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Completeness Theorem of First-Order Modal μ-calculus*

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Abstract

We prove that a natural axiom system of first-order modal μ-calculus is complete with respect to "general" models.

1 Introduction

Propositional modal μ-calculus, which is an extension of the standard propositional (multi-)modal logic **K**, has great expressive power and has been widely studied (see, e.g., [2]). The syntactical difference between modal μ-calculus and **K** is the fixed point operator "μ", which binds propositional variables. Semantics of propositional modal μ-calculus is treated through Kripke models. In each Kripke model, the set $[\![\varphi]\!]$ of possible worlds in which the formula φ is true is inductively defined as

$$[\![\neg\varphi]\!] = \mathcal{W} \setminus [\![\varphi]\!],$$

$$[\![\varphi\wedge\psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!],$$

$$[\![\Box\varphi]\!] = \{w \in \mathcal{W} \mid \forall y \in \mathcal{W}(w\mathcal{R}y \implies y \in [\![\varphi]\!])\},$$

$$[\![\mu X.\varphi(X)]\!] = \bigcap\{\alpha \in \wp(\mathcal{W}) \mid [\![\varphi(\alpha)]\!] \subseteq \alpha\},$$

where $\mathcal{W}$ is the set of possible worlds, $\mathcal{R}$ is the accessibility relation, and $\wp(\mathcal{W})$ is the power set of $\mathcal{W}$. (For simplicity, we use informal notation here.) An axiom system of propositional modal μ-calculus is obtained from that of **K** by adding the axiom and inference rule as follows:

$$\varphi(\mu X.\varphi(X)) \to \mu X.\varphi(X), \qquad \frac{\varphi(\psi) \to \psi}{\mu X.\varphi(X) \to \psi.}$$

The completeness theorem is shown in [6], which claims that a formula φ is provable in this system if and only if φ is valid in Kripke models, i.e., φ is true at any world of any Kripke model.

First-order modal μ-calculus (FOMμ, for short) is the extension of propositional modal μ-calculus with predicate symbols and first-order quantifiers. A straightforward semantics of FOMμ is Kripke models in which quantifiers are interpreted as

$$[\![\forall x.\varphi(x)]\!] = \bigcap_{d\in\mathcal{D}} [\![\varphi(d)]\!]$$

where $\mathcal{D}$ is the domain of quantification. A natural axiom system of FOMμ is obtained from the above propositional system by adding the Barcan formula ($\forall x.\Box\varphi \to \Box\forall x.\varphi$) and the usual axioms and rules for quantifiers. However, the completeness does not hold in this setting. Indeed we have

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proved that the set of valid formulas in Kripke models is not recursively enumerable (see [5]). This implies impossibility of recursive axiomatization of FOMμ that is complete with respect to Kripke models.

Then a question arises: *Find reasonable semantics to which the natural axiomatization of FOMμ is complete.* The purpose of this paper is to give an answer to this question. We introduce "general μ-model", and prove the completeness theorem: *A formula φ is provable in the system if and only if φ is valid in general μ-models.*

The difference between usual Kripke models and general μ-models appears in the definition of $[\![\mu X.\varphi(X)]\!]$. General μ-models have the definition

$$[\![\mu X.\varphi(X)]\!] = \bigcap\{\alpha \in \mathcal{Q} \mid [\![\varphi(\alpha)]\!] \subseteq \alpha\}$$

where $\mathcal{Q}$ is a fixed subset of $\wp(\mathcal{W})$. When $\mathcal{Q} = \wp(\mathcal{W})$, this is equivalent to the definition in usual Kripke models; therefore, a usual Kripke model is a special case of general μ-models. It is interesting that FOMμ and the classical second-order predicate logic (see, e.g., Ch.22 of [1] and Ch.4 of [3]) are in the same situation: the natural axiomatization is complete with respect to general models, while the set of valid formulas in usual models (special cases of general models) is not recursively enumerable.

In Section 2, we give the definitions concerning formulas. In Section 3, we introduce general μ-models and we show some basic properties on them. In Section 4, we axiomatize FOMμ in the style of sequent calculus and we prove the soundness. In Section 5, we construct the "canonical general μ-model" and we prove the completeness. From a technical point of view, "$\star$-consistency", which is introduced in Section 5 is the most important device in this paper. While the usual canonical models for standard modal logics (e.g., **K**) consist of maximally consistent sets (see, e.g., [4]), the canonical general μ-model consists of maximally $\star$-consistent sets. This enable us to carry out the proof of Main Lemma 5.5 by simple induction on the length of formulas.

2 Formulas

We define a language $\mathcal{L}$ of FOMμ.

$\mathcal{L}$ consists of the following symbols: countably many *individual parameters* (denoted by $a, b, \ldots$), countably many *individual variables* (denoted by $x, y, \ldots$), finite or countably many *predicate symbols* (denoted by $P, Q, \ldots$), countably many *propositional variables* (denoted by $X, Y, \ldots$), and logical symbols $\neg, \wedge, \Box, \forall, \mu$. The set of individual parameters (individual variables, or propositional variables) is called IndPar (IndVar, or $\mathsf{PropVar}$, respectively).

Pseudo-formulas (denoted by $\varphi, \psi, \ldots$) are constructed inductively as follows. A propositional variable is a pseudo-formula. If P is an n-ary predicate symbol and $\{t_1, t_2, \ldots, t_n\} \subseteq \mathsf{IndPar} \cup \mathsf{IndVar}$, then $P(t_1, \ldots, t_n)$ is a pseudo-formula. If φ and ψ are pseudo-formulas and $x \in \mathsf{IndVar}$, then $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\Box\varphi)$, and $(\forall x.\varphi)$ are pseudo-formulas. If φ is a pseudo-formula, $X \in \mathsf{PropVar}$, and every free occurrence of X in φ occurs positively, i.e., within the scope of an even number of negations, then $(\mu X.\varphi)$ is a pseudo-formula.

Note that the notions of free and bound occurrences of variables are as usual; for example, in the pseudo-formula

$$\big(\forall x.\mu X.\big(P(x, \underline{y}) \wedge \Box(X \wedge \underline{Y})\big)\big) \wedge Q(\underline{y}) \wedge \neg\underline{X},$$

underlined occurrences are the only free occurrences of variables. Note also that individual parameters are not bound. FPV(φ) denotes the set of propositional variables that occur free in φ.

If a pseudo-formula φ contains no free occurrences of individual variables, then φ is called a *formula*. If a formula φ contains no free occurrences of propositional variables, then φ is called a *pure-formula*. For example

- $P(a, x) \wedge \Box X$ is a pseudo-formula, but neither a formula nor a pure-formula;
- $\forall x.(P(a, x) \wedge \Box X)$ is a pseudo-formula and also a formula, but not a pure-formula;
- $\mu X.\forall x.(P(a, x) \wedge \Box X)$ is a pseudo-formula, a formula, and also a pure-formula;

where P is a binary predicate symbol, $a \in \mathsf{IndPar}$, $x \in \mathsf{IndVar}$, and $X \in \mathsf{PropVar}$.

