,1)-permutation of $\mathcal{S}$ such that*

γ *is in* $\Gamma_0 \Leftrightarrow \gamma \sim A^1$

for any formula γ in Γ.

Proof This lemma is proved by induction on the number of inference rules in $\mathcal{P}$. We only show the following three cases.

Case 1: $\mathcal{P}$ is of the form

$$\dfrac{\begin{matrix}\vdots\,\mathcal{P}_1 \\ \Theta \to \Lambda, B^1\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{P}_2 \\ C^2, \Pi \to \Sigma\end{matrix}}{(B \supset C)^3, \Theta, \Pi \to \Lambda, \Sigma} \;(\supset\to) \quad .$$

We call the above two upper sequents $(\Theta \to \Lambda, B^1)$ and $(C^2, \Pi \to \Sigma)$ by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Now, by induction hypotheses, one of the following subcases holds.

Subcase 1-1: we can get a proof:

$$\begin{array}{l} \vdots\ \mathcal{P}_1' \\ \Theta_0 \to A \end{array}$$

in LJ/ where $(\Theta_0; \Theta_1 \to A^0; \Lambda_1, B^1)$ is a (1,1)-permutation of $\mathcal{S}_1$ such that

$$\theta \text{ is in } \Theta_0 \Leftrightarrow \theta \sim A^0$$

for any formula θ in Θ. In this case, $\mathcal{P}_1'$ is the required proof $\mathcal{P}'$.

Subcase 1-2: we can get a proof:

$$\begin{array}{l} \vdots\ \mathcal{P}_2' \\ \Pi_0 \to A \end{array}$$

in LJ/ where $(\Pi_0; C^2, \Pi_1 \to A^0; \Sigma_1)$ is a (1,1)-permutation of $\mathcal{S}_2$ such that

$$\phi \text{ is in } \Pi_0 \Leftrightarrow \phi \sim A^0$$

for any formula ϕ in (C^2, Π). In this case, $\mathcal{P}_2'$ is the required proof $\mathcal{P}'$.

Subcase 1-3: we can get a proof:

$$\begin{array}{l} \vdots\ \mathcal{P}_1' \\ \Theta_0 \to B \end{array}$$

in LJ/ where $(\Theta_0; \Theta_1 \to B^1; \Lambda)$ is a (1,1)-permutation of $\mathcal{S}_1$ such that

$$\theta \text{ is in } \Theta_0 \Leftrightarrow \theta \sim B^1$$

for any formula θ in Θ, and we can get a proof:

$$\begin{array}{l} \vdots\ \mathcal{P}_2' \\ C, \Pi_0 \to A \end{array}$$

in LJ/ where $(C^2, \Pi_0; \Pi_1 \rightarrow A^0; \Sigma_1)$ is a (1,1)-permutation of $\mathcal{S}_2$ such that

ϕ is in $(C^2, \Pi_0) \Leftrightarrow \phi \sim A^0$

for any formula ϕ in (C^2, Π). Then we can get the required proof $\mathcal{P}'$ as:

$$\dfrac{\begin{array}{c}\vdots\, \mathcal{P}_1' \\ \Theta_0 \rightarrow B\end{array} \qquad \begin{array}{c}\vdots\, \mathcal{P}_2' \\ C, \Pi_0 \rightarrow A\end{array}}{B \supset C, \Theta_0, \Pi_0 \rightarrow A}\ (\supset\rightarrow) .$$

Case 2: $\mathcal{P}$ is of the form

$$\dfrac{\begin{array}{c}\vdots\, \mathcal{P}_1 \\ B^1, \Gamma \rightarrow \Sigma, C^2\end{array}}{\Gamma \rightarrow \Sigma, (B \supset C)^3}\ (\rightarrow\supset+) .$$

We call this upper sequent $\mathcal{S}_1$. Now, by induction hypothesis, one of the following subcases holds.

Subcase 2-1: we can get a proof:

$$\begin{array}{c}\vdots\, \mathcal{P}_1' \\ \Gamma_0 \rightarrow A\end{array}$$

in LJ/, where $(\Gamma_0; B^1, \Gamma_1 \rightarrow A^0; \Sigma_1, C^2)$ is a (1,1)-permutation of $\mathcal{S}_1$ such that

ϕ is in $\Gamma_0 \Leftrightarrow \phi \sim A^0$

for any formula ϕ in (B^1, Γ). Notice that $B^1 \not\sim A^0$. In this case, $\mathcal{P}_1'$ is the required proof $\mathcal{P}'$.