The expression $[\alpha/\chi]$ represents the substitution to replace the free occurrences of χ by α. For example:

$$\big(P(a,x) \wedge \Box X\big)\ [b/x]\ [(Q(a)\wedge X)/X] \ = \ P(a,b) \wedge \Box(Q(a)\wedge X).$$

$$\big(\forall x.(P(a,x) \wedge \Box X)\big)\ [b/x]\ [(Q(a)\wedge X)/X] \ = \ \forall x.(P(a,x) \wedge \Box(Q(a)\wedge X)).$$

$$\big(\mu X.\forall x.(P(a,x) \wedge \Box X)\big)\ [b/x]\ [(Q(a)\wedge X)/X] \ = \ \mu X.\forall x.(P(a,x) \wedge \Box X).$$

Let φ be a pseudo-formula, ψ be a formula, and X be a propositional variable. We say that the substitution $[\psi/X]$ *causes new binding* for φ if there is a propositional variable Y in $\mathrm{FPV}(\psi)$ such that some free occurrence of X is within the scope of μY in φ. For example, if $\varphi = \big(\mu Y.\forall x.\big(P(a,x) \wedge \Box(X\wedge Y)\big)\big) \wedge \neg X$, then the substitution $[(\neg Y)/X]$ causes new binding for φ because

$$\varphi[(\neg Y)/X] \ = \ \big(\mu Y.\forall x.\big(P(a,x) \wedge \Box((\neg \underline{Y})\wedge Y)\big)\big) \wedge \neg\neg Y$$

and the underlined Y is newly bound. A substitution is said to be *safe* for a pseudo-formula φ if it does not causes new binding for φ.

A pseudo-formula φ' is called an *α-variant* of φ if φ' is obtained from φ by renaming a bound propositional variable; that is to say, φ' is obtained from φ by replacing a sub-pseudo-formula $\mu X.\psi$ by $\mu Y.(\psi[Y/X])$ provided that Y does not occur free in ψ and that $[Y/X]$ is safe for ψ. Note that if φ' is an α-variant of φ, then φ is an α-variant of φ' because φ is obtained from φ' by replacing $\mu Y.(\psi[Y/X])$ by $\mu X.(\psi[Y/X][X/Y])$.

To describe pseudo-formulas, we use the usual abbreviations: $(\varphi\to\psi) = \neg(\varphi\wedge\neg\psi)$, and $\bot = \rho\wedge\neg\rho$ where ρ is a fixed pure-formula.

3 General μ-model

A *general μ-structure* is a 5-tuple $\langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{Q}, \mathcal{I}\rangle$ as follows.

- $\mathcal{W}$ is a nonempty set. (The set of possible worlds.)
- $\mathcal{R}$ is a binary relation on $\mathcal{W}$. (The accessibility relation.)
- $\mathcal{D}$ is a nonempty set. (The domain for individual parameters and individual variables.)
- $\mathcal{Q}$ is a subset of $\wp(\mathcal{W})$. (The domain for propositional variables.)
- $\mathcal{I}$ is an interpretation of individual parameters and predicate symbols. If a is an individual parameter, then $\mathcal{I}(a) \in \mathcal{D}$. If P is an n-ary predicate symbol and $w \in \mathcal{W}$, then $\mathcal{I}(P,w)$ is an n-ary predicate on $\mathcal{D}$.

Let $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{Q}, \mathcal{I}\rangle$ be a general μ-structure. A *valuation* on $\mathcal{S}$ is a function from $\mathsf{PropVar}$ to $\mathcal{Q}$. For a valuation $\mathcal{V}$, a propositional variable Z and a set $\alpha \in \mathcal{Q}$, the valuation $\mathcal{V}[\alpha/Z]$ is defined by

$$\mathcal{V}[\alpha/Z](X) = \begin{cases} \alpha & (\text{if } X = Z) \\ \mathcal{V}(X) & (\text{if } X \neq Z) \end{cases}$$

We extend the language $\mathcal{L}$ by adding the *names* of each elements of $\mathcal{D}$ to the set of individual parameters. The extended language is called $\mathcal{L}(\mathcal{D})$. The name of d is written as $\dot{d}$, and the interpretation of names is defined by $\mathcal{I}(\dot{d}) = d$. For a formula φ in $\mathcal{L}(\mathcal{D})$ and a valuation $\mathcal{V}$, we define $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}}$ $(\in \wp(\mathcal{W}))$ inductively as follows.

- $[\![X]\!]^{\mathcal{S}}_{\mathcal{V}} = \mathcal{V}(X)$.
- $[\![P(t_1,\ldots,t_n)]\!]^{\mathcal{S}}_{\mathcal{V}} = \{w \in \mathcal{W} \mid \mathcal{I}(P,w)(\mathcal{I}(t_1),\ldots,\mathcal{I}(t_n)) = \mathsf{true}\}$.
- $[\![\neg\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} = \mathcal{W} \setminus [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}}$.

- $[\![\varphi \wedge \psi]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} \cap [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}}$.
- $[\![\Box\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} = \{w \in \mathcal{W} \mid \forall y \in \mathcal{W}(w\mathcal{R}y \implies y \in [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}})\}$.
- $[\![\forall x.\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} = \bigcap_{d\in\mathcal{D}} [\![\varphi[\dot{d}/x]]\!]^{\mathcal{S}}_{\mathcal{V}}$.
- $[\![\mu X.\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} = \bigcap\{\alpha \in \mathcal{Q} \mid [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \subseteq \alpha\}$.

A 6-tuple $\langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{Q}, \mathcal{I}, \mathcal{V}\rangle$ is called a *general μ-model* if:

- $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{Q}, \mathcal{I}\rangle$ is a general μ-structure; and
- $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} \in \mathcal{Q}$ for any formula φ in $\mathcal{L}$.

Let φ be a formula in $\mathcal{L}$. If $[\![\varphi]\!]^{\langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{Q}, \mathcal{I}\rangle}_{\mathcal{V}} = \mathcal{W}$ for any general μ-model $\langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{Q}, \mathcal{I}, \mathcal{V}\rangle$, then φ is said to be *valid*.

From now on, $\langle \mathcal{S}, \mathcal{V}\rangle$ represents an arbitrary general μ-model where $\mathcal{S} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{Q}, \mathcal{I}\rangle$.

Lemma 3.1 (Properties of substitution) (1) *Let φ be a pseudo-formula in $\mathcal{L}(\mathcal{D})$, x be an individual variable, and t_i be an individual parameter in $\mathcal{L}(\mathcal{D})$ (i.e., $t_i \in$ IndPar or t_i is a name). If $\varphi[t_i/x]$ is a formula in $\mathcal{L}(\mathcal{D})$ and $\mathcal{I}(t_1) = \mathcal{I}(t_2)$, then $[\![\varphi[t_1/x]]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![\varphi[t_2/x]]\!]^{\mathcal{S}}_{\mathcal{V}}$.*

(2) *Let φ be a formula in $\mathcal{L}(\mathcal{D})$ and X be a propositional variable. If $X \notin \mathrm{FPV}(\varphi)$, then $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]}$ for any $\alpha \in \mathcal{Q}$.*

(3) *Let φ and ψ be formulas in $\mathcal{L}(\mathcal{D})$, and $\alpha \in \mathcal{Q}$. If $[\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}} = \alpha$ and the substitution $[\psi/X]$ is safe for φ, then $[\![\varphi[\psi/X]]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]}$.*

Proof

(1) By induction on the length of φ. If $\varphi = \forall y.\psi$ and $x \neq y$, we have

$$\begin{aligned}
[\![(\forall y.\psi)[t_1/x]]\!]^{\mathcal{S}}_{\mathcal{V}} &= [\![\forall y.(\psi[t_1/x])]\!]^{\mathcal{S}}_{\mathcal{V}} = \bigcap_{d\in\mathcal{D}} [\![\psi[t_1/x][\dot{d}/y]]\!]^{\mathcal{S}}_{\mathcal{V}} = \bigcap_{d\in\mathcal{D}} [\![\psi[\dot{d}/y][t_1/x]]\!]^{\mathcal{S}}_{\mathcal{V}} \\
&= \bigcap_{d\in\mathcal{D}} [\![\psi[\dot{d}/y][t_2/x]]\!]^{\mathcal{S}}_{\mathcal{V}} \qquad \text{(by induction hypotheses)} \\
&= \bigcap_{d\in D} [\![\psi[t_2/x][\dot{d}/y]]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![\forall y.(\psi[t_2/x])]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![(\forall y.\psi)[t_2/x]]\!]^{\mathcal{S}}_{\mathcal{V}}.
\end{aligned}$$

The other cases are similar.

(2) By induction on the length of φ. If $\varphi = \mu Y.\psi$ and $X \neq Y$, we have

$$\begin{aligned}
[\![\mu Y.\psi]\!]^{\mathcal{S}}_{\mathcal{V}} &= \bigcap\{\beta \in \mathcal{Q} \mid [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/Y]} \subseteq \beta\} \\
&= \bigcap\{\beta \in \mathcal{Q} \mid [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/Y][\alpha/X]} \subseteq \beta\} \qquad \text{(by induction hypotheses)} \\
&= \bigcap\{\beta \in \mathcal{Q} \mid [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X][\beta/Y]} \subseteq \beta\} = [\![\mu Y.\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]}.
\end{aligned}$$

The other cases are similar.

(3) By induction on the length of φ. Suppose $\varphi = \mu Y.\rho$ and $X \neq Y$. If $X \notin \mathrm{FPV}(\rho)$, we have

$$\begin{aligned}
[\![(\mu Y.\rho)[\psi/X]]\!]^{\mathcal{S}}_{\mathcal{V}} &= [\![\mu Y.\rho]\!]^{\mathcal{S}}_{\mathcal{V}} \\
&= [\![\mu Y.\rho]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \qquad \text{(by (2))}.
\end{aligned}$$

If $X \in \mathrm{FPV}(\rho)$, then $Y \notin \mathrm{FPV}(\psi)$ because $[\psi/X]$ is safe for $\mu Y.\rho$. Using (2), we have

$$\alpha = [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/Y]} \tag{†}$$

for any $\beta \in \mathcal{Q}$, and then

$$\begin{aligned}
[\![(\mu Y.\rho)[\psi/X]]\!]^{\mathcal{S}}_{\mathcal{V}} &= [\![\mu Y.(\rho[\psi/X])]\!]^{\mathcal{S}}_{\mathcal{V}} = \bigcap\{\beta \in \mathcal{Q} \mid [\![\rho[\psi/X]]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/Y]} \subseteq \beta\} \\
&= \bigcap\{\beta \in \mathcal{Q} \mid [\![\rho]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/Y][\alpha/X]} \subseteq \beta\} \qquad \text{(by induction hypotheses and (†))} \\
&= \bigcap\{\beta \in \mathcal{Q} \mid [\![\rho]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X][\beta/Y]} \subseteq \beta\} = [\![\mu Y.\rho]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]}
\end{aligned}$$

The other cases are similar. **QED**

Lemma 3.2 (Property of α-variant) *If a formula φ' in $\mathcal{L}(\mathcal{D})$ is an α-variant of φ, then $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![\varphi']\!]^{\mathcal{S}}_{\mathcal{V}}$.*

Proof It is enough to show $[\![\mu X.\psi]\!]^{\mathcal{S}}_{\mathcal{V}} = [\![\mu Y.(\psi[Y/X])]\!]^{\mathcal{S}}_{\mathcal{V}}$ provided that Y does not occur free in ψ and that $[Y/X]$ is safe for ψ.

$$\begin{array}{rcll}
[\![\mu X.\psi]\!]^{\mathcal{S}}_{\mathcal{V}} & = & \bigcap\{\alpha \in \mathcal{Q} \mid [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \subseteq \alpha\} & \\
& = & \bigcap\{\alpha \in \mathcal{Q} \mid [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X][\alpha/Y]} \subseteq \alpha\} & \text{(by Lemma 3.1 (2))} \\
& = & \bigcap\{\alpha \in \mathcal{Q} \mid [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/Y][\alpha/X]} \subseteq \alpha\} & \\
& = & \bigcap\{\alpha \in \mathcal{Q} \mid [\![\psi[Y/X]]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/Y]} \subseteq \alpha\} & \text{(by Lemma 3.1 (3))} \\
& = & [\![\mu Y.(\psi[Y/X])]\!]^{\mathcal{S}}_{\mathcal{V}}. &
\end{array}$$

QED

Lemma 3.3 (Monotonicity) *Let φ be a formula in $\mathcal{L}(\mathcal{D})$, and X be a propositional variable. Suppose that two sets $\alpha, \beta \in Q$ satisfy $\alpha \subseteq \beta$. If every free occurrence of X in φ is positive, i.e., within the scope of an even number of negations, then $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \subseteq [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X]}$. If every free occurrence of X in φ is negative, i.e., within the scope of an odd number of negations, then $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \supseteq [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X]}$.*

Proof By induction on the length of φ. We consider the case: $\varphi = \mu Y.\psi$, $X \neq Y$, and X is positive in φ. By the induction hypotheses, we have

$$[\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\gamma/Y][\alpha/X]} \subseteq [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\gamma/Y][\beta/X]}$$

for any $\gamma \in Q$. Then,

$$[\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X][\gamma/Y]} \subseteq [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X][\gamma/Y]},$$

$$\bigcap\{\gamma \in Q \mid [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X][\gamma/Y]} \subseteq \gamma\} \subseteq \bigcap\{\gamma \in Q \mid [\![\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X][\gamma/Y]} \subseteq \gamma\},$$

$$[\![\mu Y.\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \subseteq [\![\mu Y.\psi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X]}.$$

The other cases are similar. **QED**

Theorem 3.4 (Least fixed point) *Let $\mu X.\varphi$ be a formula in $\mathcal{L}(\mathcal{D})$. If $[\![\mu X.\varphi]\!]^{\mathcal{S}}_{\mathcal{V}} = \alpha$, then the following two conditions hold:*

(1) $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} = \alpha$.

(2) *If $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X]} = \beta$, then $\alpha \subseteq \beta$, where β is an arbitrary element of $\mathcal{Q}$.*

In other words, $[\![\mu X.\varphi]\!]^{\mathcal{S}}_{\mathcal{V}}$ is the least fixed point for $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\cdot/X]}$ within $\mathcal{Q}$.

Proof Let R be the set $\{\beta \in \mathcal{Q} \mid [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X]} \subseteq \beta\}$. Since $\alpha = \bigcap R$, we have the following:

$$\alpha \subseteq \beta, \text{ for any } \beta \in R. \tag{a}$$

If $\gamma \subseteq \beta$ for any $\beta \in R$, then $\gamma \subseteq \alpha$, where γ is an arbitrary element of $\wp(\mathcal{W})$. (b)

Using these, we will show (1-1) $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \subseteq \alpha$, (1-2) $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \supseteq \alpha$, and (2) above.

Proof of (1-1). Let β be an arbitrary element of R.

$$\begin{array}{rcll}
[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} & \subseteq & [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X]} & \text{(by (a) and Lemma 3.3)} \\
& \subseteq & \beta & \text{(by definition of } R\text{)}
\end{array}$$

Therefore, we have $[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \subseteq \alpha$ by (b).