Subcase 2-2: we can get a proof:

$$\begin{array}{c}\vdots\, \mathcal{P}_1' \\ \Gamma_0 \rightarrow C\end{array}$$

in LJ/, where $(\Gamma_0; B^1, \Gamma_1 \to C^2; \Sigma)$ is a (1,1)-permutation of $\mathcal{S}_1$ such that

$$\phi \text{ is in } \Gamma_0 \Leftrightarrow \phi \sim C^2$$

for any formula ϕ in (B^1, Γ). Then we can get the required $\mathcal{P}'$ as:

$$\begin{array}{ll} & \vdots\, \mathcal{P}_1' \\ & \Gamma_0 \to C \\ (w\to) & \overline{\quad B, \Gamma_0 \to C \quad} \\ (\to\supset) & \overline{\quad \Gamma_0 \to B \supset C \quad} \end{array}\ .$$

Subcase 2-3: we can get a proof:

$$\begin{array}{c} \vdots\, \mathcal{P}_1' \\ B, \Gamma_0 \to C \end{array}$$

in LJ/, where $(B^1, \Gamma_0; \Gamma_1 \to C^2; \Sigma)$ is a (1,1)-permutation of $\mathcal{S}_1$ such that

$$\phi \text{ is in } (B^1, \Gamma_0) \Leftrightarrow \phi \sim C^2$$

for any formula ϕ in (B^1, Γ). Then we can get the required $\mathcal{P}'$ as:

$$\begin{array}{ll} & \vdots\, \mathcal{P}_1' \\ & B, \Gamma_0 \to C \\ (\to\supset) & \overline{\quad \Gamma_0 \to B \supset C \quad} \end{array}\ .$$

Case 3: $\mathcal{P}$ is of the form

$$\begin{array}{ll} & \vdots\, \mathcal{P}_1 \\ & \Gamma \to \Sigma, B(a)^1 \\ (\to\forall) & \overline{\quad \Gamma \to \Sigma, \forall x B(x)^2 \quad} \end{array}\ .$$

We call this upper sequent $\mathcal{S}_1$. Now, by induction hypothesis, one of the following subcases holds.

Subcase 3-1: we can get a proof:

$$\begin{array}{c} \vdots\ \mathcal{P}_1' \\ \Gamma_0 \to A \end{array}$$

in LJ/, where $(\Gamma_0; \Gamma_1 \to A^0; \Sigma_1, B(a)^1)$ is a (1,1)-permutation of $\mathcal{S}_1$ such that

$$\gamma \text{ is in } \Gamma_0 \Leftrightarrow \gamma \sim A^0$$

for any formula γ in Γ. In this case, $\mathcal{P}_1'$ is the required proof $\mathcal{P}'$.

Subcase 3-2: we can get a proof:

$$\begin{array}{c} \vdots\ \mathcal{P}_1' \\ \Gamma_0 \to B(a) \end{array}$$

in LJ/, where $(\Gamma_0; \Gamma_1 \to B(a)^1; \Sigma)$ is a (1,1)-permutation of $\mathcal{S}_1$ such that

$$\gamma \text{ is in } \Gamma_0 \Leftrightarrow \gamma \sim B(a)^1$$

for any formula γ in Γ. Then we can get the required $\mathcal{P}'$ as:

$$\begin{array}{c} \vdots\ \mathcal{P}_1' \\ (\to \forall_{LJ})\dfrac{\Gamma_0 \to B(a)}{\Gamma_0 \to \forall x B(x)\ .} \end{array} \qquad \Box$$

Proof of theorem 5.1 By corollary 4.5, if A is a theorem of CD, there is a cut-free proof $\mathcal{P}$ of $\to A$ in c-LD. Then, since A has no negative "$\vee$", $\mathcal{P}$ contains no $(\vee\to)$. So by the previous lemma, we can get a proof of $\to A$ in LJ/. This means A is a theorem of intuitionistic logic. $\Box$

Next, we show disjunction property and existence property for CD.

Lemma 5.3 *Let $\mathcal{P}$ be a proof of*

$$\to A_1, ..., A_n$$

in LDS. Then from $\mathcal{P}$, we can construct a proof $\mathcal{P}'$ of

$$\rightarrow A'_k$$

in LDS for some k $(1 \leq k \leq n)$, where

$$\begin{array}{ll} A'_k = B \text{ or } C & \text{if } A_k = B \vee C, \\ A'_k = D(t) \text{ for some term } t & \text{if } A_k = \exists x D(x), \\ A'_k = A_k & \text{otherwise.} \end{array}$$

Proof This lemma is proved by easy induction on the number of inference rules in $\mathcal{P}$. We only notice that: since $\mathcal{P}$ is a proof of $\rightarrow \Gamma$, $\mathcal{P}$ is not an axiom, and the last inference of $\mathcal{P}$ is $(\rightarrow w)$, $(\rightarrow c)$, $(\rightarrow e)$, $(\rightarrow\wedge)$, $(\rightarrow\vee 1)$, $(\rightarrow\vee 2)$, $(\rightarrow\supset)$, $(\rightarrow\forall)$, $(\rightarrow\exists)$ or SP.CUT. □

Theorem 5.4 (disjunction and existence properties for CD) *If a formula $A \vee B$ is a theorem of CD, then either A or B is a theorem of CD. If a formula $\exists x A(x)$ is a theorem of CD, then $A(t)$ is a theorem of CD for some term t.*

Proof This is straightforward by corollary 4.5 and lemma 5.3. □

We can prove the disjunction and existence properties more easily by a model theoretic argument. But our method has an advantage of constructiveness, i.e. we can effectively construct proofs of A or B, and of $A(t)$.

Lastly, we mention the interpolation property. The problem whether Craig's interpolation theorem holds for CD or not is open ([9]). In case of classical, intuitionistic and some other logics, interpolation theorem can be proved by using cut-elimination theorem. So we hope that our cut-elimination theorems lead to interpolation theorem for CD. But it does not come true yet.

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