Proof of (1-2).

$$[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]}/X]} \subseteq [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \quad \text{(by (1-1) and Lemma 3.3)}, \tag{c}$$

$$[\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \in R \quad \text{(by (c) and definition of } R), \tag{d}$$

$$\alpha \subseteq [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\alpha/X]} \quad \text{(by (a) and (d))}.$$

Proof of (2).

$$\begin{aligned} [\![\varphi]\!]^{\mathcal{S}}_{\mathcal{V}[\beta/X]} = \beta \text{ and } \beta \in \mathcal{Q} \quad &\Longrightarrow \quad \beta \in R \quad \text{(by definition of } R) \\ &\Longrightarrow \quad \alpha \subseteq \beta \quad \text{(by (a))} \end{aligned}$$

QED

4 Sequent calculus

We axiomatize FOMμ in the style of sequent calculus. (Sequent calculi for standard modal logics **K**, **S4**, **S5**,... are found, e.g., in [4].) A *sequent* is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas in $\mathcal{L}$. We describe the initial sequents and inference rules as follows, in which $a \in \mathsf{IndPar}$, $x \in \mathsf{IndVar}$, $X \in \mathsf{PropVar}$, φ, φ', ψ are formulas in $\mathcal{L}$, ρ is a pseudo-formula in $\mathcal{L}$, and $\Gamma, \Delta, \Pi, \Sigma$ are finite sets of formulas in $\mathcal{L}$; and as usual, e.g., $(\Rightarrow \Gamma, \varphi)$ represents the sequent$(\{\} \Rightarrow \Gamma \cup \{\varphi\})$, and $\Box\Gamma$ represents the set $\{\Box\varphi \mid \varphi \in \Gamma\}$.

Initial sequents: $\varphi \Rightarrow \varphi$

Inference rules:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (Cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{ (Weakening)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \text{ (Weakening)}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \text{ (}\neg\text{ Left)} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \text{ (}\neg\text{ Right)}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi\wedge\psi, \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{ Left)} \qquad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi\wedge\psi, \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{ Left)} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\wedge\psi} \text{ (}\wedge\text{ Right)}$$

$$\frac{\rho[a/x], \Gamma \Rightarrow \Delta}{\forall x.\rho, \Gamma \Rightarrow \Delta} \text{ (}\forall\text{ Left)}$$

$$\frac{\Gamma \Rightarrow \Delta, \rho[a/x]}{\Gamma \Rightarrow \Delta, \forall x.\rho} \text{ (}\forall\text{ Right) provided that } a \text{ does not occur in the lower sequent.}$$

$$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \text{ (}\Box\text{)}$$

$$\frac{\varphi[\psi/X] \Rightarrow \psi}{\mu X.\varphi \Rightarrow \psi} \text{ (}\mu\text{ Left) provided that } [\psi/X] \text{ is safe for } \varphi.$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi[(\mu X.\varphi)/X]}{\Gamma \Rightarrow \Delta, \mu X.\varphi} \text{ (}\mu\text{ Right) provided that } [(\mu X.\varphi)/X] \text{ is safe for } \varphi.$$

$$\frac{\Gamma \Rightarrow \Delta, \forall x.\Box\rho}{\Gamma \Rightarrow \Delta, \Box\forall x.\rho} \text{ (Barcan)}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi', \Gamma \Rightarrow \Delta} \text{ (}\alpha\text{-variant) provided that } \varphi' \text{ is an } \alpha\text{-variant of } \varphi.$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi'} \text{ (}\alpha\text{-variant) provided that } \varphi' \text{ is an } \alpha\text{-variant of } \varphi.$$

We call this system Seq_μ, and we write

$$\mathrm{Seq}_\mu \vdash (\Gamma \Rightarrow \Delta)$$

if there is a proof of the sequent $\Gamma \Rightarrow \Delta$ in Seq_μ.

Example of a proof in Seq_μ.

$$\cfrac{\cfrac{\Box\mu Y.\Box Y \Rightarrow \Box\mu Y.\Box Y}{\Box\mu Y.\Box Y \Rightarrow \mu Y.\Box Y}\ (\mu\ \text{Right})\ \because \Box\mu Y.\Box Y = (\Box Y)[(\mu Y.\Box Y)/Y].}{\mu X.\Box X \Rightarrow \mu Y.\Box Y}\ (\mu\ \text{Left})\ \because \Box\mu Y.\Box Y = (\Box X)[(\mu Y.\Box Y)/X].$$

From now on, the letters $\Gamma, \Delta, \ldots$ may denote infinite sets of formulas in $\mathcal{L}$. We write

$$\Gamma \vdash \varphi$$

if there exists a finite subset Γ' of Γ such that $\mathrm{Seq}_\mu \vdash (\Gamma' \Rightarrow \varphi)$. Note that "$\vdash \varphi$" is equivalent to "$\mathrm{Seq}_\mu \vdash (\Rightarrow \varphi)$".

Lemma 4.1 *If* $\mathrm{Seq}_\mu \vdash (\psi_1, \ldots, \psi_m \Rightarrow \varphi_1, \ldots, \varphi_n)$*, then*

$$\left(\llbracket\psi_1\rrbracket^{\mathcal{S}}_{\mathcal{V}} \cap \cdots \cap \llbracket\psi_m\rrbracket^{\mathcal{S}}_{\mathcal{V}}\right) \subseteq \left(\llbracket\varphi_1\rrbracket^{\mathcal{S}}_{\mathcal{V}} \cup \cdots \cup \llbracket\varphi_n\rrbracket^{\mathcal{S}}_{\mathcal{V}}\right)$$

holds for any general μ-model $\langle \mathcal{S}, \mathcal{V}\rangle$ $(= \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{Q}, \mathcal{I}, \mathcal{V}\rangle)$, where the left-hand side is $\mathcal{W}$ if $m = 0$, and the right-hand side is empty if $n = 0$.

Proof By induction on the length of the proof of $(\psi_1, \ldots, \psi_m \Rightarrow \varphi_1, \ldots, \varphi_n)$ in Seq_μ. We divide cases according to the last inference rule of the proof, and here we show two cases (the other cases are similar).

(Case 1) The proof is of the form

$$\cfrac{\begin{array}{c}\vdots\\ \varphi[\psi/X] \Rightarrow \psi\end{array}}{\mu X.\varphi \Rightarrow \psi.}\ (\mu\ \text{Left})$$

Then,

$$\llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{V}[\llbracket\psi\rrbracket^{\mathcal{S}}_{\mathcal{V}}/X]} = \llbracket\varphi[\psi/X]\rrbracket^{\mathcal{S}}_{\mathcal{V}} \subseteq \llbracket\psi\rrbracket^{\mathcal{S}}_{\mathcal{V}} \quad \text{(by Lemma 3.1(3) and induction hypothesis)}, \qquad \text{(a)}$$

$$\llbracket\mu X.\varphi\rrbracket^{\mathcal{S}}_{\mathcal{V}} \subseteq \llbracket\psi\rrbracket^{\mathcal{S}}_{\mathcal{V}} \quad \text{(by (a) and the definition of } \llbracket\mu X.\varphi\rrbracket^{\mathcal{S}}_{\mathcal{V}}).$$

Note that the fact $\llbracket\psi\rrbracket^{\mathcal{S}}_{\mathcal{V}} \in Q$ is guaranteed by the definition of general μ-model.

(Case 2) The proof is of the form

$$\cfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi[(\mu X.\varphi)/X]\end{array}}{\Gamma \Rightarrow \Delta, \mu X.\varphi.}\ (\mu\ \text{Right})$$

We have

$$\llbracket\varphi[(\mu X.\varphi)/X]\rrbracket^{\mathcal{S}}_{\mathcal{V}} = \llbracket\varphi\rrbracket^{\mathcal{S}}_{\mathcal{V}[\llbracket\mu X.\varphi\rrbracket^{\mathcal{S}}_{\mathcal{V}}/X]} = \llbracket\mu X.\varphi\rrbracket^{\mathcal{S}}_{\mathcal{V}} \quad \text{(by Lemma 3.1(3) and Theorem 3.4)}.$$

Then the claim of this lemma is shown by the induction hypothesis. **QED**

We show the soundness of the axiom system of FOMμ.

Theorem 4.2 (Soundness) *If $\vdash \varphi$, then φ is valid, where φ is an arbitrary formula in $\mathcal{L}$.*

Proof By Lemma 4.1 ($m = 0$ and $n = 1$). **QED**

5 Canonical μ-model

The completeness is proved by using the *canonical μ-model.* For this argument, we fix an enumeration of pure-formulas and an enumeration of propositional variables in $\mathcal{L}$. The n-th pure-formula and n-th propositional variable are denoted respectively by $\mathbb{F}_n$ and $\mathbb{X}_n$. In other words, $\{\mathbb{F}_1, \mathbb{F}_2, \ldots\}$ is the set of pure-formulas in $\mathcal{L}$ and $\{\mathbb{X}_1, \mathbb{X}_2, \ldots\} = \mathsf{PropVar}$ such that $\mathbb{F}_i \neq \mathbb{F}_j$ and $\mathbb{X}_i \neq \mathbb{X}_j$ if $i \neq j$.

Given a pseudo-formula φ in $\mathcal{L}$, we define $\varphi^\star$ as follows:

$$\varphi^\star = \varphi[\mathbb{F}_{n_1}/\mathbb{X}_{n_1}][\mathbb{F}_{n_2}/\mathbb{X}_{n_2}]\cdots[\mathbb{F}_{n_k}/\mathbb{X}_{n_k}]$$

where $\{\mathbb{X}_{n_1}, \mathbb{X}_{n_2}, \ldots, \mathbb{X}_{n_k}\} = \mathrm{FPV}(\varphi)$. Since $\mathbb{F}_{n_1}, \mathbb{F}_{n_2}, \ldots, \mathbb{F}_{n_k}$ are pure-formulas, $\varphi^\star$ satisfies the following conditions:

- The definition of $\varphi^\star$ is independent of the order of the substitutions $[\mathbb{F}_{n_1}/\mathbb{X}_{n_1}], \ldots, [\mathbb{F}_{n_k}/\mathbb{X}_{n_k}]$.
- The substitution $[\mathbb{F}_i/\mathbb{X}_i]$ does not cause new binding of any variables.
- If φ is a formula, then $\varphi^\star$ is a pure-formula.

For a set Γ of formulas, $\Gamma^\star$ denotes the set $\{\varphi^\star \mid \varphi \in \Gamma\}$.

We introduce some notions about sets of formulas. In the following, Γ denotes a set of formulas in $\mathcal{L}$.

- Γ is *$\star$-inconsistent* $\overset{\text{def}}{\Longleftrightarrow}$ $\Gamma^\star \vdash \bot$.
- Γ is *$\star$-consistent* $\overset{\text{def}}{\Longleftrightarrow}$ Γ is not $\star$-inconsistent, i.e., no finite $\Delta \subseteq \Gamma^\star$ satisfies $\mathrm{Seq}_\mu \vdash (\Delta \Rightarrow \bot)$.
- Γ is *maximal* $\overset{\text{def}}{\Longleftrightarrow}$ $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$ holds for any formula φ in $\mathcal{L}$.
- Γ has *$\forall$-property* $\overset{\text{def}}{\Longleftrightarrow}$ Γ satisfies the following condition for any formula $\forall x.\varphi$ in $\mathcal{L}$:

  If $\varphi[a/x] \in \Gamma$ for any $a \in \mathsf{IndPar}$, then $(\forall x.\varphi) \in \Gamma$.
- Γ has *pure-$\forall$-$\star$-derivability* $\overset{\text{def}}{\Longleftrightarrow}$ Γ satisfies the following condition for any pure-formula $\forall x.\varphi$ in $\mathcal{L}$:

  If $\Gamma^\star \vdash \varphi[a/x]$ for any $a \in \mathsf{IndPar}$, then $\Gamma^\star \vdash \forall x.\varphi$.

Lemma 5.1 (Properties of $\star$-consistency) (1) *If a set Γ of formulas in $\mathcal{L}$ is $\star$-consistent, then at least one of the sets $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ is $\star$-consistent, for any formula φ in $\mathcal{L}$.*

(2) *If a finite set $\Gamma \cup \{\neg\forall x.\varphi\}$ of formulas in $\mathcal{L}$ is $\star$-consistent, then there exists an individual parameter a such that the set $\Gamma \cup \{\neg\forall x.\varphi, \neg\varphi[a/x]\}$ is $\star$-consistent.*

Proof (1) Easy.
(2) We take an individual parameter a that does not occur in $\Gamma^\star \cup \{(\neg\forall x.\varphi)^\star\}$. (Such an individual parameter exists because of the infinity of IndPar.) If $\mathrm{Seq}_\mu \vdash (\Gamma^\star, (\neg\forall x.\varphi)^\star, (\neg\varphi[a/x])^\star \Rightarrow \bot)$, then $\mathrm{Seq}_\mu \vdash (\Gamma^\star, (\neg\forall x.\varphi)^\star, \Rightarrow \bot)$ using the inference rules ($\forall$ Right) and others. Note that $(\neg\varphi[a/x])^\star = \neg(\varphi^\star[a/x])$ and $(\neg\forall x.\varphi)^\star = \neg\forall x.(\varphi^\star)$. **QED**

Lemma 5.2 (Properties of maximal $\star$-consistency) *If a set Γ of formulas in $\mathcal{L}$ is $\star$-consistent and maximal, then the following conditions hold for any formulas $\neg\varphi$, $\varphi\wedge\psi$, $\forall x.\varphi$ in $\mathcal{L}$, and any pure-formula ρ in $\mathcal{L}$.*

(1) *$(\neg\varphi) \in \Gamma$ if and only if $\varphi \notin \Gamma$.*

(2) *$(\varphi\wedge\psi) \in \Gamma$ if and only if ($\varphi \in \Gamma$ and $\psi \in \Gamma$).*

(3) *If $(\forall x.\varphi) \in \Gamma$, then $(\varphi[a/x]) \in \Gamma$ for any $a \in \mathsf{IndPar}$.*

(4) *$\rho \in \Gamma$ if and only if $\Gamma^\star \vdash \rho$.*

Proof Easy. Here we show only the if-part of (4).

$$\begin{aligned}\rho \notin \Gamma \quad &\Longrightarrow\ (\neg\rho) \in \Gamma \quad (\because \Gamma \text{ is maximal})\\ &\Longrightarrow\ (\neg\rho)^\star = (\neg\rho) \in \Gamma^\star \quad (\because \rho \text{ is a pure-formula})\\ &\Longrightarrow\ \Gamma^\star \vdash \neg\rho \ \Longrightarrow\ \Gamma^\star \nvdash \rho \quad (\because \Gamma \text{ is } \star\text{-consistent})\end{aligned}$$

QED

Lemma 5.3 (Properties of pure-$\forall$-$\star$-derivability) (1) *If a set Γ of formulas in $\mathcal{L}$ has pure-$\forall$-$\star$-derivability, then the set $\Gamma \cup \{\varphi\}$ also has pure-$\forall$-$\star$-derivability, for any formula φ in $\mathcal{L}$.*

(2) *If a set $\Gamma \cup \{\neg\forall x.\varphi\}$ of formulas in $\mathcal{L}$ is $\star$-consistent and has pure-$\forall$-$\star$-derivability, then there exists an individual parameter a such that the set $\Gamma \cup \{\neg\forall x.\varphi,\ \neg\varphi[a/x]\}$ is $\star$-consistent.*

(3) *If a set Γ of formulas in $\mathcal{L}$ is $\star$-consistent, maximal and having $\forall$-property, then the set $\{\varphi \mid \Box\varphi \in \Gamma\}$ has pure-$\forall$-$\star$-derivability.*

Proof (1) Let $\forall x.\psi$ be an arbitrary pure-formula.

$$\Gamma^\star \cup \{\varphi^\star\} \vdash \psi[a/x] \text{ for any } a \in \mathsf{IndPar}$$

$$\Longrightarrow\ \Gamma^\star \vdash (\varphi^\star \to \psi)[a/x] \text{ for any } a \in \mathsf{IndPar}$$

$$\Longrightarrow\ \Gamma^\star \vdash \forall x.(\varphi^\star \to \psi) \quad (\text{by pure-}\forall\text{-}\star\text{-derivability of } \Gamma)$$

$$\Longrightarrow\ \Gamma^\star \vdash \varphi^\star \to \forall x.\psi \quad (\because x \text{ does not occur free in } \varphi^\star)$$

$$\Longrightarrow\ \Gamma^\star \cup \{\varphi^\star\} \vdash \forall x.\psi.$$

(2) Suppose $\Gamma \cup \{\neg\forall x.\varphi\}$ has pure-$\forall$-$\star$-derivability.

$$\Gamma \cup \{\neg\forall x.\varphi, \neg\varphi[a/x]\} \text{ is } \star\text{-inconsistent for any } a \in \mathsf{IndPar}$$

$$\Longrightarrow\ \Gamma^\star \cup \{\neg\forall x.(\varphi^\star)\} \vdash \varphi^\star[a/x] \text{ for any } a \in \mathsf{IndPar}$$

$$\Longrightarrow\ \Gamma^\star \cup \{(\neg\forall x.\varphi)^\star\} \vdash \forall x.(\varphi^\star) \quad (\text{by pure-}\forall\text{-}\star\text{-derivability of } \Gamma \cup \{\neg\forall x.\varphi\})$$

$$\Longrightarrow\ \Gamma \cup \{\neg\forall x.\varphi\} \text{ is } \star\text{-inconsistent.}$$

(3) Let $\forall x.\psi$ be an arbitrary pure-formula.

$$\{\varphi \mid \Box\varphi \in \Gamma\}^\star \vdash \psi[a/x] \text{ for any } a \in \mathsf{IndPar}$$

$$\Longrightarrow\ \Gamma^\star \vdash \Box\psi[a/x] \text{ for any } a \in \mathsf{IndPar} \quad (\text{by the rule } (\Box))$$

$$\Longrightarrow\ (\Box\psi[a/x]) \in \Gamma \text{ for any } a \in \mathsf{IndPar} \quad (\text{by Lemma 5.2(4)})$$

$$\Longrightarrow\ (\forall x.\Box\psi) \in \Gamma \quad (\text{by } \forall\text{-property of } \Gamma)$$

$$\Longrightarrow\ \Gamma^\star \vdash \forall x.\Box\psi \quad (\text{by Lemma 5.2(4)})$$

$$\Longrightarrow\ \Gamma^\star \vdash \Box\forall x.\psi \quad (\text{by the rule (Barcan)})$$

$$\Longrightarrow\ (\Box\forall x.\psi) \in \Gamma \quad (\text{by Lemma 5.2(4)})$$

$$\Longrightarrow\ (\forall x.\psi) \in \{\varphi \mid \Box\varphi \in \Gamma\}^\star$$

$$\Longrightarrow\ \{\varphi \mid \Box\varphi \in \Gamma\}^\star \vdash \forall x.\psi.$$

QED

Lemma 5.4 (Extension of $\star$-consistent sets) (1) *If a set Γ of formulas in $\mathcal{L}$ is finite and $\star$-consistent, then there exists a set Δ of formulas in $\mathcal{L}$ such that:*

$$\Gamma \subseteq \Delta, \text{ and } \Delta \text{ is } \star\text{-consistent, maximal, and having } \forall\text{-property.} \qquad (\dagger)$$

(2) *If a set Γ of formulas in $\mathcal{L}$ has pure-$\forall$-$\star$-derivability and is $\star$-consistent, then there exists a set Δ of formulas in $\mathcal{L}$ such that the above condition ($\dagger$) holds.*

Proof (The proofs of (1) and (2) are the same except some additional description for (2), which are represented by square brackets.) Let $\varphi_1, \varphi_2, \ldots$ be an enumeration of all the formulas in $\mathcal{L}$. We define sets $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ so that each Γ_i is $\star$-consistent [and also having pure-$\forall$-$\star$-derivability], inductively as follows.

(Step 0) $\Gamma_0 = \Gamma$.

(Step k) Suppose Γ_{k-1} is already defined. We divide cases according to Γ_{k-1} and φ_k.

(Case 1) $\Gamma_{k-1} \cup \{\varphi_k\}$ is $\star$-consistent. In this case, $\Gamma_k = \Gamma_{k-1} \cup \{\varphi_k\}$.

(Case 2) $\Gamma_{k-1} \cup \{\varphi_k\}$ is $\star$-inconsistent. In this case, Lemma 5.1(1) guarantees that $\Gamma_{k-1} \cup \{\neg\varphi_k\}$ is $\star$-consistent. If $\varphi_k = \forall x.\psi$ for some x and ψ, then $\Gamma_k = \Gamma_{k-1} \cup \{\neg\varphi_k(= \neg\forall x.\psi), \neg\psi[a/x]\}$ for some $a \in \mathsf{IndPar}$ such that Γ_k is $\star$-consistent. The existence of such an individual parameter a is guaranteed by Lemma 5.1(2) or Lemma 5.3(1)(2). If $\varphi_k \neq \forall x.\psi$ for any x and ψ, then $\Gamma_k = \Gamma_{k-1} \cup \{\neg\varphi_k\}$.

[Lemma 5.3(1) guarantees that Γ_k has pure-$\forall$-$\star$-derivability.]

Then we define $\Delta = \bigcup_{i=0}^{\infty} \Gamma_i$. It is easy to show that Δ satisfies the required conditions. **QED**

We define the *canonical structure* $\mathcal{S}_c = \langle \mathcal{W}_c, \mathcal{R}_c, \mathcal{D}_c, \mathcal{Q}_c, \mathcal{I}_c \rangle$ and the *canonical valuation* $\mathcal{V}_c$ as follows.

- $\mathcal{W}_c = \{\Gamma \mid \Gamma$ is a set of formulas in $\mathcal{L}$ that is $\star$-consistet, maximal, and having $\forall$-property$\}$.
- $\Gamma \mathcal{R}_c \Delta \iff$ if $(\Box\varphi) \in \Gamma$ then $\varphi \in \Delta$, for any formula φ in $\mathcal{L}$.
- $\mathcal{D}_c = \mathsf{IndPar}$.
- $\mathcal{Q}_c = \{\mathcal{V}_c(X) \mid X \in \mathsf{PropVar}\}$, where $\mathcal{V}_c(X)$ is defined below.
- $\mathcal{I}_c(P, \Gamma)(a_1, \ldots, a_n) = \mathsf{true} \iff P(a_1, \ldots, a_n) \in \Gamma$.
- $\mathcal{I}_c(a) = a$.
- $\mathcal{V}_c(X) = \{\Gamma \in \mathcal{W}_c \mid X \in \Gamma\}$.

Note that $\mathcal{W}_c$ is nonempty because of the existence of a finite $\star$-consistent set (e.g., $\emptyset$) and Lemma 5.4(1).

Lemma 5.5 (Main lemma) *The canonical structure $\mathcal{S}_c$ and the canonical valuation $\mathcal{V}_c$ satisfy the property:*

$$\varphi \in \Gamma \textit{ if and only if } \Gamma \in [\![\varphi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}$$

for any formula φ in $\mathcal{L}$ and any set Γ in $\mathcal{W}_c$.

Proof By induction on the length of φ. We divide cases according to φ.

(Case 1) $\varphi = X$.

$$X \in \Gamma \iff \Gamma \in \mathcal{V}_c(X) \iff \Gamma \in [\![X]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}.$$

(Case 2) $\varphi = P(a_1, \ldots, a_n)$.

$$\begin{aligned} P(a_1, \ldots, a_n) \in \Gamma &\iff \mathcal{I}_c(P, \Gamma)(\mathcal{I}_c(a_1), \ldots, \mathcal{I}_c(a_n)) = \mathsf{true} \\ &\iff \Gamma \in [\![P(a_1, \ldots, a_n)]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}. \end{aligned}$$

(Case 3) $\varphi = \neg\psi$.

$$\begin{aligned} (\neg\psi) \in \Gamma &\iff \psi \notin \Gamma \quad \text{(Lemma 5.2(1))} \\ &\iff \Gamma \notin [\![\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c} \quad \text{(induction hypotheses)} \\ &\iff \Gamma \in [\![\neg\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}. \end{aligned}$$

(Case 4) $\varphi = \psi{\wedge}\rho$.

$$\begin{aligned}(\psi{\wedge}\rho) \in \Gamma &\iff \psi \in \Gamma \text{ and } \rho \in \Gamma \quad \text{(Lemma 5.2(2))}\\ &\iff \Gamma \in [\![\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c} \text{ and } \Gamma \in [\![\rho]\!]^{\mathcal{S}_c}_{\mathcal{V}_c} \quad \text{(induction hypotheses)}\\ &\iff \Gamma \in [\![\psi{\wedge}\rho]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}.\end{aligned}$$

(Case 5) $\varphi = \forall x.\psi$.

$$\begin{aligned}(\forall x.\psi) \in \Gamma &\iff \big(\psi[a/x] \in \Gamma\big) \text{ for any } a \in \mathsf{IndPar} \quad \text{(Lemma 5.2(3) and } \forall\text{-property)}\\ &\iff \big(\Gamma \in [\![\psi[a/x]]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}\big) \text{ for any } a \in \mathsf{IndPar} \quad \text{(induction hypotheses)}\\ &\iff \big(\Gamma \in [\![\psi[\dot{a}/x]]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}\big) \text{ for any } a \in \mathcal{D}_c \quad \text{(Lemma 3.1(1))}\\ &\iff \Gamma \in [\![\forall x.\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}.\end{aligned}$$

(Case 6) $\varphi = \Box\psi$. It is easy to show

$$(\Box\psi) \in \Gamma \implies \Gamma \in [\![\Box\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}$$

by the definition of $\mathcal{R}_c$ and the induction hypothesis. For the converse, it is enough to show

$$(\neg\Box\psi) \in \Gamma \implies \big(\Gamma\mathcal{R}_c\Delta \text{ and } \Delta \notin [\![\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}\big) \text{ for some } \Delta \in \mathcal{W}_c.$$

We define the set Π as

$$\Pi = \{\rho \mid \Box\rho \in \Gamma\} \cup \{\neg\psi\},$$

and suppose $(\neg\Box\psi) \in \Gamma$. Then Π is $\star$-consistent because otherwise $\Gamma^\star \vdash \Box\psi^\star$ (by the rule $(\Box)$) and Γ would be $\star$-inconsistent. Moreover, Π has pure-$\forall$-$\star$-derivability (by Lemma 5.3(1)(3)). Then we can apply Lemma 5.4(2) to Π, and we get the required set Δ. The fact $\Delta \notin [\![\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}$ is shown by the induction hypothesis.

(Case 7) $\varphi = \mu X.\psi$. We have the following.

$$\Gamma \in [\![\mu X.\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}$$

$$\iff \Gamma \in \bigcap\{\alpha \in \mathcal{Q}_c \mid [\![\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c[\alpha/X]} \subseteq \alpha\}$$

$$\iff \big(([\![\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c[\alpha/X]} \not\subseteq \alpha\big) \text{ or } \Gamma \in \alpha\big) \text{ for any } \alpha \in Q$$

$$\iff \big(([\![\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c[\mathcal{V}_c(\mathbb{X}_i)/X]} \not\subseteq \mathcal{V}_c(\mathbb{X}_i)) \text{ or } \Gamma \in \mathcal{V}_c(\mathbb{X}_i)\big) \text{ for any } i$$

$$\iff \Big(\big((\Delta \in [\![\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c[\mathcal{V}_c(\mathbb{X}_i)/X]} \text{ and } \Delta \notin \mathcal{V}_c(\mathbb{X}_i)) \text{ for some } \Delta \in \mathcal{W}_c\big) \text{ or } \Gamma \in \mathcal{V}_c(\mathbb{X}_i)\Big) \text{ for any } i$$

$$\iff \Big(\big((\Delta \in [\![\psi'[\mathbb{X}_i/X]]\!]^{\mathcal{S}_c}_{\mathcal{V}_c} \text{ and } \Delta \notin \mathcal{V}_c(\mathbb{X}_i)) \text{ for some } \Delta \in \mathcal{W}_c\big) \text{ or } \Gamma \in \mathcal{V}_c(\mathbb{X}_i)\Big) \text{ for any } i$$

(by Lemma 3.1(3) and Lemma 3.2, where ψ' is an α-variant of ψ such that the substitution $[\mathbb{X}_i/X]$ is safe for ψ')

$$\iff \Big(\big((\psi'[\mathbb{X}_i/X] \in \Delta \text{ and } \mathbb{X}_i \notin \Delta) \text{ for some } \Delta \in \mathcal{W}_c\big) \text{ or } \mathbb{X}_i \in \Gamma\Big) \text{ for any } i \qquad (\dagger)$$

(by the induction hypothesis for $\psi'[\mathbb{X}_i/X]$ and the definition of $\mathcal{V}_c(\mathbb{X}_i)$).

Using this equivalence, we prove

$$(\mu X.\psi) \in \Gamma \iff \Gamma \in [\![\mu X.\psi]\!]^{\mathcal{S}_c}_{\mathcal{V}_c}.$$

(Proof of $\Longrightarrow$) It is enough to show

$$\Big(\big(\{\mu X.\psi, \neg\mathbb{X}_i\} \subseteq \Gamma\big) \Longrightarrow \big(\{\psi'[\mathbb{X}_i/X], \neg\mathbb{X}_i\} \subseteq \Delta, \text{for some } \Delta \in \mathcal{W}_c\big)\Big) \text{ for any } i,$$

because of the above equivalence $(\dagger)$ and the fact that $\neg\mathbb{X}_i \in \Gamma(\text{or } \Delta) \iff \mathbb{X}_i \notin \Gamma(\text{or } \Delta)$; then, it is enough to show

$$\big(\{\mu X.\psi, \neg\mathbb{X}_i\} \subseteq \Gamma\big) \Longrightarrow \big(\{\psi'[\mathbb{X}_i/X], \neg\mathbb{X}_i\} \text{ is } \star\text{-consistent}\big) \qquad (\ddagger)$$

for any i, because the $\star$-consistent set $\{\psi'[\mathbb{X}_i/X], \neg\mathbb{X}_i\}$ can be extended to the set $\Delta \in \mathcal{W}_{\mathrm{c}}$ by Lemma 5.4(1). We show the contraposition of (‡):

$$\begin{aligned}
\{\psi'[\mathbb{X}_i/X], \neg\mathbb{X}_i\}^\star \vdash \bot \quad &\Longrightarrow \ \mathrm{Seq}_\mu \vdash \big((\psi'[\mathbb{X}_i/X])^\star \Rightarrow \mathbb{X}_i^\star\big) \ \Longrightarrow \ \mathrm{Seq}_\mu \vdash \big(\psi'^{\star'}[\mathbb{F}_i/X] \Rightarrow \mathbb{F}_i\big)\\
&\Longrightarrow \ \mathrm{Seq}_\mu \vdash \big(\mu X.(\psi'^{\star'}) \Rightarrow \mathbb{F}_i\big) \quad \text{(by the rule } (\mu \text{ Left}))\\
&\Longrightarrow \ \mathrm{Seq}_\mu \vdash \big((\mu X.\psi)^\star, (\neg\mathbb{X}_i)^\star \Rightarrow \bot\big) \quad \text{(by the rule } (\alpha\text{-variant}) \text{ and others)}\\
&\Longrightarrow \ \{\mu X.\psi,\ \neg\mathbb{X}_i\} \text{ is } \star\text{-inconsistent and cannot be a subset of } \Gamma;
\end{aligned}$$

where $\psi'^{\star'} = \psi'[\mathbb{F}_{n_1}/\mathbb{X}_{n_1}][\mathbb{F}_{n_2}/\mathbb{X}_{n_2}]\cdots[\mathbb{F}_{n_k}/\mathbb{X}_{n_k}]$ and $\{\mathbb{X}_{n_1}, \mathbb{X}_{n_2}, \ldots, \mathbb{X}_{n_k}\} = \mathrm{FPV}(\psi') \setminus \{X\}$.

(Proof of $\Longleftarrow$) Suppose $\Gamma \in [\![\mu X.\psi]\!]^{\mathcal{S}_{\mathrm{c}}}_{\mathcal{V}_{\mathrm{c}}}$. Since $(\mu X.\psi)^\star$ is a pure-formula, there exists a number j such that

$$(\mu X.\psi)^\star = \mathbb{F}_j = (\mathbb{X}_j)^\star.$$

Then, the above equivalence (†) guarantees that at least one of the following holds:

(1) $\big(\psi'[\mathbb{X}_j/X] \in \Delta$ and $\neg\mathbb{X}_j \in \Delta\big)$ for some $\Delta \in \mathcal{W}_{\mathrm{c}}$, where ψ' is an α-variant of ψ such that the substitution $[\mathbb{X}_j/X]$ is safe for ψ'.

(2) $\mathbb{X}_j \in \Gamma$.

However, the condition (1) cannot hold because $\{\psi'[\mathbb{X}_j/X], \neg\mathbb{X}_j\}$ is $\star$-inconsistent; this is shown by the following proof in Seq_μ :

$$\cfrac{\psi'^{\star'}[(\mu X.(\psi'^{\star'}))/X] \Rightarrow \psi'^{\star'}[(\mu X.(\psi'^{\star'}))/X]}{\psi'^{\star'}[(\mu X.(\psi'^{\star'}))/X] \Rightarrow \mu X.(\psi'^{\star'})}\ (\mu\ \text{Right})$$
$$\vdots\ (\alpha\text{-variant})$$
$$(\psi'[\mathbb{X}_j/X])^\star \Rightarrow (\mathbb{X}_j)^\star$$
$$\vdots$$
$$(\psi'[\mathbb{X}_j/X])^\star, (\neg\mathbb{X}_j)^\star \Rightarrow \bot$$

where $\psi'^{\star'} = \psi'[\mathbb{F}_{n_1}/\mathbb{X}_{n_1}][\mathbb{F}_{n_2}/\mathbb{X}_{n_2}]\cdots[\mathbb{F}_{n_k}/\mathbb{X}_{n_k}]$ and $\{\mathbb{X}_{n_1}, \mathbb{X}_{n_2}, \ldots, \mathbb{X}_{n_k}\} = \mathrm{FPV}(\psi')\setminus\{X\}$. Then the condition (2) holds, and we have $(\mu X.\psi) \in \Gamma$; otherwise $(\neg\mu X.\psi) \in \Gamma$ while $\{\mathbb{X}_j, \neg\mu X.\psi\}$ is $\star$-inconsistent. **QED**

Theorem 5.6 *The canonical structure $\mathcal{S}_{\mathrm{c}} = \langle \mathcal{W}_{\mathrm{c}}, \mathcal{R}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}, \mathcal{Q}_{\mathrm{c}}, \mathcal{I}_{\mathrm{c}}\rangle$ and the canonical valuation $\mathcal{V}_{\mathrm{c}}$ form a general μ-model; that is to say, $[\![\varphi]\!]^{\mathcal{S}_{\mathrm{c}}}_{\mathcal{V}_{\mathrm{c}}} \in \mathcal{Q}_{\mathrm{c}}$ for any formula φ in $\mathcal{L}$.*

Proof For any φ, there is a number i such that

$$\varphi^\star = \mathbb{F}_i = (\mathbb{X}_i)^\star.$$

Then we have

$$[\![\varphi]\!]^{\mathcal{S}_{\mathrm{c}}}_{\mathcal{V}_{\mathrm{c}}} = \mathcal{V}_{\mathrm{c}}(\mathbb{X}_i) \in \mathcal{Q}_{\mathrm{c}}$$

by Lemma 5.5, Lemma 5.2(1), and the fact that $\{\varphi, \neg\mathbb{F}_i\}$ and $\{\neg\varphi, \mathbb{F}_i\}$ are $\star$-inconsistent. **QED**

Now we can show the completeness of the axiom system of FOMμ.

Theorem 5.7 (Completeness for pure-formulas) *Let φ be an arbitrary pure-formula in $\mathcal{L}$. If φ is valid, then $\vdash \varphi$.*

Proof If φ is valid, then $[\![\varphi]\!]^{\mathcal{S}_{\mathrm{c}}}_{\mathcal{V}_{\mathrm{c}}} = \mathcal{W}_{\mathrm{c}}$ for the canonical model $\langle \mathcal{S}_{\mathrm{c}}, \mathcal{V}_{\mathrm{c}}\rangle$. Then we have $\vdash \varphi$; otherwise, the set $\{\neg\varphi\}$ is $\star$-consistent (note that $\varphi^\star = \varphi$) and Lemma 5.4(1) implies the existence of a set $\Gamma \in \mathcal{W}_{\mathrm{c}}$ such that $\neg\varphi \in \Gamma$—this means $[\![\varphi]\!]^{\mathcal{S}_{\mathrm{c}}}_{\mathcal{V}_{\mathrm{c}}} \neq \mathcal{W}_{\mathrm{c}}$ by Lemma 5.5. **QED**

Theorem 5.8 (Completeness) *Let φ be an arbitrary formula in $\mathcal{L}$. If φ is valid, then $\vdash \varphi$.*

Proof Suppose $\mathrm{FPV}(\varphi) = \{X_1, X_2, \ldots, X_n\}$. We change the definitions of pure-formulas and of $\psi^\star$ by considering $X_1, X_2, \ldots, X_n$ to be 0-ary predicate symbols. Then, we carry out all the arguments of this section that lead Theorem 5.7. This brings the completeness for φ. **QED**

References

[1] George S. Boolos, John P. Burgess, and Richard C. Jeffrey, **Computability and Logic, Fourth edition**, (Cambridge UP, 2003).

[2] Julian Bradfield and Colin Stirling, *Modal mu-calculi*, in **Handbook of Modal Logic** (edited by P.Blackburn, J. Van Benthem, and F. Wolter, Elsevier, 2007).

[3] Dirk van Dalen, **Logic and Structure, Third edition**, (Springer, 1997).

[4] Melvin Fitting, *Modal Proof Theory*, in **Handbook of Modal Logic** (edited by P.Blackburn, J. Van Benthem, and F. Wolter, Elsevier, 2007).

[5] Keishi Okamoto, *A First-Order Extension of Modal μ-calculus*, **AIST CVS Technical Report**, (2006).

[6] Igor Walukiewicz, *Completeness of Kozen's Axiomatization of the Propositional μ-Calculus*, **Information and Computation 157**, 142-182 (2000).